Black-Scholes equation from Gauge Theory of Arbitrage

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Abstract

We apply Gauge Theory of Arbitrage (GTA) [1] to derivative pricing. We show how the standard results of Black-Scholes analysis appear from GTA and derive correction to the Black-Scholes equation due to a virtual arbitrage and speculators’ reaction on it. The model accounts for both violation of the no-arbitrage constraint and non-Brownian price walks which resemble real financial data. The correction is nonlocal and transforms the differential Black-Scholes equation to an integro-differential one.

1 Introduction

The Black-Scholes equation [2] for prices of derivative financial instruments is probably the most beautiful rigorous result in the theory of financial analysis. It surprises not only with a simple derivation but also with its high applicability which might be even more important.

The derivation is based on several natural simplifications (see Appendix 1 for an one-page introduction to the financial derivatives). Among those the geometrical Brownian motion (or, more general, quasi-Brownian motion) model for the price of an underlying asset and the no-arbitrage constraint are most difficult to avoid. Below we stop to describe each of the assumptions in more detail.

Quasi-Brownian walks as a model for price increments keep many convenient features, such as easy-to-fit parameters [3], a developed and handleable mathematical description [5]

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which results in the existence of exact solutions obeying explicit equations [2, 6], and indeed resemble real price movements. However, the resemblance is not perfect and many deviations have been observed [7, 8, 9, 10]. These studies reveal other characteristic processes which substitute for Brownian random walks. These are the Truncated Levy Flights which are regularized Stable Levy processes slowly converging to the Brownian process at large times and large prices [11]. Another, more popular, approach to model price movements with excess kurtosis is to use the quasi-Brownian walks but with a stochastic variance (volatility) [12]. A recent study [13] shows that, at least for the S&P500 index, the volatility is distributed log-normal and follows the geometrical random walk with a good measure of accuracy. However, the last approach cannot explain scaling properties and correlations observed in financial data.

A number of approaches have been developed to correct the Black-Scholes analysis for the case of deviations from a simple geometrical Brownian motion model for prices. The well-known and widely accepted way to do it is to generalize the Black-Scholes analysis to quasi-Brownian motions with the stochastic volatility. Other approaches use various versions of risk-neutral treatment which differ by the details of the averaging procedure. A recent paper [10] presents an attempt to account for the Truncated Levy Flight nature of underlying asset price walks based on a Bouchaud and Sornette scheme [9, 14].

The present paper suggests the different approach to the problem which releases both no-arbitrage assumption and Brownian character of the price walks. As was shown in Ref [15] based on the Gauge Theory of Arbitrage (GTA) picture, the active trade behavior of speculators who profit from temporary mispricing of assets may be one of the reasons for the deviation of the probability distribution function (PDF) from the log-normal law and a possible explanation of memory effects. The origin of this is the fact that for a mispriced asset there are cash flows (inflow for a undervalued asset and outflow for an overvalued asset) which shifts the price due to a kind of supply-demand mechanism. This shift is directed and results in damping of the mispricing which, in its own, reduces the width of the PDF peak and increases the peak height. On the other hand, the arbitrageurs create an additive noise which causes the power-law wings of the PDF [16, 17]. The investigation of the influence of the directed speculations on the distribution function of prices in a very simple gauge arbitrage model demonstrated a qualitative agreement with the observed data [15]. Thus, it was demonstrated that the model distribution function of the price increments is similar to one observed in a real market and possesses the same scaling properties. This motivates us to apply developed formalism to derivative pricing and examine corrections to Black-Scholes equation due to arbitrageurs. Being responsible for both the deviation of the probability distribution function of the underlying asset prices and, at the same time, restoring virtually violated no-arbitrage constraints, speculators must cause corrections to the Brownian walks based arbitrage-free equation for prices of derivatives. The goal of the paper is to derive these corrections in a simple gauge arbitrage model which generalizes the GTA stock exchange model.

We will not go into details of GTA here and redirect the reader to Refs [1, 15]. However, to make the exposition self-contained, the main issues of GTA are sketched below.
1.1 GTA in brief

Let us remind the reader of concepts and notations of GTA [1] used in our consideration. This is field-theoretical description of the virtual arbitrage possibilities and corresponding money flows. In this framework the net present value (NPV) calculation and asset exchanges are interpreted in geometrical terms as parallel transports in some sophisticated fibre bundle space. This allows us to map the capital market theory to a theory similar to electrodynamics and then use the machinery of quantum field theory. It was shown that the free quantum gauge theory is equivalent to the assumption of log-normal walks for assets prices. In general, the theory resembles electrodynamics where particles with positive charge ("securities") and negative charges ("debts") interact with each other through the electromagnetic field (arbitrage excess return). In the case of a local virtual arbitrage opportunity money flows into the profitable security. Entering positive charges and leaving negative ones screen the profitable fluctuation and restore the equilibrium, i.e. speculators wash out the arbitrage opportunity. The starting point for GTA construction is an understanding of Net Present Value calculations as a parallel transport in some fibre bundle. The NPV states that money has a time value. This time value has to be taken into account through so-called discounting procedure. If an amount of money $F$ is to be received in $T$ years' time, the Present Value of that amount ($NPV(F)$) is the sum of money $P$ (principal) which, if invested today, would generate the compound amount $F$ in $T$ years' time (for simplicity $r$ is considered constant over the $T$ years):

$$NPV(F) \equiv P = \frac{F}{(1 + r)^T}.$$  

The interest rate involved in this calculation is known as the discount rate and the term $(1 + r)^{-T}$ is known as T-year discount factor $D_T$:

$$D_T = (1 + r)^{-T}. \quad (1)$$

Thus NPV method shows how to compare money amounts came at different moments of time [18].

