BPS Saturation from Null Reduction

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Abstract

We show that any $d$-dimensional strictly stationary, asymptotically Minkowskian solution ($d \geq 4$) of a null reduction of $d + 1$-dimensional pure gravity must saturate the BPS bound provided that the KK vector field can be identified appropriately. We also argue that it is consistent with the field equations.

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BPS solutions in supergravity theories play an important role in probing non-perturbative features of M theory and string theory. Some solutions are known to be obtained by infinitely boosting a static solution along some compactified direction. For example, let us consider the $d = 10$ Schwarzschild solution smeared along the eleventh direction:

\[ ds_{11}^2 = -\left(1 - \frac{\mu}{r^7}\right) dt^2 + \left(1 - \frac{\mu}{r^7}\right)^{-1} dr^2 + r^2 d\Omega_8 + dy^2, \] (1)

where $r^2 = x_1^2 + \cdots + x_9^2$, which is a solution of $d = 11$ supergravity (with the three-form set to zero). The global Lorentz transformation $t \rightarrow t \cosh \beta - y \sinh \beta$, $y \rightarrow -t \sinh \beta + y \cosh \beta$ boosts the solution along the $y$-direction. In the limit $\beta \rightarrow \infty$, $\mu \rightarrow 0$, $\mu e^{2\beta} \rightarrow 4Q$, this becomes

\[ ds_{11}^2 = -dt^2 + dy^2 + W(dt - dy)^2 + dr^2 + r^2 d\Omega_8, \] (2)

where $W = Q/r^7$. Compactifying $y$, one reads off a solution of type IIA supergravity:

\[
\begin{align*}
    ds_{10A}^2 &= -K^{-1/2} dt^2 + K^{1/2} (dr^2 + r^2 d\Omega_8), \\
    e^{2\phi} &= K^{3/2}, \\
    A_{\mu} &= -\delta_{\mu}^{0} WK^{-1},
\end{align*}
\] (3)

with $K = 1 + W$. This is nothing but the $d = 10$ extremal 0-brane solution [1] expressed in the isotropic radial coordinate $\tilde{r}^7 = r^7 + Q$. More complicated examples can be found in [2]. The BPS bound $M \geq c|Q|$ (with $c$ being a positive convention-dependent constant) is saturated by a Kaluza–Klein (KK) electric charge in the simplest cases, while other charged solutions can be obtained by duality symmetries.

The BPS saturation thus achieved can be intuitively understood in the following way. Suppose that we are given a static solution with energy-momentum $(d + 1)$-vector $(E, P_{\perp}, P_{\parallel}) = (M', 0, 0)$. The longitudinal momentum $P_{\parallel}$ increases as we boost it, and in the infinite-boost limit the energy-momentum approaches $(M, 0, M)$ for
some $M$. If we compactify the longitudinal direction, $P_\parallel$ becomes the KK charge, so that the solution can be viewed as a static BPS solution with mass and charge being equal.

Any infinitely boosted solution of this kind necessarily possesses a null Killing vector field, and hence is a solution of a null reduction of a higher-dimensional theory. In this letter we show that, in arbitrary dimensions $d \geq 4$, any strictly stationary, asymptotically Minkowskian solution of a null reduction of $d+1$-dimensional pure gravity must be a BPS solution\(^1\) provided that the KK vector field can be identified appropriately. In dimensions where the $d$-dimensional theory can be obtained as a bosonic sector of $N=2$ supergravity, the KK charge becomes a central charge of the superalgebra and the solution allows a Killing spinor.

