Black Hole Data via a Kerr-Schild Approach

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Abstract

We present a new approach for setting initial Cauchy data for multiple black hole spacetimes. The method is based upon adopting an initially Kerr-Schild form of the metric. In the case of non-spinning holes, the constraint equations take a simple hierarchical form which is amenable to direct numerical integration. The feasibility of this approach is demonstrated by solving analytically the problem of initial data in a perturbed Schwarzschild geometry.

04.20Ex, 04.25Dm, 04.25Nx, 04.70Bw
I. INTRODUCTION

The calculation of gravitational radiation from the inspiral of binary black holes is of paramount importance to the development of gravitational wave detection into a new form of astronomy. The strong field regime of the emission process occurring during the merger of two black holes can only be treated by numerical evolution. However, in the near future it is unclear whether computing power and techniques will be adequate to track the two black holes through many orbits, beginning with a quasi-Newtonian orbit which can be approximated by perturbation methods and ending with their merger into a single black hole. Instead, the merger radiation may have to be explored beginning with initial data in a non-perturbative setting just prior to the final orbital plunge. Numerical codes are presently being developed which promise to provide an accurate map between such initial data and the emitted gravitational waveform. However, the extent to which the merger waveform will have a characteristic signature independent of the details of the initial data is unknown, and there is some evidence that the assumptions that have to be made in the usual approach lead to data that is not physically realistic [1]. For this reason it is imperative that many different forms of black hole initial data be available.

Results on multiple black hole initial data were presented by Misner in 1963 [2]. As is well known he found an analytic solution for the special case of a hypersurface of time-symmetry. In this case the extrinsic curvature is zero so the momentum constraints are satisfied identically; and, assuming conformal flatness of the initial 3-geometry, the Hamiltonian constraint reduces to Laplace’s equation for the conformal factor. Subsequent work has been essentially a generalization of Misner’s result. The first step in treating the non-time-symmetric case was the development of the theory of conformal decomposition [3]. This then led to methods for determining initial data for one black hole with specified linear and angular momentum [4]. A procedure for “summing” such solutions has been described [5], and the procedure has been implemented numerically [6,7]. Recently, a modified version of this procedure has been formulated which avoids complications at the inner boundary for this elliptic problem by compactifying the region interior to the black holes [8]. However, only conformally flat geometries with trace-free extrinsic curvatures can be readily obtained by this technique.

Here we describe the generation of initial Cauchy data by means of a new approach, based upon adoption of an instantaneously Kerr-Schild form of the metric on the initial time slice. In this approach the time-symmetric Misner data is not expressible as a simple special case. For a single non-spinning vacuum black hole at rest, it leads to Schwarzschild initial data in Eddington-Finklestein coordinates. In the numerical treatment of the non-vacuum spherically symmetric gravitational collapse of a scalar field, the most stable Cauchy evolution algorithms for the inner boundary at the apparent horizon have been based upon a generalization of ingoing Eddington-Finklestein coordinates [11]. An analogous gauge is likely to be beneficial to stability in the two black hole case. Since Eddington-Finklestein coordinates are closely related to Kerr-Schild coordinates, the initial data generated by our approach offers the additional advantage of allowing greater flexibility in the choice of gauge conditions on the lapse and shift in the search for stable numerical algorithms for the simulation of black holes. Similar motivations have led to another independent formulation of the initial data problem which reduces to the Kerr-Schild form for the Schwarzschild or
Kerr cases [12].

The framework used is that of an instantaneously Kerr-Schild space-time, which is defined and discussed in section II. The corresponding initial data is more general than for exact Kerr-Schild space-times, and the constraint equations are derived from first principles in section III. A numerical procedure for solving the constraint equations is described in section IV; also, we show that a simple algebraic condition on the metric defines the location of the apparent horizon. We do not, in this paper, solve the numerical problem; but we show in section V that our solution procedure is feasible by formulating, and then solving analytically, the Schwarzschild perturbation problem corresponding to the “close approximation” for two black holes [13]. We conclude in section VI with comments on the future development of this research.