This last phrase points directly the geometrical interpretation which we use: discounting procedure plays the role of a "parallel transport" of an amount of money through time (though in fixed currency). The discounting factor (1) is then an element of a structural group of a fibre bundle and the discount rate coincides with the time component of the connection vector field. The "space" components of the connection are related to exchange rates and prices. Indeed, exchange rates and prices are responsible for converting money in different currencies or different securities, i.e. points of discrete "space", to the same currency (point of the space) at a fixed moment of time. They can be interpreted as elements of the structural group which "transport" the money in "space" directions and are space analogues of the discount factor. Summing up, the capital market theory has a geometrical structure which allows us to map it onto a theory of a fibre bundle.

The curvature tensor of the connection field is related to the arbitrage which is an operational opportunity to make a risk-free profit [6] with a rate of return higher than the risk-free interest rate accrued on deposit. As was derived in Refs [1], the rate of excess return on an elementary arbitrage operation (a difference between rate of return on the
operation and the risk-free interest rate) is an element of the curvature tensor calculated from the connection. It can be understood keeping in mind that the curvature tensor element is related to a difference between two results of infinitesimal parallel transports performed in different order with the same initial and final points or, in other words, a gain from an arbitrage operation. Due to this geometrical interpretation it is possible to say that the rate of excess return on an elementary arbitrage operation is an analogue of the electromagnetic field.

In the absence of any uncertainty and money flows, the only state that is realized is the state of zero arbitrage. However, if we introduce the uncertainty to the game, prices and rates move and some virtual nonequivalent possibilities to get more than less appear. Therefore we can say that the uncertainty play the same role in the developing theory as the quantization did for quantum gauge theory.

Money flow fields appear in the theory as "matter" fields which are transported by the connection (interests and exchange rates). It means that the matter fields interact through the connection. Dilatations of money units (which do not change real wealth) play the role of gauge transformation which eliminates the effect of the dilatation by a corresponding gauge transformations of the connection in the same way as the Fisher formula does for the real interest rate in the case of an inflation [18]. *The symmetry of the real wealth to a local dilatation of money units, security splits and the like is the gauge symmetry of the theory.*

An investor’s strategy is not always optimal. This is due to partially incomplete information available, partially because of an investor’s internal objectives [18]. It means that the money flows are not certain and fluctuate in the same manner as prices and rates do. So this requires a statistical description of money flows which, once again, returns us to an effective quantization of the theory.

At this stage we would like to clarify the following misunderstanding which might emerge. The arbitrage itself implies a possibility to perform an operation with a risk-free rate of return which is higher than, say, a bank deposit interest rate. In this sense buying shares cannot be considered as such operation because of assumed random walk of the share price and the corresponding risk. What do we mean then talking about the arbitrage? The randomness of the price is equivalent to a quantization as we explained in [1] and the rate of return on an (arbitrage) plaquette operation is now a quantum variable which cannot be taken as a complex number. This exactly resembles a situation with an electromagnetic field which, after the quantization, is not a number but a quantum variable. However, it does not stop us using the same name for the variable, imagining virtual quantum fluctuations and describing the influence of these fluctuations on electric charges, keeping in mind the calculation of corresponding matrix elements. In the same way we understand the arbitrage rate of return in the financial setting.

Summing up, it was shown how to map the capital market to a system of particles with positive, "securities", and negative, "debts", charges which interact with each other through an electromagnetic field, the gauge field of the arbitrage. In the case of a local virtual arbitrage opportunity, cash flows into the region of configuration space (money go in the profitable security) while "debts" try to escape from the region. This brings in positive charges and pushes out negative ones, leading to an effective screening of the profitable fluctuation. These processes restore an equilibrium and erase the arbitrage.
Formalization of the scheme drawn above leads to the lattice quantum field theory [19]. At this point the standard machinery of quantum field theory can be applied to obtain various observables such as distribution functions of the interest/exchange rates, response functions of the system and others. It may answer questions about dynamical response of a financial market, the dynamical portfolio theory and other problems.

In conclusion we want to add that notions of the (stochastic) differential geometry appeared in the context of financial modelling in paper [20]. It contains several ideas which are similar to GTA.

In this paper we want to apply the idea to derivative pricing. More precisely, we show how the standard results of Black-Scholes analysis appeared from GTA and derive correction to the Black-Scholes equation due to a virtual arbitrage and speculators reaction on it. This will model both violation of the no-arbitrage constraint and non-Brownian price walks which resemble recently observed data.

The paper is organized as follows. In next section we formulate a GTA model for a description of derivative instruments. To do this we construct base space of the theory which, in contrast to GTA model for the simplest stock exchange [15], contains a double ladder. This corresponds to simultaneously treating cash, shares and derivatives. Section 3 is devoted to an investigation of the classical limit of the action which leads to the Black-Scholes equation in the particular case of free gauge field dynamics. We show that the plaquette diagrams obtained in Section 2 have a very simple interpretation as an imbalance between the derivative and the Black-Scholes hedging portfolio. Section 4 is devoted to description of the money flow fields and the corresponding correction to an effective action for the derivative price coming from virtual speculations. In the classical limit it gives a correction to the Black-Scholes equation. The last section completes the paper with final remarks. In the appendix we give simple derivation of the Black-Scholes equation.

2 GTA model of derivatives

In this section we construct a GTA model for a share-cash-derivative system. To simplify the consideration we consider only one type of shares and the perfect capital market conditions are implied. The consideration of this paper is not restricted by any type of concrete derivative contracts. However, to simplify the consideration we will illustrate the GTA application by an analysis of European and American call options.

Shares or derivatives can be exchanged with cash and vice versa. The corresponding exchange rates are $S_i$ and $C_i$ (one share or derivative contract is exchanged on $S_i$ or $C_i$ units of cash) at some moment $t_i$, and the reverse rates (cash to share or derivative) are $S_i^{-1}$ and $C_i^{-1}$. We consider period from starting point $t = 0$ up to moment $t = T$. For the case of option we assume that $T$ is expiration time. We suppose that there exists a shortest interval of time $\Delta = T/N$ and this $\Delta$ is taken as a unit time. So, the exchange
rates $S_i$ and $C_i$ are quoted on a set of equidistant times: $\{t_i\}_{i=0}^N$, $t_i = i\Delta$ and represent the parallel transport along legs of a double ladder base graph. The interest rate for cash is $r_b$ so that between two subsequent times $t_i$ and $t_{i+1}$ the volume of cash is increased by factor $e^{r_b \Delta}$. The shares and derivatives are characterized by rates $r_1$ and $r_2$ correspondingly. These rates realize parallel transport in time direction.

