In this talk some recent results in the quantization of Chern-Simons field theories in the Coulomb gauge will be presented. In the first part, the consistency of the Chern–Simons field theories in this gauge is proven using the Dirac’s canonical formalism for constrained systems. Despite the presence of non-trivial self-interactions in the gauge fixed functional, it will be shown that the commutation relations between the fields are trivial at any perturbative order in the absence of couplings with matter fields. If these couplings are present, instead, the commutation relations become rather involved, but it is still possible to study their main properties and to show that they vanish at the tree level. In the second part of the talk the perturbative aspects of Chern–Simons field theories in the Coulomb gauge will be analysed. In particular, it will be shown by explicit computations and in a regularization independent way that there are no radiative contributions to the $n$–point correlation functions. Finally the Feynman rules in the Coulomb gauge will be derived on a three dimensional manifold with a spatial section given by a closed and orientable Riemann surface.

1 Introduction

In the recent past, the Chern–Simons (C–S) field theories $^{1,2}$ have intensively been studied in connection with several physical and mathematical applications $^{3,4}$. A convenient gauge fixing for these theories is provided by the Coulomb gauge. As a matter of fact, the presence of nontrivial interactions in the gauge fixed action allows perturbative computations. Perturbation theory is important in the calculations of the so-called link invariants $^{5,6,7,8}$ and whenever interactions are present, because in the latter case the C–S field theories are no longer exactly solvable. The advantage of the Coulomb gauge in this case is that the calculations are considerably simpler than in the covari-
ant gauges and there are no radiative corrections. Moreover, with respect to
the axial and light cone gauges, the Coulomb gauge can easily be imposed
also on manifolds with non-flat spatial sections, like for instance Riemann
surfaces. The absence of quantum contributions is a great advantage on non flat
space-times, where the computation of Feynman integrals becomes technically
difficult. Another important feature of the C–S field theories in the Coulomb
gauge is that they can be considered as two dimensional models as it will be
shown below.

Starting from the seminal works of refs. 2,9 and 10, the Coulomb gauge
has been already applied in a certain number of physical problems involving
C–S based models 4,11,12,13,14, but still remains less popular than the covariant
and axial gauges. One of the main reasons is probably the fact that there
are many perplexities concerning the use of this gauge fixing, in particular
in the case of the four dimensional Yang–Mills theories 15,16,17,18. Recently,
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analysis of the non-abelian case has been done in 22.

In this talk some recent results in the quantization of C–S field theories
in the Coulomb gauge will be presented following refs. 21,22,23 and avoiding
technical details as much as possible. In the first part of the talk the C–S field
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The absence of quantum contributions is a great advantage on non flat space-times, where the computation of Feynman integrals becomes technically difficult. Another important feature of the C–S field theories in the Coulomb gauge is that they can be considered as two dimensional models as it will be shown below.

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In this talk some recent results in the quantization of C–S field theories in the Coulomb gauge will be presented following refs. 21,22,23 and avoiding technical details as much as possible. In the first part of the talk the C–S field theories are analysed by means of the Dirac’s formalism for constrained systems. Besides some subtleties already noticed in 24, the derivation of the final Dirac brackets requires in the Coulomb gauge some care with distributions. Moreover, the final commutation relations (CR’s) between the fields are derived both in the case of pure and interacting C–S field theories. With respect to the Yang–Mills field theories the CR’s are rather involved. At a first sight, this is surprising in topological field theories with vanishing Hamiltonian and without degrees of freedom. However, at least in the pure C–S field theory, in which there are no interactions with matter fields, we show that this complexity is only apparent. As a matter of fact, taking into account the Gauss law and the Coulomb gauge fixing, the commutation relations between the gauge fields vanish identically at any perturbative order as expected. In this way the Chern–Simons field theories in the Coulomb gauge are not only perturbatively finite as has already been checked in the covariant gauges 25, but also free. This is not a priori evident, because in the Coulomb gauge the C–S functional contains non-trivial self-interaction terms. In the interacting case it is only possible to prove that the CR’s are zero at the zeroth order approximation in perturbation theory. At higher orders however they are in general different from zero and have a very complicated expression. This is probably due to the fact that C–S field theories admit states with non-standard statistics.
In the second part of this talk the radiative corrections of the Green functions are computed at any loop order and it is shown in a regularization independent way that they vanish identically. No regularization is needed for the ultraviolet and infrared divergences since, remarkably, they do not appear in the amplitudes. The vanishing of the quantum corrections is in agreement with the triviality of the commutation relations found using the Dirac’s canonical approach to constrained systems. It is important to notice that the absence of any quantum correction despite the presence of nontrivial self-interactions in the Lagrangian is a peculiarity of the Coulomb gauge that cannot be totally expected from the fact that the theories under consideration are topological, as finite renormalizations of the fields and of the coupling constants are always possible. For instance, in the analogous case of the covariant gauges, only the perturbative finiteness of the C–S amplitudes has been shown \cite{26} in a regularization independent way exploiting BRST techniques \cite{27}. Indeed, a finite shift of the C–S coupling constant has been observed in the Feynman gauges by various authors \cite{28,29}.

Finally, the Feynman rules of the C–S field theories will be derived also on a manifold whose spatial section is a Riemann surface of genus $g$.

## 2 Canonical Quantization of the C–S Field Theory in the Coulomb Gauge

### 2.1 Notations

In this Section we will use the following notations. The Lagrangian of the pure $SU(N)$ C–S field theory is given by

$$L_{CS} = \frac{s}{8\pi} \epsilon^{\mu\nu\rho} \left( A_{\mu}^{a} \partial_{\nu} A_{\rho}^{a} - \frac{1}{3} f^{abc} A_{\mu}^{a} A_{\nu}^{b} A_{\rho}^{c} \right)$$

(1)

where $s$ is a dimensionless coupling constant and $A_{\mu}^{a}$ is the gauge potential. Greek letters $\mu, \nu, \rho, \ldots = 0, 1, 2$ denote space–time indices, while the first latin letters $a, b, c, \ldots = 1, \cdots, N^2 - 1$ denote color indices. Moreover, the totally antisymmetric tensor $\epsilon^{\mu\nu\rho}$ is defined by the convention $\epsilon^{012} = 1$. The metric is given by $g_{\mu\nu} = \text{diag}(1, -1, -1)$. To derive the C–S Hamiltonian $H_{CS}$ we have to compute the canonical momenta:

$$\pi^{\mu,a}(x,t) = \frac{\delta S_{CS}}{\delta (\partial_{\mu} A_{\nu}^{a}(x,t))}$$

(2)
where \( S_{CS} = \int d^3x L_{CS} \), \( t = x^0 \) and \( \mathbf{x} = (x^1, x^2) \). A straightforward calculation shows that:

\[
H_{CS} = \int d^3x A_0^a \left( D_i^{ab} \pi^{i,b} + \partial_i \pi^{i,a} \right) \quad (3)
\]

