A two-dimensional representation of four-dimensional gravitational waves

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Abstract

The Einstein equation in $D$ dimensions, if restricted to the class of space-times possessing $n = D - 2$ commuting hypersurface–orthogonal Killing vectors, can be equivalently written as metric-dilaton gravity in 2 dimensions with $n$ scalar fields.

For $n = 2$, this results reduces to the known reduction of certain 4-dimensional metrics which include gravitational waves. Here, we give such a representation which leads to a new proof of the Birkhoff theorem for plane-symmetric space–times, and which leads to an explanation, in which sense two (spin zero-) scalar fields in 2 dimensions may incorporate the (spin two-) gravitational waves in 4 dimensions. (This result should not be mixed up with well–known analogous statements where, however, the 4–dimensional space–time is supposed to be spherically symmetric, and then, of course, the equivalent 2–dimensional picture cannot mimic any gravitational waves.)

Finally, remarks on hidden symmetries in 2 dimensions are made.

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1 Introduction

Gravitational waves and their collisions in a Friedmann universe have been discussed recently \[1\] by use of the metric

\[
\text{d} s^2 = e^M (\text{d} t^2 - \text{d} z^2) - e^{\phi}(e^\psi \text{d} x^2 + e^{-\psi} \text{d} y^2) \tag{1}
\]

where $M, \phi$ and $\psi$ depend on $t$ and $z$ only. The space-times possessing a metric of type eq. (1) can be invariantly defined by requiring that two commuting hypersurface-orthogonal space-like Killing vectors exist.

In higher-dimensional cosmology, $D$-dimensional space-times possessing $n = D - 2$ commuting hypersurface–orthogonal space–like Killing vectors are discussed, cf. \[2, 3\] and refs. cited there. They can be expressed via

\[
\text{d} s^2 = \text{d} \sigma^2 - \sum_{k=1}^{n} a_k^2(x^\alpha)(\text{d} x^k)^2 \tag{2}
\]

where

\[
\text{d} \sigma^2 = g_{\alpha\beta}(x^\alpha)\text{d} x^\alpha\text{d} x^\beta \tag{3}
\]

$\alpha, \beta \in \{0, n + 1\}$, $a_k > 0$, and $g_{\alpha\beta}$ has signature $(+ -)$.

For $D = 4$, metric (2, 3) can locally be written in the form of eq. (1) ($x^0 = t, \ x^3 = z$), because the 2-dimensional space–time $\text{d} \sigma^2$ is conformally flat, at least locally.

In the last two years, much progress, cf. refs. \[4 − 15\], has been made in dealing with 2-dimensional gravity, both classically and with its quantization. However, this progress lacks from a satisfying physical application due to the hypothetic nature of (super)-strings. So, it is often considered as toy model to study conceptual features of gravity under simplified circumstances. For instance, the evaporation of $1 + 1$-dimensional black holes \[13\] should tell something about the evaporation of $3 + 1$-dimensional ones. Analogously, the collapse of massless scalar fields in $1 + 1$-dimensional dilaton gravity \[14\] should be similar in structure to black hole formation in the $3 + 1$-dimensional case.

In the present paper, however, we go beyond a toy model: known results from 2-dimensional gravity shall be applied to higher–dimensional models.
under such circumstances where the correspondence between the low and the high dimension can be given explicitly.

In section 2, the $D$-dimensional Einstein equation for the metric (2, 3) is rewritten in a 2-dimensional form, in sct. 3 we describe the peculiarities for $D = 4$, and sct. 4 discusses the results. We enclose an appendix on the hidden symmetries in 2 dimensions.

2 The $D$-dimensional Einstein equation

The Ricci tensor of $ds^2$, eq. (2) will be denoted by $R_{AB}$, $A, B$ take the values $0, ..., n + 1$. The Einstein sum convention shall be applied to indices $\alpha, \beta$ and $A, B$ but not to $i, j, k$. Let $\phi_k = \ln a_k$, $\phi = \sum_{k=1}^{n} \phi_k$, and $\psi_k = \phi_k - \frac{\phi}{n}$. Consequently

$$\sum_{k=1}^{n} \psi_k = 0 \quad (4)$$

and eq. (2) can be rewritten as

$$ds^2 = d\sigma^2 - e^{2\phi/n} \sum_{k=1}^{n} e^{2\psi_k} (dx^k)^2 \quad (5)$$

where $\phi$ and $\psi_k$ depend on the $x^\alpha$ only.