Following Refs [1] we consider elementary arbitrage operations when an arbitrageur borrows one share at time $t_i$, sells it for $S_i$ units of cash, put the cash in the bank until time $t_{i+1}$ and, at time $(t_i+1)$, closes his short position borrowing $e^{r_1 \Delta}S_{i+1}$ units of cash and buying shares. The result of the operation for the arbitrageur will be $e^{r_b \Delta}S_i - e^{r_1 \Delta}S_{i+1}$ units of cash. The excess return on this operation is

$$Q_i^{(1)} = S_i e^{r_b \Delta} S_{i+1}^{-1} e^{-r_1 \Delta} - 1 .$$

To get this expression we discounted the amount and converted it in shares since the operation was started in the shares. Equation (2) has a form of the curvature tensor element corresponding to drawing assets through the cycle. If $Q_i^{(1)} \neq 0$ an arbitrageur can get excess return performing this or the reverse operation. The following quantity

$$R_i^{(1)} = \left( S_i^{-1} e^{r_1 \Delta} S_{i+1} e^{-r_b \Delta} + S_i e^{r_b \Delta} C_{i+1}^{-1} e^{-r_1 \Delta} - 2 \right) / \Delta$$

is used to measure the arbitrage (excess rate of return) on local cash-share operation. The absences of the arbitrage is equivalent to the equality

$$S_i^{-1} e^{r_1 \Delta} S_{i+1} e^{-r_b \Delta} = S_i e^{r_b \Delta} C_{i+1}^{-1} e^{-r_1 \Delta} = 1 .$$

The same can be done for another possible arbitrage operation. For cash-derivative plaquette it gives the following quantities

$$Q_i^{(2)} = C_i e^{r_b \Delta} C_{i+1}^{-1} e^{-r_1 \Delta} - 1 ,$$

$$R_i^{(2)} = \left( C_i^{-1} e^{r_1 \Delta} C_{i+1} e^{-r_b \Delta} + C_i e^{r_b \Delta} C_{i+1}^{-1} e^{-r_1 \Delta} - 2 \right) / \Delta .$$

Looking at (3,5) we can conclude that the arbitrage is represented in the theory by the curvature of the connection. Precisely, in the continuous limit ($\Delta \to 0$) the RHS of Eqns (3,5) converges as usual to a square of the curvature tensor element multiplied by area of plaquette

$$R_i^{(l)} = \left( Q_i^{(l)} \right)^2 / \Delta .$$

Curvature is $Q_i^{(l)}/\Delta$, the area of plaquette is proportional to the shortest time interval $\Delta$ and fixed ”space” length which is omitted.

Being represented by the curvature tensor, the notion of the arbitrage as well as the quantity $R_i^{(l)}$ are gauge invariant. All of this allows us to say that the rate of excess return on an elementary arbitrage operation is an analogue of the electromagnetic field. In the absence of uncertainty (or, in other words, in the absence of random walks of prices, exchange and interest rates) the only state realized is the state of zero arbitrage. However, if we introduce the uncertainty, prices and the rates move and some virtual arbitrage possibilities appear. Therefore, we can say that uncertainty plays the same role in the developing theory as the quantization does for the quantum gauge theory.
Further development of the consideration are based on the following assumptions about the dynamics \[1\]: gauge invariance, locality, correspondence principle, extremal action principle, limited rationality (uncertainty) and absence of correlation between excess returns on different plaquettes. Formally, these assumptions are summed up in the functional form of the probability \( P\{\{S_i, r_k\}\} \) to find a set of the exchange rates/interest rates \( \{S_i, r_k\} \) given by the expression:

\[
P\{\{S_i, r_k\}\} \sim e^{-\sum_{i,l} \beta_l R_i^{(l)}} \sim e^{-s_{\text{gauge}}},
\]

(7)

together with the statement that the integration measure is gauge invariant too. The introduced above parameters \( \beta_l \) are measures of the uncertainty of the corresponding plaquettes. The sums run over time moments \( i \) and types of the plaquettes (\( l = 1 \) for cash-share plaquettes and \( l = 2 \) for cash-derivative ones).

As it was shown in Refs[1], the assumptions which have been made are equivalent to a log-normal model for the share price walks with the distribution function

\[
P(S(T)|S(0)) = \frac{1}{\sigma S \sqrt{2\pi T}} e^{-(\ln(S(T)/S(0))-(\mu-\frac{1}{2}\sigma^2)T)/(2\sigma^2T)}.
\]

(8)

Here the volatility \( \sigma = 1/\sqrt{2\beta} \) and the average rate of share return \( \mu = r_b - r_1 \) have been introduced.

There are two points to note here. The first one is that the log-normal distribution was derived in an absence of matter fields. These matter fields can significantly change the form of the distribution function and other properties of the price random walk. Indeed, in the presence of the money flows more complicated random processes for the price motion emerge which resemble real data observations [15]. The second point concerns other types of quasi-Brownian price motions which are considered in mathematical finance. They can be also introduced in the theory, making parameters, which we keep constant for the moment, depending on price values.

Let us return to gauge fixing. Since the action \( s_{\text{gauge}} \) is gauge invariant it is possible to perform a gauge transformation which will not change the dynamics but will simplify further calculation. In lattice gauge theory [19] there are several standard choices of gauge fixing and axial gauge fixing is one of them. In the axial gauge an element of the structural group is taken as something chosen on links in the time direction and exchange rates along the ”space” direction at some particular time. This kind of gauge fixing is convenient for the model in question. Actually this gauge has been used in deriving (8).