II. INSTANTANEOUSLY KERR-SCHILD SPACE-TIMES

In the 1960s the work of Trautman [14], and of Kerr and Schild [9,10], led to the development of the theory of Kerr-Schild geometries; the theory is comprehensively reviewed in the book by D. Kramer et al [15]. Here we use notation such that Greek indices are space-time indices and Latin indices are spatial. We will often use a dot to denote a time derivative, e.g. \( \dot{V} = \frac{\partial}{\partial t} V \).

Let the scalar field \( V \) and the null vector field \( k_\alpha \) define a Kerr-Schild metric

\[
g_{\alpha\beta} = \eta_{\alpha\beta} - 2V k_\alpha k_\beta, \tag{2.1}
\]

where \( k_\alpha = (-1, k_i) \) is null. We choose the background to be a standard inertial coordinate system with space-time foliated by time slices \( t = \text{const} \) and the Minkowski metric given by \( \eta_{\alpha\beta} = \text{diag}(-1,1,1,1) \). The contravariant null vector satisfies \( k^\alpha = g^{\alpha\beta} k_\beta = \eta^{\alpha\beta} k_\beta \). For the spatial components we write \( k_\alpha = (1, k_i) \) so that \( k^i = \delta^{ij} k_j \). As an example consider the Schwarzschild geometry. In ingoing Kerr-Schild form it is

\[
V = -\frac{M}{r}, \quad k_i dx^i = -dr. \tag{2.2}
\]

Another example is the Kerr solution, which is rather long [16].

We wish to pose black hole Cauchy data at \( t = 0 \). We only require that initially the geometry be of the Kerr-Schild type. In particular, we assume that the evolution of the initial data leads to a metric of the form \( g_{\alpha\beta} + t^2 j_{\alpha\beta} \), with \( j_{ab} \) well behaved at \( t = 0 \) and \( g_{\alpha\beta} \) given by equation (2.1). The initial data at \( t = 0 \) must satisfy the constraint equations \( G_{ab} n^b = 0 \), where \( n^a \) is the unit normal to the foliation. These equations do not contain second time derivatives so that \( j_{ab} \) does not enter and we can analyze them assuming a pure Kerr-Schild form, i.e. setting \( j_{ab} = 0 \). In a \( 3 + 1 \) decomposition, these constraints become spatial equations for the initial values of \( V, k_i \) and their first time derivative, which determine the initial values of the 3-geometry and extrinsic curvature. For the family of global vacuum Kerr-Schild solutions, the null vector field \( k_\alpha \) is necessarily geodesic and shear-free. The shear-free property makes it unlikely that a globally Kerr-Schild space-time can contain more than one black hole. Here, for our instantaneously Kerr-Schild spacetimes, we impose no differential conditions on \( k_\alpha \) except those that follow from the null condition, \( \partial_\alpha (k^i k_i) = 0 \).
III. THE CONSTRAINT EQUATIONS

For completeness, we present the equations in covariant form, and also in “3 + 1” formalism with partial derivatives.

A. The covariant constraint equations

Space-time indices take on the values $\alpha, \beta, \gamma, \delta, \ldots = 0, 1, 2, 3$. The symmetric part of a tensor is $T_{(\alpha\beta)} \equiv \frac{1}{2} (T_{\alpha\beta} + T_{\beta\alpha})$, and the antisymmetric part is $T_{[\alpha\beta]} \equiv \frac{1}{2} (T_{\alpha\beta} - T_{\beta\alpha})$. The operator for the derivative along the null congruence $k^\alpha$ is $D \equiv \nabla^\alpha k^\alpha$. The acceleration vector of the null congruence $k^\alpha$ is $a^\alpha = k^\alpha,\beta k^\beta = k^\gamma k^\delta = Dk^\alpha$. The square of its norm is $a^2 \equiv a^\alpha a_\alpha$, while the twist of the congruence, $\omega$, is defined by $\omega^2 \equiv \frac{1}{2} k_{[\alpha\beta]} k^{\alpha\beta}$. We may compute the Ricci tensor in Euclidean coordinates explicitly in terms of ordinary derivatives as

