Upper bound for entropy in asymptotically de Sitter space-time

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Abstract

We investigate nature of asymptotically de Sitter space-times containing a black hole. We show that if the matter fields satisfy the dominant energy condition and the cosmic censorship holds in the considering space-time, the area of the cosmological event horizon for an observer approaching a future timelike infinity does not decrease, i.e. the second law is satisfied. We also show under the same conditions that the total area of the black hole and the cosmological event horizon, a quarter of which is the total Bekenstein-Hawking entropy, is less than \(\frac{12\pi}{\Lambda}\), where \(\Lambda\) is a cosmological constant. Physical implications are also discussed.

I. INTRODUCTION

There has been interest in space-times with a positive cosmological constant \(\Lambda\). Recent cosmological observations suggest the existence of \(\Lambda\) in our universe [1]. Also, it is widely believed that the inflation took place in the early stage of our universe, where the vacuum energy of a scalar field (inflaton) plays a roll of \(\Lambda\). Most regions in such a space-time are expected to expand as in de Sitter space-time. Some regions, however, will gravitationally collapse to form black holes, if the inhomogeneity of initial matter distribution is large. Then there will be observers who have two types of event horizons, a black hole event horizon (BEH) and a cosmological event horizon (CEH), as the observers approaching the future timelike infinity in Schwarzschild-de Sitter space-time. Throughout this paper we shall focus event horizons for such observers.

Gibbons and Hawking [2] studied thermodynamic property [3] of the event horizons in asymptotically de Sitter space-times. In particular, they found that an observer feels a thermal radiation coming from the CEH and that the entropy \(S_C\) of the CEH is equal to one quarter of its area as in the case of a BEH. Thus, the areas of the event horizons can be interpreted as the entropies, or lack of information of the observer about the regions which he cannot see.

In classical general relativity, there have been a number of studies on the nature of BEH in the asymptotically de Sitter space-time. Hayward, Shiromizu and Nakao [4] and Shiromizu, Nakao, Kodama and Maeda [5] showed that the area of a BEH in the asymptotically de Sitter space-time cannot decrease and has an upper bound \(\frac{4\pi}{\Lambda}\) if the weak cosmic censorship (WCC) [6] holds. It means that black holes cannot collide each other if the total area of them exceeds the upper bound.

Davies [7] investigated a CEH in Robertson-Walker models with \(\Lambda\) and a perfect fluid satisfying the dominant energy condition and showed that the area of the cosmological horizon cannot decrease. From this result, one may expect that in generic asymptotically de Sitter space-times the area of a CEH cannot decrease as in the case of a BEH.

Boucher, Gibbons and Horowitz [8] showed that the area of the CEH is bounded from the above by \(\frac{12\pi}{\Lambda}\) on a regular time-symmetric hypersurface. Shiromizu, Nakao, Kodama and Maeda [5] also obtained the same conclusion on a maximal hypersurface. However, one cannot say that the same conclusion holds for CEHs in a general non-stationary asymptotically de Sitter space-time, because it is highly nontrivial whether a foliation by such hypersurfaces exists and covers the relevant portion of the space-time.

The WCC is assumed in the proof of the above results as well as in the case of a BEH. An example of Schwarzschild-de Sitter space-time shows significance of this assumption, and also suggests a close relation among the area of the CEH, the WCC and positivity of the gravitational energy (mass). Fig. 1 shows the mass parameter \(m\) as a function

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of the area $A$ of the event horizon and Figs. 2(a) and 2(b) shows the Penrose diagrams for the cases of $m > 0$ and $m < 0$, respectively. One easily finds that if the WCC holds ($m > 0$) the area $A_C$ of the CEH is bounded from the above by $12\pi/\Lambda$. Indeed, one finds that the total area of the BEH and the CEH has an upper bound $12\pi/\Lambda$. On the other hand, if the WCC is violated ($m < 0$) $A_C$ is not bounded.

In this paper, we show the area theorem that the area of the CEH in an asymptotically de Sitter space-time containing a black hole cannot decrease so that the second law of thermodynamics is satisfied, and the total area of BEH and CEH is less than $12\pi/\Lambda$, hence total Bekenstein-Hawking entropy is less than $3\pi/\Lambda$, if the space-time satisfies the WCC and the energy conditions. To this end, we define a quasi-local energy in a space-time with $\Lambda$ and its monotonicity and positivity. Very roughly speaking, our analysis is a generalization of the argument of the previous paragraph to general asymptotically de Sitter space-times which are neither stationary nor spherically symmetric.