In the above equation the following convention has been used for the spatial components of the covariant derivative: \( D^a_{\mu} \equiv \partial_{\mu} \delta^a + f^{abc} A_0^c \). Finally, the nonvanishing equal time Poisson brackets (PB) among the canonical variables read as follows:

\[
\{ A_\mu^a (x,t), \pi^b_\nu (y,t) \} = \delta^{ab} g^{\mu\nu} \delta (x-y)
\]

2.2 Constraints and intermediate Dirac brackets

From eqs. (1) and (2) we obtain the following primary constraints:

\[
\begin{align*}
\varphi^{0,a} & = \pi^{0,a} \\
\varphi^{i,a} & = \pi^{i,a} - \frac{s}{8\pi} \epsilon^{ij} A_j^a \\
& \quad i = 1, 2
\end{align*}
\]

where \( \epsilon^{ij}, i, j = 1, 2 \), is the two dimensional totally antisymmetric tensor defined by \( \epsilon^{12} = 1 \). Following the Dirac procedure for constrained systems, the latter will be imposed in the weak sense: \( \varphi^{\mu,a} \approx 0 \). To this purpose, we construct the extended Hamiltonian:

\[
\tilde{H}_{CS} = H_{CS} + \int \lambda_{\mu}^{a} \varphi^{\mu,a} d^2x \quad (6)
\]

where the \( \lambda_{\mu}^a \)'s represent the Lagrange multipliers corresponding to the primary constraints \( \varphi^{\mu,a} \).

From the consistency conditions \( \dot{\varphi}^{\mu,a} = \{ \varphi^{\mu,a}, \tilde{H}_{CS} \} \approx 0 \), we obtain the secondary constraint:

\[
\mathcal{G}^a = D_i^{ab} \pi^{i,b} + \partial_i \pi^{i,a} \approx 0 \quad \text{(Gauss law)}
\]

and two relations which determine the Lagrange multipliers \( \lambda_1 \) and \( \lambda_2 \):

\[
\frac{s}{4\pi} \epsilon^{ij} \left( D_j^{ab} A_0^b - \lambda_j^a \right) \approx 0 \quad \quad i = 1, 2
\]

It is possible to see that the consistency condition \( \mathcal{G}^a \approx 0 \) does not lead to any further independent equation. The operators \( \mathcal{G}^a \) generate the \( SU(N) \) group of gauge transformations only after eliminating the second class constraints.
\( \varphi_0 \approx 0 \) of eq. (5). To this purpose, we introduce the intermediate Dirac brackets (DB's) \( \{ \ , \ \}^* \) associated to these constraints. After some calculations one finds for the intermediate DB's among the canonical variables the following expressions:

\[
\{ A^a_i(t, x), \pi^{j,b}_i(t, y) \}^* = \frac{1}{2} \delta^{ab} \delta^j_i \delta(x - y) \tag{9}
\]
\[
\{ A^a_i(t, x), A^b_j(t, y) \}^* = \frac{4\pi s}{\delta^{ab} \epsilon_{ij} \delta(x - y)} \tag{10}
\]
\[
\{ \pi^{i,a}_i(t, x), \pi^{j,b}_i(t, y) \}^* = \frac{s}{16\pi} \delta^{ab} \epsilon_{ij} \delta(x - y) \tag{11}
\]

Exploiting the DB's (9)–(11), we obtain the relations:

\[
\{ G^a_i(t, x), A^b_j(t, y) \}^* = -D_i^{ab} \delta(x - y) \tag{12}
\]
\[
\{ G^a_i(t, x), G^b_j(t, y) \}^* = -f^{abc} G^c_i(t, x) \delta(x - y) \tag{14}
\]

where \( G^a_i(t, x) = \int d^2x G^a_i(t, x) \psi^a(x) \). This shows that the \( G^a_i(t, x) \) are the generators of the \( SU(N) \) gauge transformations as desired.

2.3 Imposing the Coulomb gauge

At this point, we are left with the constraints given by eq. (4) and by the Gauss law (7). The former constraint, which is first class and involves the conjugate momentum of \( A^a_0 \), can be ignored. As a matter of fact, the field \( A^a_0 \) just plays the role of the Lagrange multiplier associated to the Gauss law in the Hamiltonian (3) and has no dynamics. From eqs. (12)–(14) it turns out that the Gauss law (7) is a first class constraint. To make it second class, we introduce the Coulomb gauge fixing:

\[
\partial_i A^i_a \approx 0 \tag{15}
\]

and the new extended Hamiltonian:

\[
\dot{H}_{CS} = \int d^2x \left[ -A_0^a \dot{G}^a_i + \frac{s}{8\pi} A^a_i \partial^i B^a + \lambda_0^a \pi^{0,a}_i \right] \tag{16}
\]

From the condition \( \{ \partial_i A^i_a, \dot{H}_{CS} \}^* \approx 0 \), we obtain an equation for \( A^a_0 \):

\[
\partial_i D_i^{ab} A^b_0 \approx 0 \tag{17}
\]
Moreover, the requirement \( \{ \partial^a D^{ab}_i A^b_0 (x), \hat{H}_{CS} \}^* \approx 0 \) determines the Lagrange multiplier \( \lambda_0 \):

\[
- \triangle \lambda_0^a - \{ \partial_i (A_i \times A_0)^a, \hat{H}_{CS} \}^* \approx 0
\]  

(18)

In the above equation the symbol \( \triangle \) denotes the two dimensional Laplacian \( \triangle = -\partial_i \partial^i \) and

\[ (A_i \times A_0)^a \equiv f^{abc} A^b_i A^c_0 \]

Another independent equation, which fixes the Lagrange multipliers \( B^a \), is provided by the requirement \( \hat{G}^a \approx 0 \):

\[ \{ G^a, \hat{H}_{CS} \}^* \approx -\frac{s}{8\pi} D^{lb}_j \partial^i B^b \approx 0 \]  

(19)

Let us notice that the above relations (7), (15) and (17)-(19) are compatible with the equations of motion of the gauge potentials:

\[
\epsilon^{ij} (D^{ab}_i A^b_j - \partial_j A^a_i) = 0
\]  

(20)

\[
D^{ab}_j A^b_0 - \partial_0 A^a_j = 0
\]  

(21)

As a matter of fact (20) is equivalent to the condition \( G^a = 0 \). Moreover, multiplying for instance eq. (21) with the differential operator \( \epsilon^k_{\,ij} \partial_k \), we obtain the relation:

\[
\partial_0 \partial^k A^a_i - \partial^k D^{lb}_j A^b_0 = 0
\]

which is consistent with the Coulomb gauge and the condition (17) on \( A^a_0 \).