$$R_{\beta\delta} = L_{\beta\delta} + 2V[(DV)k^\alpha_\alpha - 2V_a a^\alpha + D^2V + 8V^2 \omega^2]k_{\beta\delta}$$

$$+ 4V[2(DV) + V k^\alpha_\alpha]a_{(\beta\delta)} + 2V^2 a_{\beta\delta}$$

$$+ 4V^2 k_{(\beta\delta),\alpha\gamma} k^{\alpha\gamma},$$

(3.1)

where

$$L_{\beta\delta} = \frac{1}{2} \eta^{\alpha\gamma} [g_{\alpha\delta,\beta} + g_{\alpha\beta,\delta} - g_{\beta\delta,\alpha}]_{,\gamma}$$

$$= \eta^{\alpha\gamma} (V k_{\beta\delta},\alpha\gamma) - (V k^\beta k^\delta),\delta\gamma - (V k^\gamma k^\delta),\beta\gamma$$

(3.2)

and $D^2V = V_{,\alpha\beta} k^\alpha k^\beta + V_{,\delta} a^\delta$. The Ricci scalar is

$$R = -2(V k^\alpha k^\beta)_{,\alpha\beta} + 2V^2 a^2.$$  

(3.3)

The Ricci tensor may also be put in covariant form

$$R_{\beta\delta} = (V k_{\beta\delta})_{\alpha\gamma} - (V k^\alpha k_{\beta\delta})_{,\alpha\gamma} - (V k^\alpha k_{\beta\delta})_{,\alpha\delta}$$

$$+ 2V[D^2V + (V a^\alpha)_{,\alpha} - a^2V^2]k_{\beta\delta} + [2V(DV) - V^2 k^\alpha_{,\alpha}]D(k_{\beta\delta})$$

$$+ V^2 D^2 (k_{\beta\delta}) + 2V^2 a^\alpha k_{\alpha(\beta\delta)}$$

and the covariant Ricci scalar is

$$R = -2(V k^\alpha k^\beta)_{,\alpha\beta} + 2V^2 a^2.$$  

(3.4)

Using the expressions for the Ricci tensor and Ricci scalar above we may easily write out the Einstein tensor $G_{\beta\delta} = R_{\beta\delta} - \frac{1}{2} R g_{\beta\delta}$. The formulae in this section were calculated by hand and checked using the computer algebra system MathTensor.
B. The constraint equations in “3 + 1” form

It is straightforward to show that the 3-metric $\gamma_{ij}$, the lapse $\alpha$ and the shift $\beta_i$ take the following forms:

$$\gamma_{ij} = \delta_{ij} - 2V k_i k_j, \quad \alpha = \frac{1}{\sqrt{1 - 2V}}, \quad \beta_i = 2V k_i.$$  \hfill (3.5)

The ADM constraint equations are found from the extrinsic curvature $K_{ij}$, which is defined by

$$K_{ij} = \frac{1}{2\alpha} (\beta_{ij} + \beta_{ji} - \dot{\gamma}_{ij}),$$ \hfill (3.6)

where $|$ denotes covariant differentiation with respect to the metric $\gamma_{ij}$. The standard form of the ADM constraints is

$$(^3)R + (K^i_i)^2 - K^i_j K^j_i = 2\alpha^2 G^{00} = 0, \quad K^i_{ij} - (K^j_j)_i = \alpha G^0_j = 0,$$ \hfill (3.7)

where the first equation is known as the Hamiltonian constraint, and the second equation is the momentum constraint; $G^{00}$ and $G^0_j$ are components of the 4-dimensional Einstein tensor. In equation (3.7) indices are raised by the metric $\gamma^{ij}$, but we will not be using this convention below.