We follow the notation of Ref. [9] and use the units $c = G = h = k_B = 1$.

II. ASYMPTOTICALLY DE SITTER SPACE-TIME AND THE AREA LAW FOR A COSMOLOGICAL EVENT HORIZON

In this section we shall show the area theorem (Theorem 1) for a CEH in an asymptotically de Sitter space-time.

As a precise definition of an asymptotically de Sitter space-time satisfying the WCC, we assume space-time $(M, g)$ to be strongly asymptotically predictable from a partial Cauchy surface $\Sigma$ and de Sitter in the future [5], and just call it asymptotically de Sitter. In what follows, causal relationships are considered in a larger manifold $(\tilde{M}, g)$ in which $(M, g)$ is conformally embedded. Note that the future conformal infinity $I^+$ of $M$ is a spacelike hypersurface in $\tilde{M}$ [10].

We shall consider asymptotically de Sitter space-times containing a black hole and an observer whose world line $\lambda$ has a future endpoint at the “future timelike infinity.” Then $J^-(\lambda)$ consists of two components, the BEH and the CEH for the observer [2]. As the BEH can be defined by $J^-(I^+)$, the CEH can be also defined in terms of $I^+$. Namely, we define the cosmological event horizon (CEH) to be the past Cauchy horizon $H^-(I^+)$ of the future infinity.

In general, the topology of $I^+$ is not determined. However, it seems reasonable to suppose that $I^+$ is diffeomorphic to $S^2 \times (0, 1)$ if the topology of the BEH is $S^2$. In analogy of weakly asymptotically simple and empty, and future asymptotically predictable space-time (see Prop. 9.2.3 of Ref. [9]), we also assume that there is a continuous onto map $\alpha : (0, \infty) \times \Sigma \rightarrow D^+(\Sigma) \setminus \Sigma$ satisfying the following. (1) For each $t \in (0, \infty)$, $\alpha_t := \alpha(t, \cdot)$ and restriction of $\alpha$ on $(0, t) \times \Omega_t^-(\Omega_t^+ - I^+)$ are homeomorphisms, where $\Omega_t := \alpha(t) \times \Sigma$; (2) For each $t \in (0, \infty)$, $\Sigma_t$ is a Cauchy surface for $D(\Sigma)$ such that (a) $\Sigma_t^+ \subset I^+(\Sigma_t^+ - I^+)$ when $t_2 > t_1$, and (b) the edge of $\Sigma_t^+ - I^+$ in $\tilde{M}$ is a spacelike two-sphere in $I^+$. We define $W_t := \Sigma_t \cap I^+$. We have $W_{t_1} \subset W_{t_2}$ for $t_2 > t_1$ and $\bigcup_{t \in (0, \infty)} W_t = I^+$.

We also present a lemma about the topology of a CEH.

**Lemma 1** (Each component of) any sufficiently nice cut of the cosmological event horizon $H^-(I^+)$ is a topological two-sphere.

**Proof.** Since $D^-(I^+) \cap M$ is a future set in $M$, its boundary in $M$, i.e., the CEH, must be a $C^1$- embedded submanifold of $M$ (see Prop. 6.3.1 of Ref. [9]). Moreover, $\text{int}D^-(I^+)$ is simply connected because it is homeomorphic $I^+ \times \mathbb{R}$ and $I^+$ is simply connected. Thus the conclusion follows. \qed

We use the following lemma, which is shown in Ref. [9], to prove Lemma 3.

**Lemma 2** Let $\Sigma$ be a partial Cauchy surface. For any $p \in D^-(\Sigma)$, $J^+(p) \cap D^-(\Sigma)$ is compact. \hfill \Box

**Lemma 3** $D^-(I^+) = \bigcup_{t \in (0, \infty)} D^-(W_t)$.

**Proof.** Let us define a continuous function $I^+ \ni p \mapsto t \in (0, \infty)$ defined by $p \in \text{edge}(W_t)$. Because Lemma 2 implies that for any $p \in D^-(I^+)$, $J^+(p) \cap I^+$ is compact in $\tilde{M}$, there exists a maximum value for the function above. So there is a $t \in (0, \infty)$ such that $W_t \supseteq J^+(p) \cap I^+$ and hence $p \in D^-(W_t)$. Thus we have $D^-(I^+) \subseteq \bigcup_{n \in \mathbb{N}} D^-(W_n) \subseteq \bigcup_{t \in (0, \infty)} D^-(W_t)$. It follows from $D^-(I^+) \supseteq D^-(W_t)$ for each $t \in (0, \infty)$ that $D^-(I^+) \supseteq \bigcup_{t \in (0, \infty)} D^-(W_t)$. \hfill \Box

In the next step we will prove Lemma 5 by using the following Limit Curve Lemma [11].