The Ricci tensor of $d\sigma^2$ is denoted by $P_{\alpha\beta}$, and then the non-vanishing components of $R_{AB}$ are given by

$$R_{\alpha\beta} = P_{\alpha\beta} - \frac{1}{n} \phi_{,\alpha} \phi_{,\beta} - \sum_{k=1}^{n} \psi_{k,\alpha} \psi_{k,\beta} \quad (6)$$

and

$$R_{kk} = (\Box \phi_k + \phi_k^{,\alpha} \phi_{,\alpha}) e^{2\phi_k} \quad (7)$$

With $G = |\det g_{AB}|$ and $g = |\det g_{\alpha\beta}|$ we get $G = g e^{2\phi}$, so that eqs. (6,7) together give the Einstein-Hilbert Lagrangian in $D$ dimensions for metric (5) as

$$L = R\sqrt{G} = (P - \frac{n-1}{n} \Box \phi - \sum_{k=1}^{n} \psi_{k,\alpha} \psi_{k}^{,\alpha}) e^{\phi} \sqrt{g} \quad (8)$$
It should be noted that we added a suitable multiple of the divergence $(e^a \phi^a)_\alpha$ to get this simple equation.

It holds (this is a non-trivial statement): The variational derivatives of $L$ eq. (8) with respect to the 2-dimensional metric $g_{\alpha\beta}$ and $\phi$ and those $\psi_k$ fulfilling the constraint eq. (4) lead to the $D$-dimensional Einstein equation for metric (5). That every $D$-dimensional Ricci-flat space fulfills these conditions is trivial. The reverse statement, however, is non-trivial. Its proof uses the fact that by this procedure, no spurious solutions can appear.\footnote{Spurious solutions will appear e.g. if we restrict to synchronized time $t$ before the variation with respect to $g_{\alpha\beta}$ is carried out.}

Of course, the variation of $R\sqrt{G}$ with respect to $g_{AB}$ gives the $D$-dimensional Einstein tensor.

The next problem is how to deal with the constraint eq. (4). One could simply mention that it is harmless, because its validity in the initial conditions implies its validity everywhere, but one can also do it explicitly by eliminating of one of the fields $\psi_k$. For $n = 2$ we introduce $f_1$ via

$$\psi_1 = f_1/\sqrt{2}, \quad \psi_2 = -f_1/\sqrt{2}$$

For $n = 3$ we introduce 2 scalars $f_i, i = 1, 2$ (the Misner parametrization) via

$$\psi_1 = \frac{f_1}{\sqrt{6}} + \frac{f_2}{\sqrt{2}}, \quad \psi_2 = \frac{f_1}{\sqrt{6}} - \frac{f_2}{\sqrt{2}}, \quad \psi_3 = -2f_1/\sqrt{6}$$

and get

$$\sum_{k=1}^{3} \psi_{k,\alpha} \psi^\alpha_k = \sum_{i=1}^{2} f_{i,\alpha} f_i^\alpha$$

and for larger values $n$, analogous linear relations hold such that eq. (4) is identically fulfilled, and, moreover,

$$\sum_{k=1}^{n} \psi_{k,\alpha} \psi^\alpha_k = \sum_{i=1}^{n-1} f_{i,\alpha} f_i^\alpha$$

The comparison with eq. (1) and (5) for $n = 2$ requires $\psi = 2\psi_1$.

Result: Lagrangian (8) with the $n + 1$ scalar fields $\phi, \psi_k$ subject to the constraint eq. (4) is equivalent to the Lagrangian

$$L = (P - \frac{n - 1}{n} \Box \phi - \sum_{i=1}^{n-1} f_{i,\alpha} f_i^\alpha) e^\phi \sqrt{g}$$

(10)
with the \( n \) scalar fields \( \phi, f_i \).