We choose \( r_b \) to be the risk free interest rate and \( r_b - r_1, r_b - r_2 \) are the average rates of return on the share and the derivative. This means that in the situation of the double ladder base the only dynamical variables are the exchange rates (prices) as a function of time and the corresponding measure of integration is the invariant measure \( \frac{dS}{S^{dC}} \). Below we fix the price of the shares at time \( t = 0 \) taking \( S_0 = S(0) \). We also fix the exchange rate of the derivative to the share at the moment of the derivative exercise.

Let us note that quantities of our theory — exchange and interest rates are not gauge invariant but gauge covariant. So it is natural to choose the gauge in which the exchange and interest rates take their real value. In our gauge ”rate of return” on cash takes its real value \( r_b \) while average rates of return on the share and the derivative are not \( r_1 \) and
Let us return to expression (7). It is derived under the assumption that arbitrage opportunities are uncorrelated between different "space-time" plaquettes. However, there are many important problems where the correlation between returns on elementary plaquettes operations have to be taken into account. The most important such example is the portfolio theory where an optimal portfolio is constructed using correlation between assets [21, 22]. Another example is derivative pricing which is studied in the paper.

To account for the correlation we have to elaborate some details of the construction of the action (the definition of the probability finding particular configuration of exchange and interest rates). Of course we retain base principles such as gauge invariance. The configuration which has less arbitrage is more probable. But we look more precisely at the problem of independence of arbitrage operations.

The general form of the action is

\[ s_{\text{gauge}} = \sum_{\xi \zeta} Q_\xi A_{\xi \zeta} Q_\zeta / (2\Delta) \]  

where \( Q_\xi \) are local (dependent) arbitrage plaquette quantities and matrix \( A_{\xi \zeta} \) is the correlation matrix of the plaquettes. This scheme is general and can be applied to the case of many correlated assets. Expression (9) is gauge invariant and naturally generalize (7). In general, matrix \( A_{\xi \zeta} \) is not diagonal due to

To simplify the model we make a locality assumption. It means that the virtual arbitrage opportunities emerging at different times are independent. This makes matrix \( A \) diagonal in the time index. The action can be rewritten in following form:

\[ s_{\text{gauge}} = \sum_{il'l'} Q_i^{(l)} A_{il'} Q_{l'}^{(l')} / (2\Delta) \]

where \( A_{il'} \) is equal-time plaquette correlation matrix and \( l \) and \( l' \) run over all elementary plaquettes at fixed time. It is straightforward to show that in the continuous limit \((\Delta \rightarrow 0)\) the previous expression takes the following form

\[ s_{\text{gauge}} = \int_0^T \sum_{l'l'} \left( \frac{1}{S} \frac{dS_i(t)}{dt} - (r_b - r_l) \right) A_{il'}(t) \left( \frac{1}{S} \frac{dS_{l'}(t)}{dt} - (r_b - r_{l'}) \right) dt \]

This implies the following expression for the correlation matrix \( A \):

\[ A_{il'}^{-1}(t) = \left< \frac{1}{S} dS_i(t), \frac{1}{S} dS_{l'}(t) \right> / dt \equiv \sigma_{il'}^2(t) \]

(terms \((r_b - r_l)dt \) do not contribute in continuous limit). For the share-cash-derivative system we need to consider the following correlators: \( < \frac{1}{C(t)} dC(t), \frac{1}{S(t)} dS(t) > / dt, < \frac{1}{S(t)} dC(t), \frac{1}{S(t)} dS(t) > / dt, < \frac{1}{C(t)} dC(t), \frac{1}{C(t)} dC(t) > / dt. \) The first one is equal to \( \sigma^2 \equiv 1/(2\beta_1) \). The second correlator we denote as \( \alpha(t)/(2\beta_1) \) introducing the notation:

\[ \alpha(t) \equiv < \frac{1}{C(t)} dC(t), \frac{1}{S(t)} dS(t) > / < \frac{1}{S(t)} dS(t), \frac{1}{S(t)} dS(t) > \]
To calculate the last correlator $< \frac{1}{C(t)} dC(t), \frac{1}{C(t)} dC(t) > /dt$ we consider $C(t)$ as a random function of $S$. Then it can be represented as

$$< \frac{1}{C(t)} dC(t), \frac{1}{C(t)} dC(t) > /dt = \alpha^2(t)/(2\beta_1) + 1/(2\beta_2)$$

where the term $\alpha^2(t)/(2\beta_1)$ is responsible for the volatility of the derivative due to its dependence on the share price and $1/(2\beta_2)$ the derivative due to the random character of function $C$ itself. Given these parameterization with $\alpha, \beta_{1,2}$ we can find matrix $A$ from Eq(12) and get the following expression for the action (11) in this particular derivative-related setting:

$$s_{gauge} = \beta_1 \int_0^T dt \left( \frac{1}{S(t)} \frac{dS(t)}{dt} + (r_1 - r_b) \right)^2 + \beta_2 \int_0^T dt \left( \left( \frac{1}{C(t)} \frac{dC(t)}{dt} + (r_2 - r_b) \right) - \alpha(t) \left( \frac{1}{S(t)} \frac{dS(t)}{dt} + (r_1 - r_b) \right) \right)^2$$

(14)

It is interesting to note that this expression gives the main order (with respect to $\Delta$) of the lattice action

$$s_{gauge} = \sum_i (\beta_1 R_i^{(1)} + \beta_2 R_i^{(2,1)})$$

(15)

where $R_i^{(2,1)}$ is defined as

$$R_i^{(2,1)} = (Q_i^{(2)} - \alpha_i Q_i^{(1)})^2 / \Delta.$$  

(16)

Action (15) has a very simple interpretation. As we already mentioned, local cash-share and cash-derivative arbitrage operations (at the same time) clearly are not independent and we cannot use $R_i^{(1)}$ and $R_i^{(2)}$ simultaneously to characterize the independent arbitrages. Instead, we can find statistically independent combinations of $Q_i^{(1)}$ and $Q_i^{(2)}$ and define $R_i^{(2,1)}$ to use together with $R_i^{(1)}$ as independent plaquette quantities in expression (7). This returns us to the action (15). This is the action we deal with in the paper when we consider a lattice system while we use action (14) for the continuous limit.