2.4 The final Dirac brackets and their properties

It is now possible to realize that the Gauss law (7) and the Coulomb gauge fixing (15) form a set of second class constraints, so that we can impose them in the strong sense computing the final Dirac brackets \( \{ \, , \} \)\(_{DB} \). Putting

\[
\chi^a_1 = G^a \quad \chi^a_2 = \partial_i A^{i,a}
\]

with \( \alpha, \beta = 1, 2 \), and skipping all the technical details of the calculations that can be found in ref. 21, we have:

\[
\{ A^a_i (x), A^b_j (y) \} \text{\(_{DB} \)} = -\frac{4\pi}{s} S^{ab} \epsilon_{ij} \delta (x - y) + \frac{4\pi}{s} \epsilon_{ik} \partial_k D^{ic}_j (y) D^{bc}(x, y) - \frac{4\pi}{s} \epsilon_{ij} D^{ac}_i (x) \partial_k D^{cb}(x, y)
\]  

(22)
where
\[ D^a_{ic}(x) \partial_i^a D^{cb}(x, y) = \delta^{ab} \delta(x - y) \] (23)

After imposing the constraints (7) and (15) in the strong sense, the Hamiltonian \( \hat{H}_{CS} \) vanishes, but the commutation relations (CR’s) between the fields remain complicated.

Let us study the main properties of the above DB’s.

- **Antisymmetry.** The antisymmetry of the right hand side of eq. (22) is not explicit, but can be verified with the help of the relation:

\[ D^{ab}(x, y) = D^{ba}(y, x) \] (24)

The above identity is due to the fact that \( D^{ab}(x, y) \) is the Green function of the self-adjoint differential operator defined in eq. (23). Exploiting the above relation one finds that

\[ \{A^a_i(x), A^b_j(y)\}_DB = -\{A^b_j(y), A^a_i(x)\}_DB \] (25)

as expected

- **Consistency with the Coulomb gauge constraint.** The CR’s (22) are consistent with the Coulomb gauge, i.e.: 

\[ \{A^a_i(x), \partial^i A^b_j(y)\}_DB = \{\partial^i A^a_i(x), A^b_j(y)\}_DB = 0 \]

- **Covariance under the Poincaré group of transformations.** The proof that the C–S theory in the Coulomb gauge is invariant under the Poincaré group is not trivial due to the complicated CR’s (22). A good strategy consists in evaluating the CR’s among the generators of the Poincaré group using the intermediate DB’s (9)–(11). In this way one finds that the Poincaré algebra is not closed due to “extra” terms which are proportional to the constraints. For instance, for the generators of the time and the space translations we obtain the following result:

\[ \{P_0, P_k\}^* = \int d^2 x A^a_0 \partial_k G^a \]

where \( G \) is given in (7). Clearly, all these unwanted terms disappear after imposing the final DB’s (22) and the CR’s between the generators of the Poincaré group can be recovered.

7
• Interactions. For simplicity we have considered here pure Chern–Simons field theories. However, we stress that the form of the CR’s remains unchanged also adding to the lagrangian interactions of the kind $L^I = \int d^3 x A_\mu J^{\mu, a}$, where $J^{\mu, a}$ is a current associated to matter fields. The only differences occur in equations (7), (8) and (17)–(19), in which $J^{\mu, a}$ will appear as an external source. For instance the Gauss law (7) is modified as follows:

$$D^{ab}_{i} \pi^{i,b} + \partial_j \pi^{j,b} + J^a_0 \approx 0 \quad (26)$$

2.5 The abelian case

The case of a Chern–Simons field theory with abelian gauge group $U(1)$ is particularly instructive in order to understand the meaning of the CR’s. Indeed, in this case the Green function $D(x,y)$ has the following simple expression:

$$D(x,y) = -\frac{1}{2\pi} \log |x - y| \quad (27)$$

Let $U_\mu$ denote the abelian gauge fields. Substituting the right hand side of equation (27) in (22) and replacing the DB’s with quantum commutators, we obtain:

$$[U_i(t,x), U_j(t,y)] = 0$$

As a consequence the fields $U_\mu$ do not propagate. This result is in agreement with the fact that the theory is topological so that the fields have no dynamics. Indeed, exploiting the Gauss law, the Coulomb gauge fixing and eqs. (8), (17)–(19), it is easy to see that the only possible solution of the equations of motion is $U_i = U_0 = \lambda_\mu = B = 0$. On the other side, the triviality of the CR’s holds also in the presence of interactions, i. e. when the theory becomes no longer topological and the solutions of the equations of motion $U_\mu, \lambda_\mu, B$ are in general different from zero.

2.6 The non-abelian case

In non-abelian C–S field theories the equations of motion of the constraints are nonlinear and can be solved only using a perturbative approach. At the zeroth order, the Green function $D(x,y)$ is given again by eq. (27). Thus the CR’s (22) are zero at this order. At higher orders, however, the right hand side of eq. (22) is in general different from zero, apart from the case in which there are no interactions. The vanishing of the CR’s for the pure C–S theories, proven at any perturbative order in the coupling constant $\frac{1}{s}$ in $2^{11}$, is in agreement with the fact that these theories are topological. Finally, we notice that the
CR’s (22) are particularly complicated with respect to the usual Yang–Mills field theories. This is probably related to the fact that field theories coupled to a C–S term exhibit a non-abelian statistics.

2.7 Final Remarks

• The C–S theories become in the Coulomb gauge two dimensional models. Only the fields $A^a_i$, for $i = 1, 2$, have in fact a dynamics, which is governed by the commutation relations (22). Moreover, the latter do not contain time derivatives, so that the time can be considered as an external parameter.

• If no interactions with matter fields are present, the CR’s (22) vanish at any perturbative order. Thus the C–S field theories in the Coulomb gauge are not only finite, but also free. A natural question that arises at this point is if analogous conclusions can be drawn for the covariant gauges. For this reason it would be interesting to repeat the procedure of canonical quantization developed here also in this case.

• If the interactions with other fields are switched on, the CR’s (22) still remain trivial in the abelian case. Thus, if we quantise the C–S theory replacing the DB’s with quantum commutators, we obtain that

$$[A_a^i(x), A_b^j(y)] = 0 \quad (28)$$

In the non-abelian case the above relation is valid only at the zeroth order in the coupling constant $\frac{1}{s}$, while at higher orders the CR’s do not vanish and are rather complicated. Let us notice that there is no contradiction between eq. (28) and the fact that, starting from the Lagrangian (1), it is possible to derive a non-zero propagator for the C–S fields. In fact, going from the Hamiltonian normal-ordered formalism to the Lagrangian time-ordered formalism it is know that contact terms may arise, which contain distributions in the time variable. Indeed, the components of the propagator computed in the next section will have exactly the form of contact terms of this kind $^{31}$.

• Finally, we have shown that the CR’s (22) are perfectly well defined and do not lead to ambiguities in the quantization of the C–S models in the Coulomb gauge. They are consistent with the constraints and the Poincaré covariance of the theory. Moreover, in the pure C–S field theory the CR’s vanish at any perturbative order in $\frac{1}{s}$.  