**Lemma 4** (Limit Curve Lemma) Let $\gamma_n : (-\infty, \infty) \rightarrow M$ be a sequence of inextendible non-spacelike curves (parametrized by arc length in $g_{\mathbb{R}}$ which is a complete Riemannian metric). Suppose that $p \in M$ is an accumulation point of the sequence $\{\gamma_n(0)\}$. Then there exist an inextendible non-spacelike curve $\gamma$ such that $\gamma(0) = p$ and subsequence $\{\gamma_m\}$ which converges to $\gamma$ uniformly (with respect to $g_{\mathbb{R}}$) on compact subsets of $\mathbb{R}$. 

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Lemma 5 For any generator \( \lambda \) of \( H^-(I^+) \), parametrized with respect to \( g_R \)-arc length, there exists a sequence \( \{ \lambda_n \} \) of null geodesics in \( D^-(I^+) \), parametrized with respect to \( g_R \)-arc length, such that (1) \( \{ \lambda_n \} \) converges uniformly to \( \lambda \) with respect to \( h \) on compact subsets of \( \mathbb{R} \), and (2) each \( \lambda_n \) generates an achronal set.

Proof. Let \( p \) be a point of \( \lambda \) which is not the endpoint. Any neighborhood \( U \) of \( p \) contains a point of \( D^-(I^+) \). It follows from Lemma 3 that there exist a \( n_0 \in \mathbb{N} \) such that \( U \cap D^-(W_{n_0}) \neq \emptyset \) hence \( U \cap H^-(W_{n_0}) \neq \emptyset \) for all \( n \geq n_0 \). Then one can construct a sequence \( \{ p_n \} \) such that \( p_n \in H^-(W_{m_n}) \) and \( p_n \rightarrow p \), where \( (W_{m_n}) \) is a subsequence of \( \{ W_n \} \). Letting \( \lambda_n \) be the generator of \( D^-(W_{m_n}) \) through \( p_n \), one has from Lemma 4 that there exists an inextendible non-spacelike \( C^0 \)-curve \( \gamma \) through \( p \) such that \( \{ \lambda_n \} \) converges to \( \gamma \) uniformly on compact subsets of \( \mathbb{R} \). However, because \( \{ \lambda_n \} \) can have its accumulation points only on \( H^-(I^+) \), \( \gamma \) must lie on \( H^-(I^+) \). Since \( \gamma \) is a non-spacelike curve through \( p \) and is lying on \( H^-(I^+) \), it must coincide with \( \lambda \).

Finally we present the following area theorem of the CEH.

Theorem 1 (Area law for a CEH) In an asymptotically de Sitter space-time with a piecewise smooth CEH satisfying the weak energy condition, \( A(H^-(I^+) \cap \Sigma_{t_2}) \geq A(H^-(I^+) \cap \Sigma_{t_1}) \) for \( t_2 > t_1 \), where \( A(S) \) denotes the area of a two-surface \( S \).

Proof. Piecewise smoothness of the CEH implies that there are a finite number of pairwise disjoint smooth submanifolds \( U_i \)’s such that the CEH is \( \bigcup_i U_i \). It suffices to show that the expansion \( \theta \geq 0 \) on each \( p \in \mathbb{int} U_i \) because each \( U_i \) is foliated by future inextendible null geodesic generators. For any point \( p \in \mathbb{int} U_i \) for some \( i \) there is an open set \( V \ni p \) diffeomorphic to \( S \times \mathbb{R} \) where \( S \) is a locally spacelike two-surface containing \( p \) with compact closure. By Lemma 4 and compactness of \( \mathbb{S} \) there is a sequence of diffeomorphisms \( \phi_n : V \rightarrow V_n \subset H^-(W_n) \) such that (1) each \( \phi_n(S) \) is spacelike, (2) each \( \phi_n \) preserves the foliations by null geodesic generators, and (3) \( \phi_n(V) \) converges uniformly to \( V \) on compact subsets of \( \mathbb{S} \times \mathbb{R} \). Suppose the expansion \( \theta \) of future-directed null geodesic generators of the CEH was negative at \( p \). Then by the continuity of \( \theta \) there would be some \( n \) such that the expansion \( \theta_n \) of generators of \( V_n \) was negative at \( \phi_n(p) \). From the weak energy condition the generator from \( \phi_n(p) \), since it is future complete, would have a conjugate point of \( \phi_n(S) \) (see Prop. 4.4.6 of Ref. [9]). This contradicts achronality of \( H^-(W_n) \).