This kind of reduction seems to be new, and it shall be applied as follows: For 2-dimensional nonlinear gravity \( L = f(P) \sqrt{g} \), a generalized Birkhoff theorem [16] states that every solution of the field equation possesses an isometry. In [17], nonlinear gravity is shown to be conformally equivalent (after suitable field redefinitions) to dilaton metric theories in 2 dimensions following from

\[
L = D(\phi) P - V(\phi) - Z(\phi) (\nabla \phi)^2 + Y(\phi) \square \phi
\]

(11)

so that this Birkhoff theorem holds also in all these theories, cf. [4, 9, 13]. In [18] it is explained, that this theorem is a consequence of the fact that the traceless part of the Ricci tensor identically vanishes in 2 dimensions. Let us now apply this Birkhoff theorem: If we restrict to solutions with vanishing fields \( f_i \), then Lagrangian (10) has the structure (11), and one isometry can be found. The vanishing of \( f_i \) means that all functions \( a_k \) in eq. (2) coincide, and so we may conclude:

If all functions \( a_k \) in metric (2) are equal, and \( ds^2 \) (2) is \( D \)-dimensionally Ricci-flat, then besides the \( n \) isometries

\[
\frac{\partial}{\partial x^k},
\]

one further isometry exists, its Killing vector\(^2\) is proportional to \( \epsilon^{\alpha \beta} P_{\alpha \beta} \), where \( \epsilon^{\alpha \beta} \) is the Levi-Civita tensor in \( d\sigma^2 \).

3 The 4-dimensional Einstein equations

If we restrict the results of sect. 2 to \( D = 4 \), then we recover a couple of known results. Nevertheless, it proves useful to point them out in our general approach.

The vanishing of the \( f_i \) is equivalent to put \( \psi = 0 \) in eq. (1), i.e., one has plane-symmetric space–times. The Birkhoff theorem for them has already

\(^2\)In case \( \epsilon^{\alpha \beta} P_{\alpha \beta} = 0 \) in a whole region, then \( P = \text{const.} \), and we have not only one but three additional isometries.
been proven in [19, 20]. Our approach has the advantage that the additional Killing vector can be explicitly given without the need to specify a coordinate system. The analogous procedure for spherical instead of plane symmetry can be found in [21].

Gravitational waves in our Universe – even if not yet surely identified – represent a serious object of experimental undertakings. Therefore, metrics like eq. (1) deserve to be carefully understood also theoretically. The controversy [22, 23] about “what can be learned from metric–dilaton gravity in two dimensions for the theoretical understanding of gravitational waves in four dimensions?” can now be resolved by mentioning the above deduced equivalence of the 4-dimensional Einstein equation for eq. (1) to the 2-dimensional metric dilaton theory eq. (10) for $n = 2$.

Let us add some related results: In chapter 15 of [24] the reduction of metric (1) to a 2-dimensional formulation is given (and, by the way, without requiring hypersurface-orthogonality, one additional scalar field is needed), but they did it only from the point of view how to simplify the search for exact 4-dimensional solutions; they did not give the relation to dilaton gravity in 2 dimensions.

In [25, 26], Geroch discussed the solutions of Einstein’s equation in the presence of one or two Killing vectors. Especially, in appendix A of [26] he gave a general construction to the reduced 2-dimensional formulation if two commuting Killing vectors exist. The main result of [25, 26] was the construction of new solutions of the Kerr–Schild type

$$\hat{g}_{ij} = g_{ij} + v_i v_j$$

from a given one $g_{ij}$. This “hidden symmetry” of Einstein’s equation has been applied in [27, 28, 29] for quantization issues. Both [27] and [28] use also a reduction of the 4-dimensional Einstein equation with two independent Killing vectors to a two-dimensional model, but the details are quite different, so that no direct comparison can be made. More detailed: [27] studies the stationary axisymmetric case with the Ernst equation, and [28] uses the complex Ashtekar variables.
Now let us return to the 4-dimensional metrics of type eq. (1) possessing two commuting hypersurface-orthogonal space-like Killing vectors. These Killing vectors represent the translations into $x$- and $y$-direction. This space-time is called plane-symmetric if it possesses also the rotations in the $x$-$y$-plane as isometries. Clearly, this is the case if and only if $\psi$ in eq. (1) represents a constant.