We have carried out an algebraic computation using REDUCE to evaluate the constraints for the instantaneous Kerr–Schild metric. Before we give the results, which are obtained after some simplification of the REDUCE output, some general comments are in order. The calculation is carried out in the Euclidean background metric where the components of covariant and contravariant vectors are identical. Thus it is possible to write all indices as suffices, and we choose to do this because it greatly simplifies the computer algebra. A repeated latin suffix means summation with respect to the metric $\delta^{ij}$. These conventions will also apply throughout the rest of this section, and in section IV A. Some simplification rules were used in the computer algebra, explicitly $k_i k_i = 1$, $k_i k_{ij} = 0$ and $k_i \dot{k}_i = 0$. The quantity $z_i$ is defined to be a vector orthogonal to $k_i$, i.e. $z_i k_i = 0$; thus $z_i$ is not uniquely defined but instead lies in a 2-D surface. The analogue of the 4-acceleration $a^\alpha$ is denoted by $A_j$ defined by $A_j = k_j, i k_i$.

The momentum constraint quantity $\alpha G^0_j k_j$ is denoted by $m_k$, and is

$$0 = m_k = \frac{-\sqrt{-2V + 1}}{2V - 1} \left[ k_{i,j} k_i V - \dot{k}_i A_i V - 2A_{j,i} V^2 - k_{j,i} k_i V - 2\dot{k}_{j,j} V^2 + \dot{k}_{j,j} V \\
-V_{i,j} k_i k_j + V_{j,i} + \dot{V} k_{j,i} - (4V + 1)V_j A_j + 2A_j A_j V^2 - k_{j,j} V_i k_i \\
-\dot{k}_j((4V - 1)V_j - 2A_j V^2) \right].$$ \hfill (3.8)

In order to eliminate $\dot{V}$, the Hamiltonian constraint is not expressed in the standard ADM form. Rather, we use $G^0_{0j}$, which is
\[ G_{00} = -\alpha^2 G_{00} + \alpha 2 V m_k. \] (3.9)

Denoting \( G_{00} \) by \( h_2 \), we find

\[
0 = h_2 = \left[ \frac{-1}{2(2V - 1)^2} \right] (V_i A_i + 2k_{ij} k_{ji} V - 8 A_{jj} V^3 + 4 k_{ij} k_{ij} V^2 - 4 k_{i} A_i V^2 + 4 k_{ij} k_{ij} V^2 ) (2V - 1) \\
+2(2V - 1)^2 - V_i V_j k_{ij} V - \left\{ 2 \left[ (k_{ij} V + 2V_i k_i) k_{ij} - (k_{ij} k_{ij} V + 1) - V_{jj} + V_{ij} k_i k_j \right] + 2 k_{ij} V^2 + k_{ij} k_{ij} V^2 \right\} \\
+2(4V - 1)(V - 1) A_j A_j V^2 - (32V^3 - 16V^2 - 8V + 3) V_j A_j \\
-4(4V_j V - 2V_j - A_j V)(2V - 1) k_j V \right]. \] (3.10)

The momentum constraint quantity \( \alpha G_{j}^0 z_j \) is denoted by \( m_z \), and is

\[
0 = m_z = \frac{\sqrt{(-2V + 1)}}{(2V - 1)^2} \left[ \left( V_i k_{j,i} - \dot{V} A_i + \dot{V}_i \\
+ k_{ij} V - k_{ii,j} k_{ij} - (k_{j,j} V + V_j k_j)(2V - 1) \dot{k}_i \right) (2V - 1) \\
+2(V_j k_j) V - 2(V_j k_j) V + V_{ij} k_j - (k_{j,j} V + V_j k_j)(2V - 1) k_i \\
-/[2(V_j - A_j V^2) - (2V - 1) k_j V ](2V - 1) k_{i,j} \\
-/[2(k_j + A_j)(2V - 1) V^2 - V_j k_{j,i} \right] z_i. \] (3.11)

**IV. ON THE SOLUTION OF THE CONSTRAINT EQUATIONS**

**A. The surface orthogonal case**

In the case of a black hole with spin, such as a Kerr black hole, \( k_i \) is proportional to \( \nabla_i \phi + c_i \) with \( c_i \) tangent to the surfaces given by \( \phi = \text{const} \). While the analysis of the solution of the constraint equations in this general case is quite complicated it simplifies drastically if \( c_i = 0 \) so that \( k_i \) is orthogonal to 2-surfaces \( S \), with

\[
k_i = \nabla_i \phi / |\nabla \phi|. \] (4.1)
The constraints may then be solved by carrying out a hierarchy of integrals along the rays normal to the surfaces $S$. (Note that difficulties may arise at points where $\nabla_i \phi = 0$.) Here, we confine our attention to this surface orthogonal case. It includes the description of boosted black holes.