9
3 Perturbative Analysis of the C–S Field Theory in the Coulomb Gauge

3.1 Derivation of the Feynman rules

In this section we consider the following gauge fixed C–S action:

\[
S_{CS} = S_0 + S_{GF} + S_{FP}
\]

(29)

with

\[
S_0 = \frac{s}{4\pi} \int d^3 x \epsilon^{\mu\nu\rho} \left( \frac{1}{2} A_\mu^a \partial_\nu A_\rho^a - \frac{1}{6} f^{abc} A_\mu^a A_\nu^b A_\rho^c \right)
\]

(30)

\[
S_{GF} = \frac{is}{8\pi\lambda} \int d^3 x \left( \partial_i A^i \right)^2
\]

(31)

and

\[
S_{FP} = i \int d^3 x \tau^a \partial_i \left( D^i [A] c^a \right)
\]

(32)

In (31) \( \lambda \) is a real gauge fixing parameter. With respect to the previous section the metric is now Euclidean and is of the form \( g_{\mu\nu} = \text{diag}(1,1,1) \). Moreover, the covariant derivatives are now defined as follows

\[
D_\mu^a [A] = \partial_\mu \delta^{ab} - f^{abc} A^c_\mu
\]

In eq. (29) the Coulomb gauge constraint is weakly imposed and the proper Coulomb gauge fixing

\[
\partial_i A^i = 0 \quad i = 1, 2
\]

(33)

of the previous section is recovered setting \( \lambda = 0 \) in eq. (31).

From (29) the components of the gauge field propagator \( G_{\mu\nu}^{ab}(p) \) in the Fourier space are given by:

\[
G_{ij}^{ab}(p) = -\delta^{ab} \frac{4\pi \lambda p_ip_j}{s} \frac{p^4}{p^4}
\]

(34)

\[
G_{0j}^{ab}(p) = \delta^{ab} \left( \frac{4\pi}{s} \epsilon_{0jk} \frac{p^k}{p^4} - \frac{4\pi \lambda p_j p_0}{s} \frac{p^4}{p^4} \right)
\]

(35)

\[
G_{0i}^{ab}(p) = -\delta^{ab} \left( \frac{4\pi}{s} \epsilon_{0jk} \frac{p^k}{p^4} + \frac{4\pi \lambda p_0 p_j}{s} \frac{p^4}{p^4} \right)
\]

(36)

\[
G_{00}^{ab}(p) = -\delta^{ab} \frac{4\pi \lambda p_0^2}{s} \frac{p^4}{p^4}
\]

(37)
where $\mathbf{p}^2 = p_1^2 + p_2^2$. Let us notice that the variable $p_0$ appears only in the longitudinal contributions to the propagator and disappears after choosing the proper Coulomb gauge. Also the ghost propagator $G_{gh}^{ab}(p)$ is independent on $p_0$:

$$G_{gh}^{ab}(p) = \frac{\delta^{ab}}{p^2} \quad (38)$$

Finally, the three-gluon vertex and the ghost-gluon vertex are respectively given by:

$$V_{\mu_1\mu_2\mu_3}^{a_1 a_2 a_3} (p,q,r) = -\frac{is}{3\sqrt{4\pi}} (2\pi)^3 f^{a_1 a_2 a_3} \epsilon_{\mu_1\mu_2\mu_3} \delta^{(3)} (p + q + r) \quad (39)$$

and

$$V_{gh i_1}^{a_1 a_2 a_3} (p,q,r) = -i(2\pi)^3 (q)_{i_1} f^{a_1 a_2 a_3} \delta^{(3)} (p + q + r) \quad (40)$$

In the above equation we have only given the spatial components of the ghost-gluon vertex. From eq. (32), it is in fact easy to realize that in the Coulomb gauge its temporal component is zero. As we see, the presence of $p_0$ remains confined in the vertices (39)–(40) and it is trivial because it is concentrated in the Dirac $\delta$-functions expressing the momentum conservations. As a consequence, the CS field theory can be considered as a two dimensional model in the proper Coulomb gauge.

3.2 Potential divergences

At this point, we study the divergences that may arise in the computation of the Feynman diagrams. The potential divergences are of three kinds: ultraviolet, infrared and spurious.

1. Ultraviolet divergences (UV). The naive power counting gives the following degree of divergence $\omega(G)$ for a given Feynman diagram $G$:

$$\omega(G) = 3 - \delta - E_B - \frac{E_G}{2} \quad (41)$$

with $^b$

(a) $\delta =$ number of momenta which are not integrated inside the loops

(b) $E_B =$ number of external gluonic legs

(c) $E_G =$ number of external ghost legs

$^b$We use here the same notations of ref. $^{32}$
Eq. (41) shows that UV divergences are possible in the two and three point functions, both with gluonic or ghost legs. Moreover, there is also a possible logarithmic divergence in the case of the four point interaction among two gluons and two ghosts. In principle, we had to introduce a regularization for these divergences but in practical calculations this is not necessary. As a matter of fact, we will see below that there are no UV divergences in the quantum corrections of the Green functions.

2. Infrared (IR) divergences. The pure C–S field theories are known to be free of infrared divergences so that there is no need to discuss them.

3. Spurious divergences. These singularities appear because the propagators \((34)-(38)\) are undamped in the time direction and are typical of the Coulomb gauge. To regularize spurious divergences of this kind, it is sufficient to introduce a cutoff \(\Lambda_0 > 0\) in the domain of integration over the variable \(p_0\):

\[
\int_{-\infty}^{\infty} dp_0 \rightarrow \int_{-\Lambda_0}^{\Lambda_0} dp_0 \quad (42)
\]

The physical situation is recovered in the limit \(\Lambda_0 \rightarrow \infty\). As we will see, this regularization does not cause ambiguities in the evaluation of the radiative corrections at any loop order. In fact, the integrations over the temporal components of the momenta inside the loops turn out to be trivial and do not interfere with the integrations over the spatial components.

3.3 Perturbative analysis at one loop order

In this Section we compute the \(n\)–point correlation functions of C–S field theories at one loop order. From now on, we choose for simplicity the proper Coulomb gauge, setting \(\lambda = 0\) in eq. (31). In this gauge the gluon-gluon propagator has only two nonvanishing components:

\[
G_{j0}(p) = -G_{0j}(p) = \delta^{ab} \frac{4\pi}{s} \epsilon_{ijk} p^k \quad (43)
\]

The following observation greatly reduces the number of diagrams to be evaluated:

Observation: Let \(G^{(1)}\) be a one particle irreducible (1PI) Feynman diagram containing only one closed loop. Then all internal lines of \(G^{(1)}\) are either ghost or gluonic lines.
A proof of the above observation can be found in ref. 22. An important consequence is that, at one loop, the only non–vanishing diagrams occur when all the external legs are gluonic. Hence we have to evaluate only the diagrams describing the scattering among \( n \) gluons.