Corollary If the assumptions of Theorem 1 hold and every future incomplete null geodesics terminates in a strong curvature singularity of Królik [12], then every generator of the CEH is future complete.

Proof. From the proof of Theorem 1, the expansion of each null geodesic generator cannot be negative. This contradicts the condition of the strong curvature singularity.

III. QUASI-LOCAL ENERGY IN SPACE-TIMES WITH \( \Lambda \)

We define a quasi-local energy \( E(S) \) in a space-time with \( \Lambda \) and examine its monotonicity and positivity, which we will use to show the existence of an upper bound for entropy (Theorem 2) in Sec. IV.

Let us introduce Hayward’s double null formalism [13], namely, smooth foliations of null three-hypersurfaces labeled by \( \xi \) such that each intersection of two hypersurfaces of constant \( \xi \) is a closed spacelike two-surface. We have the evolution vector \( u_\xi = \partial / \partial \xi \), the normal one-forms \( n_\xi = -d\xi \), the metric \( h = g + e^{-f(n_+ n^- + n^- n_+)} \) induced on the two-surface, the projection \( \lambda \) on the two-surface, the shift vectors \( r_\xi = \lambda u_\xi \), and the null normal vectors \( l_\xi = u_\xi - r_\xi \). The expansions \( \theta_\xi \), the shears \( \sigma_\xi \), and the twist \( \omega \) on a two-surface are defined as

\[
\theta_\xi = \frac{1}{2} h^{-1} : \mathcal{L}_\xi h, \\
\sigma_\xi = \mathcal{L}_\xi h - \theta_\xi h, \\
\omega = \frac{1}{2} e^f h \cdot [l_+, l_-],
\]

where \( \mathcal{L}_\xi \) represents the Lie derivatives along the vector fields \( l_\xi \), and a dot and a colon denote single and double contraction, respectively. The quasi-local energy is defined in each embedded spatial two-surface \( S \) as

\[
E(S) := \frac{1}{8\pi} \sqrt{\frac{A}{16\pi}} \int_S \mu \left( \mathcal{R} + e^f \theta_+ \theta_- - \frac{2\Lambda}{3} \right),
\]
where \( A, \mathcal{R}, \) and \( \mu \) represent the total area of \( S \), the Ricci scalar on \( S \), and the area 2-form on \( S \), respectively. This is the Hawking energy with the last term added in the integrand. Physically, \( E(S) \) is the gravitational energy subtracted by the energy due to the cosmological constant \( \Lambda \), so that it is considered as the energy of the matter fields. In Schwarzschild-de Sitter space-time \( E(S) \) coincides with the mass parameter \( m \). In spherically symmetric space-times with dust \( E(S) \) coincides with the mass function [14]. In space-times without \( \Lambda \) our quasi-local energy \( E(S) \) reduces to the Hawking energy.

The Einstein equations are given by

\[
e^{-f}L^\pm(e^f\theta^\pm) + \frac{1}{2}\theta^\pm^2 + \frac{1}{4}\|\sigma^\pm\|^2 = -8\pi\phi^\pm,
\]

where \( \phi^\pm = T(l^\pm, l^\pm) \) and \( \rho = T(l^+, l^-) \) for the energy tensor \( T \), and \( \mathcal{D} \) is the covariant derivative with respect to \( h \).