Formally, the ansatz used is that $\phi$ is given everywhere on a 3-D manifold with Euclidean metric $\delta_{ij}$. The vector field $k_i$ is defined by equation (4.1), and is everywhere orthogonal to 2-surfaces $S$ (which are equipotentials of $\phi$). Also, $V$ and $\dot{k}_i$ are given on a particular equipotential $S_0$. The solution procedure assumes that $k_i$ is defined everywhere, and also that $k_{i,i} \neq 0$ everywhere. In multiple black hole problems it is likely that these conditions could be violated at isolated points, and special procedures would need to be developed for integrating through such points.

The equations $h_2 = 0$, $m_k = 0$ and $m_z = 0$ appear complicated, but it should be remembered that we know terms involving $k_i$ and any spatial derivative, e.g. $A_{i,i}$. In addition, we know terms involving $V$, $\dot{k}_i$ and any spatial derivative within $S_0$. In particular, the following terms are known on $S_0$ from the given data:

- $V_i A_i$ and $V_j k_j$. Since both $A_i$ and $\dot{k}_j$ are orthogonal to $k_i$ they both lie in $S_0$, and therefore the $V$-derivative in both terms is calculable from data on $S_0$.
- $V_{ij} - V_{ji} k_i k_j$ and $-k_j V_j k_i V_i + (V_j)^2$. In both cases the off-$S_0$ derivatives cancel out, and so the terms are calculable from data on $S_0$.
- $\dot{k}_i,i$. Since $\dot{k}_i$ is orthogonal to $k_i$, it follows that $\dot{k}_{i,i}$ can be expressed in terms of derivatives of $\dot{k}_i$ within $S_0$.

Equation (3.10) ($h_2 = 0$) is a linear equation for $k_i V_i$, and may be written schematically as

$$\partial_k V = \mathcal{O}_1 (D_1),$$

(4.2)

where $\mathcal{O}$ represents an operator on $S_0$, and $D$ represents the data given on $S_0$; $\partial_k$ means $k_i \partial_i$. Next, $\dot{V}$ is found explicitly from equation (3.8) ($m_k = 0$):

$$\dot{V} = \mathcal{O}_2 (D_2, \partial_k V).$$

(4.3)

Finally, equation (3.11) ($m_z = 0$) is regarded as a linear equation to determine $\dot{k}_{i,j} k_j z_i$ (Noting that $z_i \partial_i$ means a derivative within $S_0$, it is straightforward to see that all terms are expressible entirely in terms of known data):

$$z_i \partial_k \dot{k}_i = \mathcal{O}_3 (D_3, \partial_k V, \dot{V}).$$

(4.4)

Because $z_i$ can take two independent directions, equation (4.4) gives two components of $\dot{k}_{i,j} k_j$. The remaining component follows from the condition that $\dot{k}_i k_i = 0$ so that $\dot{k}_{i,k_i} = -k_{i,j} k_{j,i}$, and therefore $\dot{k}_i k_j k_i = -k_{i,j} k_{j,i}$.

In this way $\dot{V}$ and the $\partial_k$ derivatives of $V$ and $\dot{k}_j$ are found at each point of $S_0$. Thus the solution may be extended to the “next” 2-surface $S$, and, in the absence of singularities, to the whole manifold.
B. Multiple black holes

For a single (unboosted) Schwarzschild black hole, \( k_i \) is a radial field which can be obtained by the equipotentials of the function \( \phi = 1/r \). Thus the choice \( \phi = \sum_i M_i/r_i \) generates Schwarzschild-like equipotentials about each site where \( r_i = 0 \) and becomes a candidate for data for (unboosted) multiple black holes with masses \( M_i \). By giving appropriate data for \( V \) on \( S_0 \) we shall show that this does produce the necessary apparent horizons.