All said above is valid for any derivative contract. A particular type of derivative is reflected by details of the construction of the base graph of the theory. We consider here in more detail European and American call options. Other kinds of derivatives require other base constructions which, however, are straightforward.

We start with the European call option (see Appendix 1 for the definition). As we said before, the ”space”-time graph for it and the underlying share is the double ladder. The only new element here, comparing with the model of Ref [15] adopted for two types of shares, is the additional link at the expiration date between the option and cash and the share with the direction chosen to describe the possibility to change this option and $E$ units of cash on the share. (Fixed exchange rate on this link gives missing gauge fixing condition.)

The graph for the American call option and the underlying share contains the double ladder and the directed links from the option and cash to share at any intermediate times before the expiration time $T$. These links are the same as that for European option.
The links which are responsible for the swap of the derivative to the share, i.e. exercising of the option, generate new plaquettes on the "space"-time graph. Corresponding plaquette quantities $R_i'$ have to be taken into account. These terms depend very much on concrete conditions of the derivative contract. We keep the terms terms aside and rewrite (7) in the form

$$P(\{S_i,r_k\}) \sim e^{-s_{\text{gauge}}} - \sum_i \beta_i R_i' .$$

Below we do not concentrate on the primed plaquettes. They contain directed links and contribute to the boundary conditions only. However, to make complete analytical and numerical analysis they are important and have to be retained. We return to this point in section 3 where we derive the boundary conditions. In the next section we show how this formalism reproduces all the results of standard derivative pricing theory.

3 Derivation of the Black-Scholes equation

In this section we obtain the Black-Scholes equation as the equation of the saddle-point in the quasi-classical limit of the gauge theory in absence of money flows.

Let us return to Eqn(14) for the action in the continuous ($\Delta \to 0$) limit. The first term in the RHS of this equation corresponds to geometrical random walks and provides a background for the derivation.

Being a derivative instrument from the share price, the price of the derivative has to be correlated or even defined by the share price. That is why it is natural and more convenient for our purposes to write $C(t)$ as a some unknown function of $S(t)$. Since we integrate over all $C(t)$ (or over all functions $C(t,S)$) this does not mean any loss of generality. We use the fact that $C$ is a function of $S$ and the property of the geometrical random walk of the underlying asset to obtain a compact expression for the parameter $\alpha(t)$ (13):

$$\alpha = S \frac{\partial C}{C} \frac{\partial}{\partial S} .$$

To explain this we use the following fact for the geometrical Brownian motion known as Ito’s lemma:

$$df(t,S) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} dt + o(dt)$$

where $f(t, S)$ is an arbitrary function of $t$ and $S$ (the function is supposed to be smooth) [6]. Using this equality and the fact $<dS/S,dS/S> \to S^{-2} <dS,dS>$ we obtain Eqn(18).

Now let us turn our attention to the second term in Eqn(14). Using expression (22) and Ito’s lemma (19) we end with the following action term:

$$\beta_2 \int dt \left( \frac{1}{C} \frac{\partial C}{\partial t} + \frac{1}{2} S^2 \frac{\partial^2 C}{\partial S^2} - r_v (1 - \frac{S}{C} \frac{\partial C}{\partial S}) \right)^2 .$$

This term corresponds to a virtual arbitrage account.

The classical limit for the action, i.e. $\beta_2 \to \infty$, reduces the functional integration over functions $C(t,S)$ to the contribution from the classical trajectory only and this
trajectory is defined by the Black-Scholes equation for the price of the financial derivative:

\[
\frac{\partial C}{\partial t} + \sigma^2 \frac{S^2}{2} \frac{\partial^2 C}{\partial S^2} - r_b (C - S \frac{\partial C}{\partial S}) = 0 ,
\]

since, as it is easy to see, the rate \( r_2 \) on the derivative whose price is defined by (20) is give by the expression

\[
r_2 = r_1 < \frac{S \partial C}{C \partial S} >
\]

The equation does not depend on a particular type of the derivative which is encoded in the boundary conditions. For a comparison we give standard no-arbitrage derivation of the appendix.

### 3.1 Boundary conditions

In this subsection we consider boundary conditions which characterize a particular derivative. Here we treat only European and American calls but it is straightforward to generalize the consideration. To this end we return to the construction of the base graph in section 2.

Let us first consider in more detail the European call option. At the expiration time \( T \) the option can be sold or bought for \( C(T) \) units of cash but it can also be exchanged (together with \( E \) units of on the share. to effective scrap of the call. These opportunities create new arbitrage possibilities which we take into account by introducing the \( \beta^\prime R_N^\prime \) term in action (17). There exists a new arbitrage operation which is available at expiration time \( T \): and \( E \) currency (directed link on the base graph), sell share for \( S(T) \) units and buy the portfolio again. This gives us a plaquette excess return

\[
Q_N^\prime = (S(T) \theta (S(T) - E) + E \theta (E - S(T)))(C(T) + E)^{-1} - 1 , \]

and for the plaquette quantity \( R_N^\prime \) we get

\[
R_N^\prime = \frac{E \theta (E - S(T)) + S(T) \theta (S(T) - E)}{C(T) + E} + \frac{C(T) + E}{E \theta (E - S(T)) + S(T) \theta (S(T) - E)} - 2 . \]

In the quasi-classical (no-arbitrage) limit we have \( \beta^\prime \rightarrow \infty \) which gives the following boundary condition for the European call option from Eqn(23):

\[
C(T) = (S(T) - E) \theta (S(T) - E) . \]

If we neglect this arbitrage possibilities at the time \( T \), we can use this equation as a gauge fixing condition and need not consider additional links and boundary plaquettes. This approximation is quite realistic and clearly simplifies the scheme.