Let us examine the monotonicity of \( E(S) \) on an outgoing null hypersurface \( \xi_- = \text{constant} \) (the monotonicity on an ingoing null hypersurface \( \xi_+ = \text{constant} \) or on a spacelike hypersurface can be argued similarly.) The derivative of the energy \( E(S) \) along the outgoing direction \( l^+ \) is

\[
8\pi L^+ E = \sqrt{\frac{A}{16\pi}} \int_S \mu \theta^+ \int_S \mu (\mathcal{R} + e^f\theta^+_\theta^-) - \int_S \mu \theta^- \left( \frac{1}{4}\|\sigma^-\|^2 + 8\pi\phi^- \right) - \int_S \mu \theta^+ \left( \frac{1}{2}\mathcal{R} + \frac{1}{2}e^f\theta^+_\theta^- + \mathcal{D} \cdot \left( \frac{1}{2}\mathcal{D} f + \omega \right) \right)

= \frac{1}{2}\mathcal{D} f + \omega \right| - 8\pi e^f \rho. \quad (3.7)
\]

We assume that the matter fields satisfy the dominant energy condition, \( \phi^+ \geq 0 \) and \( \rho \geq 0 \), and take a foliation of the hypersurface \( \xi_- = \text{constant} \) by spatial two-surfaces \( S \). The energy \( E(S) \) is non-decreasing in the outgoing null direction\( (\theta^+, \theta^- \leq 0) \), \( L^+ E \geq 0 \), if

\[
\langle \theta^+ \rangle(f) \geq \langle \theta^+_+ F \rangle \quad (3.8)
\]

on each \( S \), where

\[
F := \mathcal{R} + e^f\theta^+_\theta^- + 2\mathcal{D} \cdot \left( \frac{1}{2}\mathcal{D} f + \omega \right), \quad (3.9)
\]

\[
\langle \cdot \rangle := \frac{\int_S \mu \cdot}{\int_S \mu}. \quad (3.10)
\]

We remark that each term of \( F \) except the third term is invariant under rescaling of the outgoing null normal \( l^+ \).

An example of the foliations satisfying Eq. (3.8) is one with \( F = \text{constant} \), which we can take by the rescaling of \( l^+ \). Another example is the uniformly expanding foliation [15].

IV. UPPER BOUND FOR THE AREA

In this section we will show that the total area of the BEH and the CEH is bounded in asymptotically de Sitter space-times (Theorem 2).

We define the apparent horizons according to Hayward [16]. A marginal surface is a spatial two-surface \( S \) on which \( \theta^+_+ = 0 \) or \( \theta^- = 0 \). A black hole apparent horizon (BAH) is the closure \( T_B \) of a hypersurface \( T_B \) foliated by marginal surfaces on which \( \theta^+_+ = 0, \theta^- < 0 \) and \( L^+ \theta^+_+ < 0 \). A cosmological apparent horizon (CAH) is \( T_C \) foliated by marginal
surfaces on which \( \theta_- = 0, \theta_+ > 0 \) and \( \mathcal{L}_+ \theta_- > 0 \). Here the coordinates \( \xi_\pm \) are taken so that they are constant on each of the above spatial two-surfaces.

Hayward, Shiromizu and Nakao \cite{4} showed that the area of a BAH has an upper bound \( 4\pi/\Lambda \). They also showed that the area of a BEH is less than \( 4\pi/\Lambda \) by implicitly assuming the existence of the limit two-surface \( \mathcal{S} \) of the BEH, though its physical meaning is not clear (see Appendix). Instead, one can reach the same conclusion under a physically reasonable condition, strongly future asymptotically predictability (or WCC) in an “extended” sense \cite{17}. It states that singularities are hidden inside not only a BEH but also a BAH. More precisely, the closure of the domain of dependence of a partial Cauchy surface contains not only \( \mathcal{I}^+ \) and the BEH but also the outermost part of the BAH, i.e., (i) there exist \( t > 0 \) and a subset \( T_B^+ \) of \( T_B \), foliated by marginal surfaces, such that \( H^-(T_B^+) \cap J^+(\Sigma_t) \supseteq \mathcal{I}^-(\mathcal{I}^+) \cap J^+(\Sigma_t) \) and \( [I^-(T_B^+) \cap \mathcal{I}^+(\Sigma_t)] \subseteq D^+(\Sigma_t) \). We have the following proposition, whose proof we give in Appendix.

**Proposition 1** In an asymptotically de Sitter space-time satisfying condition (i) above and the weak energy condition, the area of a black hole event horizon (BEH) is less than \( 4\pi/\Lambda \).