In order to describe boosted black holes we begin with the null vector \( k_\alpha = -\nabla_\alpha (t + r) \) for a single Schwarzschild black hole. Under a boost with velocity \( v \) in the \( z \)-direction, \( t \rightarrow (t + vz)/\sqrt{1 - v^2} \) and \( z \rightarrow (z + vt)/\sqrt{1 - v^2} \), we have

\[
72x644 \rightarrow \frac{t + vz + \sqrt{(1 - v^2)(x^2 + y^2)} + (z + vt)^2}{\sqrt{1 - v^2}}.
\]

Then, for a single boosted Schwarzschild black hole, initial data for \( k_i \) at \( t = 0 \) can be obtained from the equipotentials of \( \hat{\phi} = \{ vz + [(1 - v^2)(x^2 + y^2) + z^2]^{1/2} \}^{-1} \)

Construction of a solution corresponding to multiple black holes requires a numerical integration procedure. For simplicity, we restrict our discussion to an analysis of the binary case \( \phi = M_1/r_1 + M_2/r_2 \), although the above considerations make it apparent how to generalize to the case of multiple, boosted black holes. In this simple case, the equipotentials consist of small disjoint (topological) spheres which merge in a “figure-eight” to form a common set of outer spheres. The hard problems arise at the bifurcation point of the “figure-eight”. We could avoid this problem by considering the case where the two holes are close so that they are surrounded by a common apparent horizon. This would suffice to determine data for the exterior spacetime without going down to the bifurcation point. The problem with doing this is that we would probably not be able to include the merger radiation this way and that is the most interesting feature. Nevertheless, it seems advisable to tackle the close binary case first.

C. Apparent horizons

In order to make sure that the data proposed above really represents black hole data we must demonstrate the existence of apparent horizons. Consider the equipotential 2-surfaces \( S \) given by \( \phi = \text{const} \). For the reasons given in the introduction, we are interested in ingoing Kerr-Schild coordinates so that \( k_\alpha \) is chosen to be the incoming null vector normal to \( S \). The divergence of \( k_\alpha \) is the same when measured either in the full Kerr-Schild metric or in the background Minkowski metric. Thus it is apparent from the asymptotically spherical shape of the equipotentials on the approach to each \( r_i = 0 \) site that \( \rho_{\text{in}} \) (the incoming divergence of \( S \)) goes uniformly negative.

The outgoing null normal to \( S \) is given by \( \ell_\alpha = 2t_\alpha - (1 + 2V)k_\alpha \), where \( t_\alpha = (1, 0, 0, 0) \) is the time translation Killing vector of the Minkowski background. It has normalization \( \ell_\alpha k_\alpha = -2 \). Let \( (\ell_\alpha, k_\alpha, m_\alpha, \bar{m}_\alpha) \) form a null tetrad. We choose normalization \( m_\alpha \bar{m}_\alpha = 2 \) (with respect to both the Kerr-Schild and Minkowski metrics). The complex spatial vector \( m_\alpha \) is tangent to \( S \). The outward divergence of \( S \) is given by \( \rho_{\text{out}} = m^\alpha \bar{m}_\beta \nabla_\alpha \ell_\beta \). We have
\[ 2m(\alpha \bar{m} \beta) \nabla_\alpha t_\beta = m(\alpha \bar{m} \beta) \mathcal{L}_t g_{\alpha \beta} = m(\alpha \bar{m} \beta) \mathcal{L}_\eta g_{\alpha \beta} = 0, \quad (4.6) \]

since \( t^\alpha \) is a Minkowski Killing vector. As a result, \( \rho_{\text{out}} = -(1 + 2V) \rho_{\text{in}}. \)

Where \( |V| \) is small, \( \rho_{\text{in}} \) is negative and \( \rho_{\text{out}} \) is positive, as for a sphere in Minkowski space. For the Schwarzschild solution \( V = -M/r \) and the (apparent) horizon forms at \( V = -1/2 \). Our result shows this holds in general: An equipotential with \( V = -1/2 \) is marginally trapped and determines the position of the apparent horizon on the initial time slice.