According to [19, 20], plane-symmetric solutions of the Einstein equation possess a further symmetry. Now, let us give a new proof of this statement using the results of 2-dimensional gravity (the analogous procedure for spherical instead of plane symmetry has been carried out in ref. [17, 18]): For $n = 2, D = 4$, eq. (8) reduces to

$$L = (\frac{1}{2} \Box \phi - \frac{1}{2} \psi_\alpha \psi^\alpha) e^\phi \sqrt{g}$$  \hspace{1cm} (12)$$

For $\psi = \text{const.}$, $L$ eq. (12) has the requested structure as dilaton-metric theory without extra scalar field, and so we may apply the generalized Birkhoff theorem mentioned at the end of sect. 2 giving rise also to an additional isometry for the solutions (1) if $\psi = 0$. Of course, here it repeated only a known result, but this shall show how the method can be applied. The potential applications will induce the following: Let the dilaton-metric 2-dimensional theory with one additional scalar field $\psi$ ($L$ eq. (12)) possess a certain class of solutions, then they give rise to analogous solutions of the 4-dimensional Einstein equation metric (1), and all such solutions - including gravitational waves - will be reached by this procedure. The next step of application might be that any satisfactory quantization of $L$ eq. (12) may be directly transformed to a corresponding quantization of 4-dimensional gravity (cf. also [11] for this).

The correspondences of this type discussed up to now had been essentially restricted to spherically symmetric gravity in 4 dimensions, just excluding the gravitational waves from the beginning. But quantization of gravity should include gravitational waves. We circumvented this problem by changing from spherical symmetry to the symmetry of eq. (1).

To elucidate the formalism at a concrete example, let us now consider a
special class of pp-waves (plane-fronted waves with parallel rays) which can be written in the form of eq. (1) with vanishing $M$, i.e., $d\sigma^2$ is flat, and the gradients of $\phi$ and $\psi$ are parallel and lightlike. With these additional assumptions, eq. (1) may be written as

$$ds^2 = 2dudv - e^{\phi(u)}(e^{\psi(u)}dx^2 + e^{-\psi(u)}dy^2)$$  \hspace{1cm} (13)$$

It holds: Eq. (13) represents a gravitational wave, i.e., the Ricci tensor of $ds^2$ vanishes, if and only if

$$\phi'' + \frac{1}{2}\phi'^2 + \frac{1}{2}\psi'^2 = 0$$  \hspace{1cm} (14)$$

where the dash denotes $\frac{d}{du}$. (And it is non-flat iff additionally $\phi'' + \phi'\psi' \neq 0$.)

On the other hand, if we insert $d\sigma^2 = 2dudv$ into the field equation following from the 2-dimensional Lagrangian $L$ eq. (12) and require that $\phi$ and $\psi$ depend on $u$ only, then again just eq. (14) remains to be solved. (By the way, it follows already from the field equation that the gradients of $\phi$ and $\psi$ are lightlike.) All the polynomial curvature invariants of metric (13) can be expressed as polynomial invariants of the 2-dimensional system (12), i.e., as $2\phi, 2\psi, \psi_{jk}\phi_{jk}$ etc. The proof that all of them identically vanish can be directly performed in the 2-dimensional picture.

### 4 Discussion

One of the arguments why metric-dilaton gravity in two dimensions should not be able to represent a 4–dimensional gravitational wave goes as follows: The scalars in the 2–dimensional picture have spin zero, and so the spin 2–graviton cannot be correctly incorporated by them.

This argument can be outruled as follows: If one restricts the calculation from eq. (4) via eqs. (8), (9) till eq. (10) to $n = 2$, then one can see that the scalar $\psi$ of eq. (1) is a scalar in the 2-dimensional point of view only.

In the 4–dimensional picture one has to consider the $\psi_1 - \psi_2$–plane (where $\psi = 2\psi_1 = -2\psi_2$) and one has to consider the following part of the metric

$$e^{\psi_1}dx^2 + e^{-\psi}dy^2 = e^{2\psi_1}dx^2 + e^{2\psi_2}dy^2$$
One can see: A rotation in physical space in the $x - y$-plane by $90^\circ$ corresponds to $\psi_1 \rightarrow -\psi_1$ and $\psi_2 \rightarrow -\psi_2$; this represents a rotation by $180^\circ$ in the $\psi_1 - \psi_2$-plane. So the spin is calculated as $180^\circ/90^\circ = 2$, and the correct tensor representation is maintained for the spin 2-gravitational wave.