Now we will show that the total area of BEH and CEH has an upper bound \( 12\pi/\Lambda \) by making use of Proposition 1. We require the following conditions. (ii) There exists \( t_0 > 0 \) such that the cross section of \( \mathcal{J}^-(\mathcal{I}^+) \cap \Sigma_t \) is smooth one connected component and the topology is \( S^2 \); (iii) There exists a marginal surface \( \Sigma_t \) with \( \theta_\pm = 0 \) whose topology is \( S^2 \) in each \( \Sigma_t(t \geq t_0) \) and surrounds \( \mathcal{J}^-(\mathcal{I}^+) \cap \Sigma_t \); (iv) \( I^- (T_C) \cap J^+(\Sigma_t) = (I^- (\mathcal{I}^+) - D^-(\mathcal{I}^+)) \cap J^+(\Sigma_t) \); (v) any null geodesic generator of \( \mathcal{E} \) is future complete. (vi) Matter fields satisfy the dominant energy condition.

This implies that matter field satisfies the weak energy condition (see e.g., Ref. \cite{9}.) \( \Box \) (vii) There exists a foliation satisfying Eq. (3.8) on each outgoing null hypersurface \( \mathcal{J}^-(\Sigma_t) \) inside the CAH. Condition (iv) is similar to condition (i) above.

**Lemma 6** For an arbitrary small positive value, \( \epsilon_1 \), there is an acasual hypersurface \( \Sigma_{t_1}(t_1 > t_0) \) such that for any closed spacelike two-surface \( \mathcal{S}_B \) of \( \mathcal{J}^-(\mathcal{I}^+) \cap J^+(\Sigma_{t_1}) \) the quasi-local energy \( E(\mathcal{S}_B) \) satisfies

\[
E(\mathcal{S}_B) \geq \frac{1}{8\pi} \sqrt{\frac{A(\mathcal{S}_B)}{16\pi}} \left( 8\pi - \epsilon_1 - \frac{2\Lambda}{3} A(\mathcal{S}_B) \right) > 0. \quad (4.1)
\]

**Proof.** Consider each null geodesic generator \( L_+ \) of the BEH. By condition (v), (vi) and the Raychaudhuri equation (3.5), \( \lim_{\xi \to \infty} \theta_+ = 0 \) is satisfied, where \( \xi \) is an affine parameter of \( l_+ \). \( \lim_{\xi \to \infty} \int_E \mu \epsilon^j \theta_+ \theta_- = 0 \) is also satisfied because the area of a BEH has an upper bound. Therefore there is a \( \Sigma_{t_1} \) such that for any closed spacelike two-surface \( \mathcal{S}_B \) of \( \mathcal{J}^-(\mathcal{I}^+) \cap J^+(\Sigma_{t_1}) \), \( \int_E \mu \epsilon^j \theta_+ \theta_- \) is larger than \( -\epsilon_1 \), where \( \epsilon_1 \) is an arbitrary small positive value. From Eq. (3.4) and Prop. 1 one can get the desired result by using the Gauss–Bonnet theorem and condition (ii).

**Lemma 7** \( \mathcal{J}^-(T_C) \cap J^+(\Sigma_{t_0}) = (\cup_j \mathcal{J}^-(\Sigma_{t_j})) \cap J^+(\Sigma_{t_0}). \)

**Proof.** For any point \( p \in \mathcal{J}^-(T_C) \cap J^+(\Sigma_{t_0}) \) there is a point \( q \in J^+(p) \cap T_C \). Then there is \( \Sigma_t \ni q \) so that \( p \in \mathcal{J}^-(\Sigma_t) \).

**Theorem 2** If an asymptotically de Sitter space-time satisfies the conditions (i)-(vii) above, \( A_B := \lim_{\xi_+ \to \infty} A(\mathcal{S}_B) \) and \( A_C := \lim_{\xi_- \to \infty} A(\mathcal{S}_C) \) satisfy

\[
A_B + A_C + \sqrt{A_B A_C} \leq \frac{12\pi}{\Lambda}. \quad (4.2)
\]