Let us turn to the American call option. It can be exchanged for the share at any time up to \( T \) by paying additional \( E \) units cash. In this case the base graph contains links from the option to the share (but not backwards) at all time points. It allows two possible arbitrage operations. The first operation is similar to considering operation with a European option at exercise time \( T \). Now this operation is available at any moment
One can borrow the portfolio consisting of an option and \( E \) currency units, exchange it on the share, sell the share for \( S(t) \) units of money and buy the portfolio again. The excess return on this operation is

\[
Q'_t = S(t)(C_A(t) + E)^{-1} - 1,
\]

and for the boundary plaquette quantity we get

\[
R'_t = \frac{S(t)}{C_A(t) + E} + \frac{C_A(t) + E}{S(t)} - 2. \tag{25}
\]

In close analogy with the European option in the quasi-classical (the absence of the arbitrage) limit we get the following boundary condition for the American call option:

\[
C_A(t) = S(t) - E \tag{26}
\]

Now we have to determine moment when American option exercised. To this end we consider another possible arbitrage operation. Let an arbitrageur have portfolio of an option and \( E \) units of cash at some time \( t \). He can exchange the portfolio for a share and keep it up to time \( t + dt \) or hold the portfolio and exchange it on a share at time \( t + dt \). The expected return on the portfolio is \((C_A(t)(r_b - r_2) + Er_b)/(C_A(t) + E)\) so at time \( t + dt \) an arbitrageur will have

\[
\left[ 1 + \frac{C_A(t)(r_b - r_2) + Er_b}{C_A(t) + E} dt \right] \frac{C_A(t) + E}{C_A(t + dt) + E}
\]

portfolios of an option and \( E \) cash units which he can exchange for the equal number of shares. The expected return on share is \((r_b - r_1)\). Therefore the excess return on this operation is

\[
Q''_t = \left[ 1 + \frac{C_A(t)(r_b - r_2) + Er_b}{C_A(t) + E} dt \right] \frac{C_A(t) + E}{C_A(t + dt) + E} \left[ 1 + (r_b - r_1) dt \frac{S(t)}{S(t + dt)} \right]^{-1},
\]

and for the boundary plaquette quantity we obtain the expression:

\[
R''_t = Q''_t + (Q''_t)^{-1} - 2. \tag{27}
\]

In the no-arbitrage limit we have

\[
Q''_t - 1 = 0
\]

at the exercise time. This gives us second boundary condition for the American call:

\[
\frac{\partial C_A(t, S)}{\partial S} = 1 \tag{28}
\]

To prove this expression we have to use the no-arbitrage condition \( Q''_t - 1 = 0 \), expression

\[
\frac{S}{C} \frac{\partial C}{\partial S} = r_2
\]
and to take into account first boundary condition (26).

So finally we obtain the following boundary conditions for the American call option [6]:

\[
\frac{\partial C_A(t, S)}{\partial S} = 1, \quad C_A(t, S) = S(t) - E
\]

The equations determine the time when the option is exchanged for the share and the corresponding payoff. It is easy to show from the plaquette analysis that in absence of dividends American call is never exercised early and, hence, is equivalent to European call but we do not stop for it.

Summing up, we have shown that in the absence of money flows and in the quasi-classical limit (i.e. suppression of the arbitrage operations with the derivatives) the Black-Scholes equation emerges as the equation for the saddle-point which is provided with appropriate boundary conditions.

3.2 Connection with Black-Scholes analysis

In conclusion we want to clarify the connection with the original Black-Scholes analysis (see Appendix). Let us return back to the formula for \( R^{(2,1)} \) (16):

\[
R^{(2,1)}_i = (Q^{(2,1)}_i)^2 / \Delta
\]

with

\[
Q^{(2,1)}_i = Q^{(2)}_i - Q^{(1)}_i \alpha_i .
\]

To first order in \( \Delta \) the last expression can be rewritten as

\[
Q^{(2,1)}_i = C_i S_i^{\alpha_i} e^{\alpha_i r_1 \Delta e^{(1-\alpha_i) r_2 \Delta e^{-r_2 \Delta S}} C_{i+1}^{-1}} - 1 .
\]

It is easy to give a simple interpretation of the last expression. Indeed, it is not difficult to see that it describes the following money circle:

\[
C \rightarrow USD \rightarrow (1 - \alpha)USD \oplus \alpha S \rightarrow USD \rightarrow C .
\]

The portfolio comprising shares and cash (in parts \( \alpha \) and \( 1 - \alpha \)) emerging at the intermediate state is the Black-Scholes hedging portfolio for the derivative and the hedging relation. From this point of view the plaquette \( Q^{(2,1)}_i \) represents the arbitrage fluctuations in hedging portfolio-derivative plaquette. This returns us back to the treatment of the expression

\[
(\frac{\partial C}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} - r_\delta (C - S \frac{\partial C}{\partial S}))/ (C - S \frac{\partial C}{\partial S})
\]

as the arbitrage excess return on the infinitesimal operation [6].

4 Money flows and correction to the Black-Scholes equation

Now let us turn to money flow fields. These fields represent cash-debt flows in the market. The importance of the cash-debt flows for our consideration is explained by their role in
a stabilization of market prices. Indeed if, say, some asset price creates a possibility to get a bigger return than from other assets (with similar risk), then an effective cash flow appears, directed to these more valuable shares. This causes restoration of equilibrium due to the demand-supply mechanism. The same picture is valid for debt flows if there is a possibility for debts restructurisation. As we will see all these features find their place in the GTA framework.