Thus, since \( V \) is free data on some choice of equipotential \( S_0 \) (which might consist of disjoint pieces), it is straightforward to give data that guarantees the existence of black holes. Note that the location of apparent horizons is considerably simpler here than in the standard approach. This suggests that the apparent horizon itself be chosen as the inner equipotential to start the integration process.

**V. PERTURBATIVE CALCULATION**

We use the above methods to find initial data for a *surface orthogonal* problem, specifically a perturbed Schwarzschild geometry.

**A. The close approximation**

As described in Sec. IVB, two non-spinning black holes initially at rest can be modeled in terms of the potential \( \phi = M_1/r_1 + M_2/r_2 \), where \( r_1 = |x^i - x_i^1| \) and \( r_2 = |x^i - x_i^2| \). Without loss of generality, we can take the line connecting the black holes to be the \( z \)-axis, with the origin located according to the center of mass condition \( M_1 z_1 + M_2 z_2 = 0 \). We let \( a = z_2 - z_1 \) and \( M = M_1 + M_2 \), and use standard spherical polar coordinates \((r, \theta, \varphi)\). The close approximation is defined by the condition

\[ \epsilon = \frac{a}{M} \ll 1. \quad (5.1) \]

We find the expansion

\[ \phi = \frac{M}{r} + \frac{\epsilon^2 M_1 M_2 M P_2}{r^3} \quad (5.2) \]

with \( O(\epsilon^3) \) remainder, where

\[ P_2 = (3 \cos^2 \theta - 1)/2. \quad (5.3) \]

The approximation is valid when \( r > \max(z_1, z_2) \), and improves with increasing \( r \). To this order of approximation, the covariant components of \( k \) in standard spherical coordinates are

\[ (k_r, k_\theta, k_\varphi) = (-1, -\frac{3\epsilon^2 M_1 M_2 \cos \theta \sin \theta}{r}, 0). \quad (5.4) \]

The ansatz on the initial hypersurface \( t = 0 \) is then equation (5.4) above, together with
\[ V = -M/r + \epsilon^2 v(r)P_2 \quad (5.5) \]

\[ \dot{V} = \epsilon^2 v_T(r)P_2 \quad (5.6) \]

\[(\dot{k}_r, \dot{k}_\theta, \dot{k}_\phi) = (0, 3\epsilon^2 k_T(r) \cos \theta \sin \theta, 0). \quad (5.7)\]

The functions \(v_T(r)\) and \(k_T(r)\) are, in a 4-dimensional sense, time derivatives; but since the whole analysis is carried out in a spacelike 3-D manifold, formally we regard \(v_T(r)\) and \(k_T(r)\) as independent functions on the manifold rather than as derivatives. The objective now is to determine the functions \(v(r), v_T(r)\) and \(k_T(r)\).

### B. The algebraic computations

The computation of the constraint equations has been done in two different ways, and using two different algebra systems (Maple and REDUCE). In one calculation we put the ansatz above into the equations reported in section III B, and in the other calculation the ansatz is put into the full 4-dimensional Einstein equations and \(G^0_\alpha\) is found. Both procedures give identical results, and this serves as a partial confirmation that our computer algebra programs are correct.

The constraint equations \(m_k = 0, \ h_2 = 0\) and \(m_z = 0\) were evaluated, and to order zero in \(\epsilon\) they were satisfied identically; and to order \(\epsilon^2\) the angular terms cancelled out and the equations involved \(r\) only. After some manipulation we find:

\[ v_T(r) = -\frac{1}{r^3} \left[ 6k_T(r)M^2 + 3Mrk_T(r) + \frac{6M_1M_2M}{r^2} \{r - M\} + 3v(r)r^2 \right], \quad (5.8) \]

\[ v'(r) = -\frac{1}{r^3} \left[ 4v(r)r^2 + \frac{6M_1M_2M}{r^2} \{r - M\} + 6k_T(r)M^2 \right], \quad (5.9) \]

\[ k'_T(r) = \frac{1}{Mr^2} \left[ 2Mrk_T(r) - \frac{6M_1M_2M}{r} - v(r)r^2 \right], \quad (5.10) \]

where \(\dot{}\) represents \(\frac{d}{dr}\). Equations (5.9) and (5.10) constitute a system of two first-order equations for \(v'(r)\) and \(k'_T(r)\); once they have been solved \(v_T(r)\) is found explicitly from equation (5.8). The system (5.9) and (5.10) may be re-expressed as one second-order equation:

\[ v''(r)r^5 + 5r^4v'(r) - 6r^2Mv(r) - \frac{6M_1M_2M}{r} \{3r + 2M\} = 0. \quad (5.11) \]
C. Behavior of the solution