This can be done as follows. First of all, we consider the diagrams with internal gluonic lines. After suitable redefinitions of the indices and of the momenta, it is possible to see that their total contribution is given by:

\[
V_{i_1...i_n}^{a_1...a_n} (1; p_1, ..., p_n) = C \left[ -i (2\pi)^3 \right] n! \frac{n! (n - 1)!}{2} \delta^{(2)} (p_1 + ... + p_n) \tag{44}
\]

\[
f^{a_1 b_1' c_1'} f^{a_2 b_2' c_2'} ... f^{a_n b_n' c_n'} \int d^2 q_1 \frac{q_1^{i_1} ... q_n^{i_n} + q_1^{j_1} ... q_n^{j_n} + ... + q_n^{i_n - 1} q_1^{i_1}}{q_1^2 ... q_n^2} \tag{45}
\]

where \( C = (2\Lambda_0)^{2n} \) is a finite constant coming from the integration over the zeroth components of the momenta and

\[
q_2 = q_1 + p_1 + p_n + p_{n-1} + ... + p_3
\]

\[
\vdots
\]

\[
q_j = q_1 + p_1 + p_n + p_{n-1} + ... + p_{j+1}
\]

\[
\vdots
\]

\[
q_n = q_1 + p_1
\]

for \( j = 2, ..., n - 1 \).

The case of the Feynman diagrams containing ghost internal lines is more complicated. After some work, it is possible to distinguish two different contributions to the Green functions with \( n \) gluonic legs that cannot be reduced into one by renaming indices and momentum variables:

\[
V_{i_1...i_n}^{a_1...a_n} (2a; p_1, ..., p_n) = -C \left[ -i (2\pi)^3 \right] n! \frac{n! (n - 1)!}{2} \delta^{(2)} (p_1 + ... + p_n) f^{a_1 b_1' c_1'} f^{a_2 b_2' c_2'} ... f^{a_n b_n' c_n'} \int d^2 q_1 \frac{q_1^{i_1} ... q_n^{i_n}}{q_1^2 ... q_n^2} \tag{46}
\]

and

\[
V_{i_1...i_n}^{a_1...a_n} (2b; p_1, ..., p_n) = -C \left[ -i (2\pi)^3 \right] n! \frac{n! (n - 1)!}{2} \delta^{(2)} (p_1 + ... + p_n) f^{a_1 b_1' c_1'} f^{a_2 b_2' c_2'} ... f^{a_n b_n' c_n'} \int d^2 q_1 \frac{q_1^{i_1} ... q_n^{i_n}}{(q_1^2 ... q_n^2)^2} \tag{47}
\]

13
In the above equations the variables $q_2, \ldots, q_n$ and the constant $C$ are the same as in eq. (44).

As it is possible to see from eqs. (44),(46) and (47), the only nonvanishing components of the $n$ points functions are those for which all tensor indices $\mu_1, \ldots, \mu_n$ are spatial. We notice here that eq. (47) has been obtained after a shift of the integration variable $q_1$. However, it is not difficult to verify that the right hand sides of eqs. (44)–(47) are neither IR nor UV divergent for $n \geq 3$, so that a shift of $q_1$ is not dangerous. At this point we can sum equations (44),(46) and (47) together. It is easy to realize that the total result is zero, i.e.:

$$V_{a_1 \ldots a_n}^{a_1 \ldots a_n}(1; p_1, \ldots, p_n) + V_{a_1 \ldots a_n}^{a_1 \ldots a_n}(2a; p_1, \ldots, p_n) + V_{a_1 \ldots a_n}^{a_1 \ldots a_n}(2b; p_1, \ldots, p_n) = 0 \quad (48)$$

For $n \geq 3$ this result is regularization independent since the Feynman integrals are IR and UV convergent.

Only the case $n = 2$ needs some more care and will be treated separately. After a few calculations one obtains that for $n = 2$ the total contribution to the gluonic propagator to one loop is given by:

$$V_{ij}^{ab}(1; p_1, p_2) + V_{ij}^{ab}(2a; p_1, p_2) + V_{ij}^{ab}(2b; p_1, p_2) =$$

$$(2\pi)^6 (2\Lambda_0)^2 N \delta^{ab} \delta^2(p_1 + p_2) \int d^2q \frac{\left[q_i(p_1)_j - q_i(p_1)_i\right]}{q^2(q + p_1)^2} \quad (49)$$

The integrand appearing in the rhs of (49) is both IR and UV finite. Moreover, a simple computation shows that the integral over $q$ in (49) is zero. As a consequence, there are no contributions to the Green functions at one loop.

### 3.4 The higher order radiative corrections

Now we are ready to consider the higher order corrections of the $n$ points Green functions. At two loop, a general Feynman diagram $G^{(2)}$ can be obtained contracting two legs of a tree diagram $G^{(0)}$ with two legs of a one loop diagram $G^{(1)}$. As previously seen, the latter have only gluonic legs and their tensorial indices are all spatial. Consequently, in order to perform the contractions by means of the propagator (43), there should exist one component of $G^{(0)}$ with at least two temporal indices, but this is impossible. To convince oneself of this fact, it is sufficient to look at fig. (1) and related comments. The situation does not improve if we build $G^{(0)}$ exploiting also the ghost-gluon vertex (40), because it has no temporal component. As a consequence, all the Feynman graphs vanish identically at two loop order. Let us notice that it is possible
to verify their vanishing directly, since the number of two loop diagrams is relatively small in the Coulomb gauge and one has just to contract the space-time indices \( \nu_i, i = 1, \ldots, n \), can be temporal.

The proof that also higher order diagrams vanish can be done by induction. First of all, a diagram with \( N + 1 \) loops \( G^{(N+1)} \) has at least one subdiagram \( G^{(N)} \) containing \( N \) loops. Supposing that \( G^{(N)} \) is identically equal to zero because it cannot be constructed with the Feynman rules (38)-(40) and (43), also \( G^{(N+1)} \) must be zero. As we have seen above, there are no Feynman diagrams for \( N = 2 \). This is enough to prove by induction that C–S field theories have no radiative corrections in the Coulomb gauge for any value of \( N \).

3.5 Final remarks

- In the Coulomb gauge the C–S field theories do not have quantum corrections at any loop order, has as been shown by explicit computations.

- IR and UV divergences are absent in the calculations of the Feynman diagrams. Only spurious singularities are present, related to the fact that the propagators are undamped in the time direction. They are similar to the singularities observed in the four dimensional Yang–Mills field theories\(^{15}\), but in the C–S case appear in a milder form. In fact, after introducing the regularizarion (42) and integrating over the time component of the momenta in a given amplitude, the total contribution at any loop order reduces to an overall constant factor. The remaining calculations consist of finite two dimensional Feynman integrals over the space variables. As a consequence, the results obtained here are regularization independent.