Following Refs [1] we can formulate the dynamics for the cash-debts flows basing on several assumptions, such as gauge invariance of the dynamics, an investor’s wish to maximize his return and his limited rationality. This gives us the following functional integral representation for the matrix element of an evolution operator in the coherent state representation of the money flows in the case of our double ladder base graph ($r_0 \equiv r_b$):

$$< \tilde{\psi}_N, \tilde{\chi}_N | \hat{U}(t + N\Delta, t) | \psi_0, \chi_0 > = \int \prod_{k=0, i=1}^{2, N-1} d\tilde{\psi}_{k,i} d\psi_{k,i} d\tilde{\chi}_{k,i} d\chi_{k,i} e^{(s_1 + s_1' + a_b)} ,$$ \hspace{1cm} (30)

with the actions for cash and debt flows:

$$s_1 = \sum_{k=0, i=0}^{2, N-1} (\tilde{\psi}_{k, i+1} e^{\beta r_k \Delta} \psi_{k, i} - \tilde{\psi}_{k, i} \psi_{k, i}) + \sum_{i=0}^{N-1} \left( (1 - t_c) \beta C_i \tilde{\psi}_{0, i+1} \psi_{2, i} \right) + (1 - t_c) \beta C_i \tilde{\psi}_{2, i+1} \psi_{0, i} + (1 - t_c) \beta S_i \tilde{\psi}_{0, i+1} \psi_{1, i} + (1 - t_c) \beta S_i \tilde{\psi}_{1, i+1} \psi_{0, i} ,$$ \hspace{1cm} (31)

$$s_1' = \sum_{k=0, i=0}^{2, N-1} (\tilde{\chi}_{k, i+1} e^{\beta r_k \Delta} \chi_{k, i} - \tilde{\chi}_{k, i} \chi_{k, i}) + \sum_{i=0}^{N-1} \left( (1 + t_c) \beta C_i \tilde{\chi}_{0, i+1} \chi_{2, i} \right) + (1 + t_c) \beta C_i \tilde{\chi}_{2, i+1} \chi_{0, i} + (1 + t_c) \beta S_i \tilde{\chi}_{0, i+1} \chi_{1, i} + (1 + t_c) \beta S_i \tilde{\chi}_{1, i+1} \chi_{0, i} .$$ \hspace{1cm} (32)

Here $t_c$ is a relative transaction cost and the action $s_1$ represents exchanges along boundary plaquettes and is contract-dependent. Using this expression it is easy to obtain the transition probability in the occupation number representation simply by integrating over $\tilde{\psi}, \psi, \tilde{\chi}, \chi$ variables. Let us consider a transition to the state with $n_0(n_0')$ cash units in the long (short) position, $n_1(n_1')$ stock units in the long (short) position, $n_2(n_2')$ derivatives units in the long (short) position at time $t_1 = t + N\Delta$ from the state $(m_k, m_k')$ at the original time $t$. The long position corresponds to possessing assets and the short position implies possessing liabilities ("debts"). The probability for this transition has been derived in [15]:

$$P((\{n_k\}, \{m_k\}), (\{n'_k\}, \{m'_k\}), t_1, t) = S^{\beta (m_1 - m_1' - n_1 + n_1')} (t) C^{\beta (m_2 - m_2' - n_2 + n_2')} (t)$$

$$\int d\psi d\tilde{\psi} d\chi d\tilde{\chi} < \tilde{\psi}_N, \tilde{\chi}_N | \hat{U}(t + N\Delta, t) | \psi_0, \chi_0 > e^{-\psi_N \psi_0} \prod_{k=0}^{2} \tilde{\psi}_{k, 0} \chi_{k, 0} \psi_{k, N} \chi_{k, N} / n_k ! n'_k ! .$$ \hspace{1cm} (33)

Expressions (15,17,30,33) form the complete set of necessary equations to describe the dynamics and mutual influence of interest/rate prices and money fields.

We now derive the correction to the effective action of the gauge field due to the presence of money flows. Since, as it was shown in the previous section, the Black-Scholes
equation appears in the quasi-classical limit of the free gauge theory, the correction to the effective action leads in the same limit to a correction to the Black-Scholes equation.

Let us suppose that the initial asset configuration at time \( t \) is given by the probability distribution \( F(\{m_k\}, \{m'_k\}) \). Then, as it follows from section 2, the correction to the effective action for the gauge field is equal to:

\[
\delta s_g = \ln \sum_{\{n_k\}, \{n'_k\}, \{m_k\}, \{m'_k\}} P(\{n_k\}, \{m_k\}, \{n'_k\}, \{m'_k\}, 0, T) F(\{m_k\}, \{m'_k\})
\]

where the function \( P(\{n_k\}, \{m_k\}, \{n'_k\}, \{m'_k\}, 0, T) \) is given by Eqns(30 - 33). to many investment horizons is straightforward but cumbersome.

The correction to the effective action of the prices’ gauge field transforms the Black-Scholes equation for the classical trajectory to the following equation for the trajectory \( C(t) \) minimizing the action:

\[
\min_{C(t)} \left( \int dt \left( \frac{1}{C} \frac{\partial C}{\partial t} + \frac{1}{C} \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial^2 S} - r_b \left( 1 - \frac{S}{C} \frac{\partial C}{\partial S} \right)^2 \right) - \frac{1}{\beta_2} \delta s_g \right).
\] (34)

Then the Black-Scholes equation is formally substituted by the equation:

\[
\frac{\delta}{\delta C(t)} \left( \int dt \left( \frac{1}{C} \frac{\partial C}{\partial t} + \frac{1}{C} \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial^2 S} - r_b \left( 1 - \frac{S}{C} \frac{\partial C}{\partial S} \right)^2 \right) - \frac{1}{\beta_2} \delta s_g \right) = 0 \tag{35}
\]

with the same boundary conditions. However, we think that the former form of the problem (Eqn(34)) is more convenient for numerical solution while the latter (Eqn(35)) cannot provide much for analytical inside since even in the more simple situation of the GTA model of stock exchange [15] an analytical approach hardly produce valuable results.

However we can state general properties of the correction which can be derived in the same way as was done for the correction to the effective action for the price of shares [15]. It was shown there that the correction from money flows has two important properties:

1. The correction vanishes at large time intervals;

2. The correction disappears in a limit of completely noisy traders who do not consider their potential profit as a motivation for transactions, i.e. in the limit \( \beta \to 0 \);

3. The correction is governed by a number of traders (read as money available for arbitrage operations), i.e. the correction disappears when no money is available for arbitrage operations.

This results in the convergence of the price movement process to the geometrical Brownian walks at large time scale which is well-know from real data observation. "Physically" it means that for large times the arbitrage is washed out (if there are arbitrageurs and they prefer to get more than less), no-arbitrage constraint holds firm and the Black-Scholes description is correct.