**Remark.** In particular, the area \( A_C \) of the CEH is less than \( 12\pi/\Lambda \).

**Proof.** For any closed spacelike two-surface \( \mathcal{S}_C \) of \( H^- (\mathcal{I}^+) \cap D^+(\Sigma) \) there exists a partial Cauchy surface \( \Sigma_{S_C} \) containing \( \mathcal{S}_C \). Consider a sequence of marginal surfaces \( \Sigma_{S_n} (n \in \mathbb{N}) \) defined above and define \( N^+_n \) and \( N^-_n \) as the null hypersurfaces generated by the future-directed outgoing and ingoing null geodesic generators of \( \mathcal{J}^-(\Sigma_{S_n}) \), respectively. Denote the spacelike two-surface \( N^-_n \cap \Sigma_{S_n} \) by \( K_n \). From the condition (iv) and Lemma 7 it follows that \( \lim_{n \to \infty} A(K_n) = A(\mathcal{S}_C) \). The expansion \( \theta_- \) of \( L_- \) is non-negative in the future direction between \( \Sigma_{S_n} \) and \( K_n \) because \( \mathcal{L}_-(\epsilon^j \theta_-) \leq 0 \) there, as implied by the Raychaudhuri equation (3.5) of \( L_- \) and condition (vi), and by \( \theta_+ = 0 \) on each \( \Sigma_{S_n} \). Thus, as in the proof of Prop. 1, there exists \( n_1 \) for an arbitrary small positive value \( \epsilon_2 \) such that for all \( n > n_1 \)

\[
A(\mathcal{S}_C) - \epsilon_2 \leq A(\Sigma_{S_n}) \quad (4.3)
\]
is satisfied.

Consider outgoing null hypersurfaces $\hat{N}_n^+ := N_n^+ \cap J^+(\Sigma_{t_n}) \cap J^+(\Sigma_{S_C})$. From the condition (iv) and Lemma 7, for any neighbourhood $U$ of $J^-(I^+) \cap J^+(\Sigma_1) \cap J^+(\Sigma_{S_C})$, there is $n_2 > n_1$ such that for $n > n_2$ each $\hat{N}_n^+$ intersect $U$.

For $n > n_2$, take spacelike two-surfaces $Q_n$ in $N_n^+ \cap U$. The sequence $\{Q_n\}$ converges to a spacelike two-surface $S_B$ of $J^-(I^+) \cap J^+(\Sigma_{t_n}) \cap J^+(\Sigma_{S_C})$. By the continuity of $E(Q_n)$, for an arbitrary small $\epsilon_3 > 0$ there is $n_3 > n_2$ such that $E(S_B) - \epsilon_3 \leq E(Q_n)$ for each $n > n_3$. By condition (vii), the energy $E(S)$ is non-decreasing from $Q_n$ to $S_n$ on $N_n^+$. Thus $E(S_B) - \epsilon_3 \leq E(S_n)$ for each $n > n_3$. By Lemma 6 and Eq.(3.4) for $S_n$, this inequality can be rewritten as

$$A(S_B) + A(S_n) \leq \frac{12\pi}{\Lambda} - \sqrt{A(S_B)A(S_n)} + O(\epsilon_1) + O(\epsilon_3).$$

Since $C$ is an arbitrary two-surface of $H^-(I^+)$, one gets the desired result by taking limit $\epsilon_1, \epsilon_3 \to 0$. 

\section*{V. CONCLUSIONS AND DISCUSSION}

We have shown in Theorem 1 that in an asymptotically de Sitter space-time the area $A(S_C)$ of the CEH is non-decreasing if the WCC and the weak energy condition hold. This means that the area law of event horizons holds not only for a BEH but also for a CEH hence it also applies to the total area of event horizons (total Bekenstein-Hawking entropy, i.e. a quarter of the total area of the BEH and the CEH). Next we have shown in Theorem 2 that the final values of the areas satisfy $A_B + A_C + \sqrt{A_B A_C} \leq 12\pi/\Lambda$. This means that the final values of entropies $S_B := A_B/4$ of the BEH and $S_C := A_C/4$ of the CEH satisfy

$$S_B + S_C + \sqrt{S_B S_C} \leq 3\pi/\Lambda.\quad (5.1)$$

In particular, the total entropy is bounded from the above by $3\pi/\Lambda$ in an asymptotically de Sitter space-time. We note that the inequality in Theorem 2 is stronger than the previous result and conjecture which state that $A(S_B) \leq 4\pi/\Lambda$ and $A(S_C) \leq 12\pi/\Lambda$.

As discussed in Ref. [2], a BEH is unstable against the Hawking radiation, while a CEH is stable. Physically this suggests that all asymptotically de Sitter space-times approach de Sitter space-time. This is consistent with the inequality (5.1) which states that for a fixed $\Lambda$ the total entropy attains its maximum in de Sitter space-time, although the quantum effects were not taken into account in the derivation of the inequality. This curious correspondence suggests that the inequality is another law of EH thermodynamics in asymptotically de Sitter space-times.