- The vanishing of the quantum contributions described in Section 3 is a peculiarity of the Coulomb gauge that does not strictly depend the fact that the C–S field theories are topological. In fact, finite renormalizations of the fields and of the coupling constant \( s \) are always possible as it happens in the case of the covariant gauges. An analogous situation.

Figure 1: This figure shows that in an arbitrary tree diagram \( T_{\nu_1\nu_2\ldots\nu_{n-1}\nu_n} \) constructed in terms of the gauge fields propagator (43) and the three gluon vertex (39), only one component in the space-time indices \( \nu_i, i = 1, \ldots, n \), can be temporal.
in which there are no radiative corrections occurs in the light cone gauge in the presence of a boundary. In that case, radiative corrections arise in principle due to the interactions of the fields with the boundary but all the related Feynman diagrams vanish identically.

- The C–S field theories in the proper Coulomb gauge can be considered as two dimensional models. This has been shown in the previous section and has been confirmed here by the fact that the dependence on the time component of the momenta in the propagators and vertices is trivial.

- Contrary to what happens using the covariant gauges or the axial gauges, the Coulomb gauge can easily be applied also when space-times with non-trivial spatial section are considered, like for instance a Riemann surface. The absence of radiative corrections particularly useful in this case, where the momentum representation does not exist and this the evaluation of Feynman diagrams becomes forbiddenly difficult.

- The C–S field theories can be considered as a good laboratory in order to study the possible remedies of pathologies that appear in similar ways in the more complicated four dimensional gauge field theories. For example, it would be interesting to apply to the Yang–Mills case the regularization introduced here for the spurious singularities. Let us notice that a different regularization has been recently proposed in.

4 Chern–Simons field theories in the Coulomb Gauge on Curved Space–Times

In this section we consider a manifold \( M_3 \) with a Robertson-Walker metric and Euclidean signature of the kind

\[
g_{00} = 1 \quad g_{zz}(z, \bar{z}, t) = g_{\bar{z}z}(z, \bar{z}, t) = a(t) h(z, \bar{z}) \quad g^{zz} g_{zz} = 1
\]  

\[(50)\]

\( g^{zz} \) is the metric on a Riemann surface \( \Sigma_g \) of genus \( g \) and \( z \) and \( \bar{z} \) are local coordinates on \( \Sigma_g \): We suppose that \( a(t) > 0 \) for each values of the time \( t \).

Thus \( M_3 \) correspond to an expanding universe having the Riemann surface \( \Sigma_g \) as spatial section.

The gauge fixed Chern-Simons action (29) becomes in complex coordinates \( S_{CS} = S_{\text{free}} + S_{\text{int}} \), where:

\[
S_{\text{free}} = \int_{M_3} d^2z dt \, 2i \left( A_0^a \partial_z A_z^a + A_z^a \partial_0 A_0^a + A_0^a \partial_\bar{z} A_\bar{z}^a - \text{c.c.} \right)
\]
\begin{align}
S_{\text{int}} &= \int_{M_3} d^2 z dt \left[ \epsilon_{\mu\nu\rho} f^{abc} A_a^{\mu} A_b^{\nu} A_c^{\rho} - f^{abc} \bar{c}^a (A_b^z \partial_z + A_b^{\bar{z}} \partial_{\bar{z}}) c^c \right] \\
S_{\text{tot}} &= \int_{M_3} d^2 z dt \left[ \bar{c}^a (A_b^z \partial_z + A_b^{\bar{z}} \partial_{\bar{z}}) \right] + \frac{1}{2} \bar{z} \lambda (\partial_z + \partial_{\bar{z}}) \left[ \bar{c}^a (A_b^z \partial_z + A_b^{\bar{z}} \partial_{\bar{z}}) c^c \right]
\end{align}

and \( d^2 z = \frac{1}{2i} dz \wedge d\bar{z} \). The factor \( 2i \) in eq. (51) comes from the form of the \( \epsilon^{\mu\nu\rho} \) tensor in complex coordinates. In fact, the Levi-Civita tensor \( \epsilon_{\mu\nu\rho} \) becomes in these coordinates:

\[
[\epsilon]^{0z\bar{z}} = -2ig^{z\bar{z}} a^{-1}(t)
\]

All the other components can be obtained from eq. (53) permuting the indices \( 0, z, \bar{z} \) and changing the sign according to the order of the permutation. In this Section it will be useful to denote a sum over the complex indices with the first letters of the Greek alphabet \( \alpha, \beta, \gamma \) and so on. For example, the Coulomb gauge condition becomes now

\[
\partial_{\alpha} A_{\alpha}^a = 0.
\]

Using the metric (50) to rise and lower the indices, this equation reads:

\[
\partial_z A_a^z + \partial_{\bar{z}} A_a^{\bar{z}} = 0
\]

Eq. (54) does not contain the metric explicitly. This means that the Coulomb gauge condition is compatible with the transition functions at the intersections of the open sets covering the Riemann surface \( \Sigma_g \). Therefore eq. (54) is globally valid on \( M_3 \). The gauge fields \( (A_a^z, A_a^{\bar{z}}, A_a^0) \) are connections on the trivial principal bundle

\[
P(M_3, SU(N)) = M_3 \otimes SU(N)
\]

This bundle is trivial due to the fact that \( SU(N) \) is a simply connected Lie group. One can show as in the flat case that the Coulomb gauge (54) is a good gauge fixing without Gribov ambiguities \( ^{35} \) at least in the perturbative approach (see ref. \( ^{23} \) for details). We are now ready to compute the propagators of the gauge fields

\[
G_{\mu\nu}^{ab}(z, w; t, t') = < A_{\mu}^a (z, \bar{z}, t) A_{\nu}^b (w, \bar{w}, t) >
\]

where now \( \mu, \nu = 0, z, \bar{z} \). The equations satisfied by the above propagator are:

\[
-4i \partial_z G_{z0}^{ab}(z, w; t, t') + 4i \partial_{\bar{z}} G_{\bar{z}0}^{ab}(z, w; t, t') = \frac{8\pi}{s} \delta^{ab} \delta_{z\bar{z}}^{(2)} (z, w) \delta(t - t')
\]

\[
-4i \partial_z G_{0w}^{ab}(z, w; t, t') + 4i \partial_{\bar{z}} G_{\bar{z}w}^{ab}(z, w; t, t') - 2\frac{a^{-1}(t)}{\lambda} \partial_z \left[ g^{zz} \partial_z G_{zw}^{ab}(z, w; t, t') + g^{z\bar{z}} \partial_{\bar{z}} G_{z\bar{z}}^{ab}(z, w; t, t') \right] = \frac{8\pi}{s} \delta^{ab} \delta_{z\bar{z}}^{(2)} (z, w) \delta(t - t')
\]
Another equation can be obtained from (56) permuting the indices $z$ and $\bar{z}$ and substituting the index $w$ with $\bar{w}$. There are still other relations relating the various components of the propagators together:

$$
\partial_z G_{ab}^{\bar{z}0}(z, w; t, t') = 0 \quad (57)
$$

$$
-4i \partial_\alpha G_{00}^{ab}(z, w; t, t') + 4i \partial_0 G_{00}^{ab}(z, w; t, t') - \frac{a^{-1}(t)}{\lambda} \partial_\alpha [g^{zz} \partial_z G_{z0}^{ab}(z, w; t, t') + g^{zz} \partial_z G_{0z}^{ab}(z, w; t, t')] = 0 \quad (58)
$$

where $\alpha = w, \bar{w}$. Eq. (57) implies that the propagators $G_{\bar{z}0}^{ab}(z, w; t, t')$ and $G_{z0}^{ab}(z, w; t, t')$ do not have transverse components.. Finally we have:

$$
-4i \partial_\bar{z} G_{00}^{ab}(z, w; t, t') + 4i \partial_0 G_{00}^{ab}(z, w; t, t') - \frac{a^{-1}(t)}{\lambda} \partial_\bar{z} [g^{\bar{z}z} \partial_{\bar{z}} G_{\bar{z}0}^{ab}(z, w; t, t') + g^{\bar{z}z} \partial_{\bar{z}} G_{0\bar{z}}^{ab}(z, w; t, t')] = 0 \quad (59)
$$

Again it is possible to get another independent relation from eq. (59) interchanging the two indices $z$ and $\bar{z}$ and substituting $\bar{w}$ with $w$. Eqs. (55–59) are the equivalent of the equations defining the propagator in the flat case. However, they are still incomplete, because in deriving them we have neglected the zero mode contributions. In fact, we should remember that due to a theorem stating that the total charge on a Riemann surface (like in any other two dimensional compact manifold) is always zero, an isolated $\delta$ function $\delta^{(2)}(z, w)$ is not allowed. Therefore, in the right hand sides of eqs. (55–56) there must be also terms containing zero modes, whose expressions will be uniquely determined below. Since it is very difficult to solve equations (55–59) for any value of $\lambda$, we choose here the proper Coulomb gauge taking the limit $\lambda \to 0$. In this case drastic simplifications occur, so that the above equations reduce to the following two relations:

$$
\partial_z G_{20}^{ab}(z, w; t, t') = \partial_{\bar{z}} G_{\bar{2}0}^{ab}(z, w; t, t') = \frac{4\pi i}{s} \delta^{ab} \delta^{(2)}(z-w)\delta(t-t') + \text{zero modes} \quad (60)
$$

$$
\partial_z G_{20}^{ab}(z, w; t, t') + \partial_0 G_{20}^{ab}(z, w; t, t') = 0 \quad (61)
$$

These equations describe exactly the main requirement of the Coulomb gauge, i.e. the fact that only the transverse fields in the two dimensional spatial section $\Sigma_0$ of $M_3$ propagate. The transverse fields in complex coordinates satisfy in fact the following condition: $A_a^a = (A_z^a) = -A_z^a$. The solution of eqs. (60) and (61) is provided by the following Green functions:

$$
<A_a^a(z, t) A_0^b(w, t') = \frac{2\pi i}{s} \delta^{ab} \partial_z K(z, w) \delta(t-t') \quad (62)
$$
and

\[ < A^a_z(z, t) A^b_\bar{z}(w, t') > = - \frac{2\pi i s}{i} \delta^{ab} \partial_z K(z, w) \delta(t - t') \]  

(63)

where \( K(z, w) \) is the usual propagator of the scalar fields on a Riemann surface satisfying the equations (see ref. 36 for more details):

\[ K(z, w) = \delta^{(2)}(z, w) + \frac{g_{zz}}{\int_{\Sigma} d^2 u g_{uu}} \]  

(64)

\[ \partial_z \partial_\bar{z} K(w, z) = - \delta^{(2)}(z, w) + \bar{\omega}_i(\bar{z}) \left[ \text{Im } \Omega \right]^{-1}_{ij} \omega_j(w) \]  

(65)

\[ \int_{\Sigma} d^2 z g_{zz} K(z, w) = 0 \]  

(66)

In eq. (65) the \( \omega_i(z)dz, i = 1, \ldots, g, \) denote the usual holomorphic differentials and \( \Omega_{ij} \) represents the period matrix. It is important to stress here that \( K(z, w) \) is a singlevalued function on \( \Sigma_g \). Using the propagators (62) and (63) it is easy to see that eq. (61) is trivially satisfied. Therefore, the Coulomb gauge requirement (54) is fulfilled and the above defined propagators describe exactly the transverse components of the gauge fields. Still there is an ambiguity in the solutions (62) and (63) due to the zero mode sector of the fields \( A^a_z \) and \( A^a_\bar{z} \). In order to remove this ambiguity, we have to require that the above propagators are singlevalued along the nontrivial homology cycles of the Riemann surface. Otherwise, the propagators are not well defined on \( M_3 \), but in one of its coverings. Therefore, the propagators should obey the following relations:

\[ \oint_\gamma dz < A^a_z(z, t) A^b_\bar{z}(w, t') > = \oint_\gamma d\bar{z} < A^a_\bar{z}(z, t) A^b_z(w, t') > = 0 \]  

(67)

along any nontrivial homology cycles \( \gamma \). Due to the properties of singlevaluedness of the Green function \( K(z, w) \), eq. (67) is trivially satisfied by the propagators given in eqs. (62) and (63). In this way these two propagators are well defined and also the freedom in the zero mode sector is removed. Now we insert their expressions in eq. (60) in order to get the exact form of the zero mode terms appearing in the right hand side of this equation:

\[ \partial_z G_{\gamma 0}^{ab}(z, w; t, t') - \partial_\bar{z} G_{\gamma 0}^{ab}(z, w; t, t') = \]

\[ \frac{4\pi i s}{i} \delta^{ab} \delta^{(2)}(z, w) \delta(t - t') + 4\pi i s \int_{\Sigma} d^2 u g_{ua} \delta(t - t') \]  

(68)
The fact that the propagators in the Coulomb gauge must obey eq. (67) can be understood also decomposing the fields by means of the Hodge decomposition of the gauge fields in a coexact, exact and harmonic part:

\[ A_a^z = i \partial_z \phi_a^z + \partial_z \rho_a^z + A_{a,\text{har}}^z \]  
\[ A_{\bar{a}}^z = i \partial_{\bar{z}} \phi_{\bar{a}}^z + \partial_{\bar{z}} \rho_{\bar{a}}^z + A_{\bar{a},\text{har}}^z \]  
\phi^a \text{ and } \rho^a \text{ represent two real scalar fields. The above decomposition is allowed since the gauge invariance has been completely fixed by the choice of the Coulomb gauge, at least in the perturbative approach, and the } G^-\text{bundle } P(M_3, SU(N)) \text{ is trivial as we previously remarked. In the Coulomb gauge, the only components of the gauge fields which are allowed to propagate are the coexact differentials, i.e. the 1–forms obtained differentiating the scalar fields } \phi^a \text{ in eqs. (69) and (70). Therefore, the requirement (67) is a pure consequence of the fact that the coexact forms have vanishing holonomies around the nontrivial homology cycles.}