We state the central result of the paper as a derivation of the Black-Scholes equation and the corresponding corrections. However we want to emphasize that to account for both the virtual arbitrage and the money flows one needs to calculate the functional integral over money fields and prices keeping all action terms, in particular boundary plaquettes.
5 Conclusion

In conclusion, in the paper we have shown how to adapt the Gauge Theory of Arbitrage to apply it to derivative pricing. This can be done by constructing base underlying shares not only for cash but also for each other at some prespecified moments of time. The framework is general enough to describe both European and American derivatives with various payoffs. It was demonstrated that in a quasi-classical limit in absence of money flows, the treatment reproduces the Black-Scholes equation with appropriate boundary conditions. Formal expression for the corrections to the equation in the presence of money flows is also obtained.

All listed above results are obtained analytically. However, any further developments give rise to some quite complicated calculations. We believe that only computer calculation can demonstrate the agreement (or disagreement) of the theory with financial data and further development of the proposed approach to the financial modelling should be based on computer simulation. Indeed, as we mentioned before, even in the much easier case of cash-and-shares system only results of numerical calculation can be successively compared with real statistics. The situation becomes more extreme with the derivative pricing. On the other hand, the numerical algorithms to be used in this context essentially the same as ones, which have been used in lattice gauge theory with matter fields.

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Appendix 1

In this appendix we give a simple introduction to the financial derivatives and an almost rigorous derivation of the Black-Scholes equation using standard no-arbitrage arguments.

A financial derivative is a financial instrument whose value is defined by values of other (underlying) financial variables. These underlying variables may be prices of stocks, bonds or other derivatives, exchange rates, market indices and so on. Some particular derivatives (as warrants) have been known for centuries, but after the introduction in 1973 of exchange-traded derivatives on stocks in US trading in derivatives became a really huge industry (for practical aspects see, for instance, [3, 4]). Most popular derivatives are call and put options, futures while swaps, forwards and other derivatives are also important for practitioners. Let us give a simple definition of these derivatives.

The call option gives a right but not an obligation to buy a certain number of shares (or any prespecified underlying asset) at fixed (strike) price \( E \). If the right can be used at the final moment of (expiration) time \( T \) only, the call option is called European. In contrast, the American call option provides the right to buy the underlying asset at the strike price at any time before the expiration time \( T \). The put options guarantee the same rights not for buying but for selling of the share. Since the American option gives
an investor more freedom it is more valuable. In general, the price of the options are nonzero and depend on the price of the underlying asset. The only time when the option price can be zero is the expiration time subject to the case when the underlying asset price is less than $E$ (for call options).

A futures contract is an agreement between two parties to buy or sell an asset at a certain time in the future for a certain price. This agreement is a security and can be traded. The price of the agreement clearly depends on the underlying asset price and can be positive as well as negative. The negative price means that to close the position an additional amount should be paid. There is a number of methods to estimate derivative prices and hedge them (which is even more important from practical point of view). The most standard is Black-Scholes analysis which is based on no-arbitrage condition and quasi-Brownian character of the underlying price. Below we assume that the underlying asset is a share and reproduce simple version using the geometrical Brownian process for the underlying share price:

$$dS/S = \mu dt + \sigma dW$$

where $dW$ is the Wiener process, $\mu$ is an average rate of return on the share and $\sigma$ is a standard deviation of the return (so called volatility). Though our derivation is not strictly speaking rigorous, a formal derivation can be found in [5]. The derivative price $C$ is determined by the share price $S$. The share price at some moment in the future depends only on the current share price, not on the share price in the past (i.e. we assume a form of market efficiency). We also assume that the derivative price $C$ is not influenced by other factors. This means that $C(t, S)$ is nonstochastic function of the stochastic parameter $S$ which means that for given $t$ and $S$, $C(t, S)$ has a definite value. Using Ito’s lemma (19) for the geometrical Brownian process, we get following equation describing the derivative price movement

$$dC(t, S) = \left(\frac{\partial C}{\partial t} + \frac{S^2 \sigma^2}{2} \frac{\partial^2 C}{\partial S^2}\right) dt + \frac{\partial C}{\partial S} dS + o(dt) .$$

RHS of the equation contains $dS$ so option price movement is stochastic. However, we can construct a portfolio from the derivative and the underlying shares which will be risk-free. Indeed, let us consider portfolio of the derivative and $\partial C/\partial S$ shares in the short position. Price movement of the portfolio $\Pi$ is

$$d\Pi = \left(\frac{\partial C}{\partial t} + \frac{S^2 \sigma^2}{2} \frac{\partial^2 C}{\partial S^2}\right) dt + o(dt) .$$

Here we omit the term $Sd(\partial C/\partial S]$ which describes a change of the portfolio structure but not the portfolio value. Since in the last equation there is no longer $dS$ term, the portfolio is risk-free and cannot grow faster than the risk-free interest rate on a bank deposit. This latter statement is known as the no-arbitrage condition (the violation of the This gives us the following

$$\frac{\partial C}{\partial t} + \frac{S^2 \sigma^2}{2} \frac{\partial^2 C}{\partial S^2} = \left(C - S\frac{\partial C}{\partial S}\right) r_b$$

which is the famous Black-Scholes equation. It describes any derivatives and the specific of a concrete derivative is encoded in boundary conditions only. For example, the boundary
condition for the futures contract is

\[ C(S,T) = S - E \]

while for the European call it takes the form:

\[ C(S,T) = (S - E)\theta(S - E). \]

Other types of boundary conditions and various modifications of the Black-Scholes equations can be found in [4].

The last note here concerns transaction costs. As we consider continuous time limit we neglect \( o(dt) \) in formula (36). This means that the portfolio of share and cash (hedging portfolio) have to be rearranged continuously. This leads to infinite transaction costs. In this case we have to keep finite time steps and can not neglect \( o(dt) \). This results in impossibility of perfect hedging portfolio construction.

**References**