It is of interest to pursue connections of the present result with the cosmic no hair conjecture [2,18]. Here we consider a weaker version of the conjecture which states that a space-time with $\Lambda$ has a future asymptotic region rather than is recollapsing, i.e., the space-time is future asymptotically de Sitter. Since the areas of the EHs have universal bound (i.e., are bounded by numbers which depend only on $\Lambda$) and the areas are expected to become larger when matter falls into them, one expects that the amount of matter which falls into the EHs has a universal bound. So, in collapse of an isolated object, if most of the matter falls into either the BEH or the CEH, i.e., if there will be no heavy shell-like “star” surrounding the black hole, one can expect that the total initial energy of the matter should be bounded by a number which only depends on $\Lambda$. This may provide a criterion for the existence of the future asymptotic region of the space-time, that is, a criterion for the validity of the cosmic no hair conjecture.

To solve the problems above, it is very important to know the property of the total entropy $S_T$ of the universe, i.e., the sum of the entropy of the EHs and that of the matter between the EHs. We conjecture that in asymptotically de Sitter space-time $S_T$ is non-decreasing, i.e., the generalized second law of thermodynamics holds, and also $S_T$ is bounded.

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APPENDIX A: PROOF OF PROPOSITION 1

In their proof of the theorem of the upper bound for the area of the BEH in Ref. [4] Hayward, Shiromizu and Nakao implicitly assumed that there is a limit two-surface of the BEH on which quantities such as $L \lesssim \theta_+$ are continuous, i.e., independent of how one approaches the “timelike infinity” $i^+$. However, this is physically not very well motivated and is highly nontrivial in general. Here, we shall drop the assumption above and prove a slightly modified version of the theorem.

Proof of Proposition 1. It is enough to show that the area of a BEH in $J^+(\Sigma)$ has an upper bound because the area does not decrease in the future direction as shown in Ref. [5]. Let us consider a sequence of marginal surfaces $\mathcal{S}_n$ with $\theta_+ = 0$ on the BAH $T_B$ and take a sequence of subsets $T_n$ of $T_B$ such that $T_{n-1} \subset T_n$, edge$(T_n) = \mathcal{S}_n$, $\bigcup_{n \in \mathbb{N}} T_n = T_B$. Consider spacelike two-surfaces $T := J^-(I^+) \cap \Sigma_t$ and $T_n := H^-(T_n) \cap \Sigma_t$ for some (sufficiently large) fixed $t$. We can observe $D^-(T'_n) = \bigcup_{n \in \mathbb{N}} D^-(T_n)$, by replacing $I^+$ in Lemma 3 with $T'_n$. This together with condition (i) implies that the sequence $T_n$ converges to $T$. The expansion $\theta_+$ of each null geodesic generator $l_+$ of $H^-(T_n)$ is non-negative in the future direction between $T_n$ and $\mathcal{S}_n$ because $L_+ (\epsilon^\theta_+) \leq 0$ there, as implied by the Raychaudhuri equation (3.5) and the weak energy condition, and by $\theta_+ = 0$ on each $\mathcal{S}_n$. Thus,

$$A(T_n) \leq A(\mathcal{S}_n) \leq \frac{4\pi}{\Lambda},$$  \hspace{3cm} (A1)

where the second inequality is obtained by integrating Eq. (3.6) multiplied by $e^{\ell}$ on marginal surface $\mathcal{S}_n$ and using the Gauss-Bonnet theorem [4]. Since the sequence $T_n$ converges to $T$, for arbitrary small $\epsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that $A(T_n)$ with $n > n_0$ satisfies

$$A(T) - \epsilon \leq A(T_n).$$  \hspace{3cm} (A2)

From inequalities (A1) and (A2) we have

$$A(T) - \epsilon \leq \frac{4\pi}{\Lambda}.$$  \hspace{3cm} (A3)

Since this holds for any $\epsilon$ we have

$$A(T) \leq \frac{4\pi}{\Lambda}.$$  \hspace{3cm} (A4)
FIG. 1. The mass parameter $m$ of a Schwarzschild-de Sitter solution for a fixed $\Lambda$ is related to the area $A$ of event horizons as $m = (A/16\pi)^{1/2}(1 - \Lambda A/12\pi)$. $A_b$, $A_c$ are the areas of a BEH and a CEH, respectively.

FIG. 2. Penrose diagrams of Schwarzschild-de Sitter space-times with mass parameters (a) $m > 0$, and (b) $m < 0$, respectively.