Let us notice that the zero mode term appearing in the right hand side of eq. (68) is totally irrelevant. To eliminate it it is sufficient to introduce new gauge fields, let say \( \tilde{A}_a^z, \tilde{A}_{\bar{a}}^z \), differing from the old ones by the fact that they are normalized to zero at a point \((0, 0)\) of the Riemann surface:

\[ \tilde{A}_a^z(z, \bar{z}, t) = A_a^z(z, \bar{z}, t) - A_a^z(0, 0, t) \]  
\[ \tilde{A}_{\bar{a}}^z(z, \bar{z}, t) = A_{\bar{a}}^z(z, \bar{z}, t) - A_{\bar{a}}^z(0, 0, t) \]  

Using the above new fields it is easy to check that the second term in the right hand side of eq. (68), which is a zero mode contribution, cancels out.

We finish this Section providing the explicit form of the other correlation functions of Chern-Simons field theory. The propagator of the ghost fields becomes:

\[ G_{\delta \theta}^{ab}(z, w; t, t') = \delta^{ab} K(z, w) \delta(t - t') \]  

The vertex coming from the cubic interaction between the gauge fields reads instead:

\[ V_{z_{1,00}}^{abc}(z_1, z_2, z_3; t, t', t'') = \frac{2 \pi^2 i \delta}{3} \int_{\Sigma_g} d^2 z f^{abc} \partial_{z_1} K(z_1, z) \partial_{z_2} K(z_2, z) \partial_{z_3} K(z_3, z) - \partial_{z_1} K(z_2, z) \partial_{z_2} K(z_3, z) \delta(t - t'') \delta(t' - t'') \]  

On \( M_3 \) this implies that the new fields are normalized to zero along the whole line of the time. This is possible to do since the three dimensional manifold is flat in the time direction.
The simple integration in the variable \( t \) has been already carried out in the above expression of the vertex. The component \( V_{\overline{z}_1\overline{z}_0}^{abc}(z_1, z_2, z_3; t, t', t'') \) of the vertex can be simply obtained replacing the derivative \( \partial_{\overline{z}_1} \) in the above equation with its complex conjugate. Finally, the vertex describing the interaction between ghost and gauge fields has only one component which is given by:

\[
V_{gh}^{abc}(z_1, z_2, z_3; t, t', t'') = \frac{2\pi i}{sa(t)} \int_{M_3} d^2 z f^{abc} K(z_1, z) \left[ \partial_{z} K(z_2, z) \partial_{\overline{z}} K(z_3, z) - \partial_{\overline{z}} K(z_2, z) \partial_{z} K(z_3, z) \right] \delta(t - t') \delta(t' - t'')
\]

It is easy to check that the above expressions of the vertices are real as it should be.

5 Conclusions

In summary, our study indicates that the Coulomb gauge is a convenient and reliable gauge fixing, especially in the perturbative applications of C-S field theory. Let us remember that, despite of the fact that the theory does not contain degrees of freedom, the perturbative calculations play a relevant role, for instance in the computation of knot invariants \( ^5, ^6, ^7, ^8, ^29 \). Contrary to what happens in the covariant gauges, where it becomes more and more difficult to evaluate the radiative corrections as the loop number increases \( ^29, ^6, ^34 \), in the Coulomb gauge only the tree level contributions to the Green functions survive. This feature is particularly useful in the case of non-flat manifolds, where the momentum representation does not exist. As an application, the Feynman rules of C-S field theories on Riemann surfaces have been derived in section 4. Moreover, the analysis performed using the Dirac's formalism has shown that the the CR’s (22) are perfectly well defined and do not lead to ambiguities in the quantization of the C–S models in the Coulomb gauge. In particular, it has been verified the consistency of the CR’s (22) with the constraints and the covariance of the theory under the Poincaré group.

Despite of these positive results, there are still many open questions concerning the use of the Coulomb gauge. For instance we have seen that, in this gauge, the C–S theories become two dimensional models, so that it is lecit to ask how it is possible to compute three dimensional link invariants. At the lowest order, where the link invariant is the simple Gauss invariant, one can check that the results obtained in the Coulomb gauge are consistent with those obtained in the covariant gauge (see appendix).

Another problem already mentioned is the derivation of the gauge field propagator starting from the commutation relations (22).
Finally one would also apply the prescription (42) also to the more complicated case of the four dimensional Yang–Mills field theories.

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Appendix: Wilson loop in the abelian case

Let us consider for instance in the abelian case the vacuum expectation value

\[ \langle W(C) \rangle = \langle e^{i \int_C dx^\mu A_\mu} \rangle \]

for a single closed loop \( C \). At the lowest order we have:

\[ \langle W(C) \rangle \sim -\frac{i}{s} \mathcal{P} \phi(C) \]  

(73)

where

\[ \phi(C) = \frac{1}{4\pi} \int_C dx^0 \int_C dy^i \epsilon_{ij} \partial^j \log|x - y|\delta(x^0 - y^0) \]

If the loop \( C \) lies on a plane, it is easy to see that the Wilson loop (73) is trivial. If the loop is not planar, using Stokes theorem, we obtain:

\[ \phi(C) = \frac{1}{4\pi} \int_C dx^0 \int_{\Sigma_0} d^2 y \delta^{(3)}(x - y) \]  

(74)

where \( \Sigma_0 \) is the projection on the plane \( x_1, x_2 \) of a surface \( \Sigma \) spanned by the loop \( C \). This result is to be compared with what we would obtain in the covariant gauge:

\[ \phi(C)_{\text{cov}} = \frac{1}{4\pi} \int_C dx^0 \int_{\Sigma_0} d^2 S_\mu \delta^{(3)}(x - y) \]

where now \( d^2 S_\mu \) is the infinitesimal area element on the surface \( \Sigma \). Introducing a framing 5 with framing contour \( C_f \) in equation (74) we have:

\[ \phi_f(C) = \frac{1}{4\pi} \int_{C_f} dx^0 \int_{\Sigma_0} d^2 y \delta^{(3)}(x - y) \]

After a few calculations one finds that the above integral is exactly a Cauchy integral counting how many times the loop \( C_f \) is intersecting the loop \( C \), which is exactly the Gauss link invariant as expected.
References

31. F. Ferrari and I. Lazzizzera, work in progress.