In this section we are concerned with finding the behavior of the solution of equation (5.11). Making the transformation
\[
r = \frac{24M}{x^2}, \quad v(r) = x^4 u(x),
\]
equation (5.11) becomes
\[
x^2 u_{,xx} + xu_{,x} - u(x^2 + 16) - \frac{M_1 M_2}{6912 M^2} (36x^2 + x^4) = 0.
\]
(5.13)

The solution to equation (5.13) is
\[
 u(x) = C_1 I_4(x) + C_2 Y_4(ix) + \frac{M_1 M_2}{M^2} (w(x) - \frac{x^2}{2304} - \frac{I_4(x) \log x}{72}),
\]
(5.14)
where \(I_4\) and \(Y_4\) are standard Bessel functions \([17]\) and \(i = \sqrt{-1}\). The function \(w(x)\) may be written as
\[
w(x) = w_6 x^6 + w_8 x^8 + w_{10} x^{10} + \ldots
\]
(5.15)
and the coefficients in the series are found from the following recurrence relation applied to \(n = 6, 8, 10, \ldots\), and using the initial condition \(w_4 = 0\):
\[
(n^2 - 16) w_n = w_{n-2} + \frac{n}{36(n^2 - 2)!} \left( \frac{n}{2} + 2 \right)! 2^n.
\]
(5.16)

Using equation (5.12) we may now find \(v(r)\); then equations (5.9) and (5.8) lead to \(k_T(r)\) and \(v_T(r)\).

The leading term in the inhomogeneous part of the solution (5.14) behaves as \(x^2\) as \(x \to 0\), which means that \(v(r)\) would behave as \(r^{-3}\) as \(r \to \infty\); this is the expected quadrupole behavior at infinity. The constants \(C_1\) and \(C_2\) represent gauge freedom, as is evident from the fact that they persist in the Schwarzschild case \((M_1 = M \text{ and } M_2 = 0)\). \(Y_4\) behaves as \(x^{-4}\) near \(x = 0\), so that if \(C_2 \neq 0\) then \(v(r)\) would behave as a constant as \(r \to \infty\). While this is possible, equation (5.9) then indicates that \(k_T(r)\) would be singular as \(r \to \infty\). Thus it is convenient to fix \(C_2 = 0\). \(I_4\) behaves as \(x^4\) near \(x = 0\), so that \(v(r)\) would behave as \(r^{-4}\) as \(r \to \infty\). \(C_1\) could be chosen so as to fix the position of the apparent horizon: for example, \(C_1\) could be chosen so that \(v(2M) = 0\), which would mean that the position of the apparent horizon is not perturbed relative to the Schwarzschild case.

VI. CONCLUSION

In this paper we have shown how to express the initial value problem of general relativity in a way that can be solved using initial data that is instantaneously Kerr-Schild. We have used this approach to formulate, and solve analytically, the initial value problem of a perturbed Schwarzschild geometry. It may be interesting to see if the difference between
this initial data and that obtained from the conformally flat method affects results about the perturbative evolution of a black hole [13]. We have shown that the apparent horizon is described in a very simple way, \( V = -\frac{1}{2} \); thus the generation of initial data for black holes by the Kerr-Schild approach would automatically give the initial location of the apparent horizon.

Clearly the Kerr-Schild approach to the initial value problem is very much in an early stage of development, particularly in comparison with the conformally flat method that was introduced in 1963 [2]. This paper is just a first step towards developing this new approach. The next steps, which are currently under investigation, will be the construction of a code to solve numerically equations (3.8), (3.10) and (3.11), and the generalization of our approach to be able to include problems with spin, where \( k_i \) is not orthogonal to the 2-surfaces \( S \).

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