Let us finish by mentioning some of the results of refs. [3 - 12]: They discuss exact classical solutions of dilaton-metric theories of gravitation [6], of theories with $L = P^k \sqrt{g}$ [3], and of theories with torsion [5, 7]. In 2 dimensions we have the peculiarity that the torsion tensor $T^\alpha_{\beta\gamma}$ is equivalent to the pseudovector field

$$T^\alpha = T^\alpha_{\beta\gamma} \varepsilon^{\beta\gamma}$$

where $\varepsilon^{\beta\gamma}$ is the covariantly constant antisymmetric unit-pseudotensor. (Of course, the interpretations may differ.)

A lot deal of work has already be done to determine the global behaviour, their horizons and singularities of the corresponding solutions, cf. [3,4,8,9,10,11,15]. In 2 dimensions, the behaviour of the space-time is much easier to understand because the space-time is flat if and only if the curvature scalar vanishes, whereas in 4 dimensions (e.g. a gravitational wave of the form of eq. (1)) a space-time may be non-flat with all polynomial curvature invariants being identically zero. By field redefinitions and/or conformal transformations many of the variants of 2-dimensional gravity theories become equivalent (every paper [3-16] uses such transformations). For theories $L = F(P) \sqrt{g}$ with non-linear $F$ and, according to this equivalence, also for dilaton-metric theories without extra scalar fields, i.e.

$$L = e^\phi [P - \Box \phi + V(\phi)] \sqrt{g}$$

a generalized Birkhoff theorem holds (deduced in [16] for $L = F(P) \sqrt{g}$ and in [4,30] for the other cases): every solution possesses a symmetry.

The results presented here should motivate to direct future research of 2-dimensional models to such Lagrangians which have really a known 4-dimensional counterpart.
Appendix on hidden symmetries

Hidden symmetries of 2–dimensional models are discussed in [27 – 31] from several points of view. Geometrically, they have three origins: The local conformal flatness of all 2-spaces; the fact, that locally, \( P \) is a divergence; and, the fact that the traceless part of \( P_{ij} \) vanishes. The latter has already been mentioned to be the origin of the Birkhoff-type theorem. It should be mentioned that for the conformal equivalence of \( L = f(P)\sqrt{|g|} \) and \( \hat{L} = \phi \hat{P} - V(\phi) \) the necessary conformal factor is a well-defined function of the curvature scalar \( P \), so this conformal equivalence of theories is not a consequence of the overall conformal flatness.

For any constants \( a, b \) with \( a \neq 0 \) let

\[
f_{a,b}(P) = af(P) + bP
\]

In this picture, the “new-conformal extra symmetry” introduced in [31] is simply a variation of the constants \( a \) and \( b \). Here we will show that even within the set of classical vacuum solutions\(^3\), this symmetry is not so innocuous as it seems from the first glance. To this end we define \( f^{(\epsilon)}(P) = P^{1+\epsilon} \) and ask for the limiting theory as \( \epsilon \to 0 \).

If always \( a = 1, \ b = 0 \), then we get

\[
\lim_{\epsilon \to 0} f^{(\epsilon)}(P) = P
\]

If we put, however, \( a = \frac{1}{\epsilon}, \ b = -\frac{1}{\epsilon} \), i.e., \( f^{(\epsilon)}_{ab}(P) = \frac{1}{\epsilon}(P^{1+\epsilon} - P) \) then

\[
\lim_{\epsilon \to 0} f^{(\epsilon)}_{ab}(P) = P \ln P
\]

This means: If we factorize the set of Lagrangians with respect to this \( a \)–\( b \)–symmetry (15), then it is no more a Hausdorff space.

From a more geometric point of view this can be explained as follows: \( P\sqrt{-g} \) in two dimensions gives no contribution to the field equation from

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\(^3\)Of course, every element of the 2-parameter family (15) of Lagrangians gives rise to the same set of vacuum solutions.
two different reasons: First, $P\sqrt{-g}$ is locally a divergence. Second, $\int P\sqrt{-g}$ is a conformally invariant action, and 2-spaces are locally conformally flat. [On the contrary in four dimensions, no polynomical curvature invariant exists which is simultaneously a local divergence and which gives rise to a conformally invariant action.] And the coincidence of these two properties produces the additional $P \cdot \ln P$ similar to ordinary linear differential equations where the solution $e^{\lambda x}$ is accompanied by $e^{\lambda x} \cdot x$ if the eigenvalue $\lambda$ is a double one.

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