Let us consider a pseudo-Riemannian manifold admitting a pair of commuting Killing vector fields one of which is assumed to be null. We start with the following parameterization of the vielbein (see [3] for the general framework of null reduction):

\[
E_M^{\hat{A}} = \begin{bmatrix}
E_m^a & u_m & SC_m \\
0 & u_w & SC_w \\
0 & 0 & S
\end{bmatrix}
\]  \hspace{1cm} (4)

with a flat lightcone metric

\[
\eta_{\tilde{A}\tilde{B}} = \begin{bmatrix}
\delta_{ab} & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}.
\]  \hspace{1cm} (5)

We shall use the following conventions. Capital letters $M, N, \ldots$ and $\tilde{A}, \tilde{B}, \ldots$ (as well as $A, B, \ldots$ below) will denote curved and flat indices, respectively, in $d+1$ dimensions. Upon dimensional reduction they are split into transversal and longitudinal indices, i.e., $M = (m, w, v)$ and $\tilde{A} = (a, +, -)$ where $a, b, \ldots, m, n, \ldots = 1, \ldots, d - 1$. By \(^{1}\) or at least an extremal solution in some sense; anticipating an application to supersymmetric theories we will employ the former terminology throughout the paper.
the above choice of parameterization the local $SO(d,1)$ Lorentz invariance is broken down to $SO(d-1) \times SO(1,1)$. This is in contrast to dimensional reduction with a null Killing vector alone [3] for which the residual tangent space symmetry is the inhomogeneous Lorentz group $ISO(d-1)$.

The $wv$ part of the metric reads
\[
\begin{pmatrix}
G_{ww} & G_{wv} \\
G_{vw} & G_{vv}
\end{pmatrix} = \begin{pmatrix}
2S u_w C_w & S u_w \\
S u_w & 0
\end{pmatrix}.
\]

The two Killing vectors corresponding to the dimensional reduction are taken to have components $\omega^M = (0, \ldots, 0, 1, 0)$ and $\xi^M = (0, \ldots, 0, 1)$, so that
\[
\omega \equiv \omega^M \partial_M = \partial_w, \quad \xi \equiv \xi^M \partial_M = \partial_v,
\]
respectively, and $\xi$ is indeed null.

To identify the KK vector field we change tangent space lightcone coordinates $\tilde{A} = (a, +, -)$ into standard Minkowski coordinates $A = (a, 0, d)$ by applying a similarity transformation to $\eta_{\tilde{A}\tilde{B}}$ so that it becomes
\[
\eta_{AB} = \begin{pmatrix}
\delta_{ab} & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Consequently, the vielbein changes into
\[
E_M^A = \begin{pmatrix}
E_{m}^a & \frac{1}{\sqrt{2}}(u_m - SC_m) & \frac{1}{\sqrt{2}}(u_m + SC_m) \\
0 & \frac{1}{\sqrt{2}}(u_w - SC_w) & \frac{1}{\sqrt{2}}(u_w + SC_w) \\
0 & -\frac{1}{\sqrt{2}}S & \frac{1}{\sqrt{2}}S
\end{pmatrix}.
\]

To make contact with ordinary dimensional reduction w.r.t. a spacelike Killing vector $\partial_x$ let us now perform a change of coordinates $w = pt + rx, v = qt + sx$ with some constants $p, q, r, s$ such that $\Delta := ps - rq \neq 0$. Taking also the freedom of a local $SO(1,1)$ transformation into account, the $tx$ part of the vielbein becomes
\[
\begin{pmatrix}
E_t^0 & E_t^d \\
E_x^0 & E_x^d
\end{pmatrix} = \begin{pmatrix}
p & q \\
r & s
\end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix}
u_w - SC_w & u_w + SC_w \\
-S & S
\end{pmatrix} \begin{pmatrix}
cosh \theta & \sinh \theta \\
\sinh \theta & \cosh \theta
\end{pmatrix}.
\]
where \( \rho := r[2Su_w(C_w+s/r)]^{1/2} \) and we have chosen \( \theta \) such that \( e^{2\theta} = (rC_w+s)S/ru_w \). The constants \( r, s \) must satisfy \( s/r > -\text{inf} C_w \) because \( \theta \) is real. Besides we take \( r > 0 \) so that we may identify \( \rho \) with (the exponential of) the dilaton. The \( tx \) part of the \((d+1)\)-metric then reads

\[
\begin{bmatrix}
G_{tt} & G_{tx} \\
G_{xt} & G_{xx}
\end{bmatrix} = \begin{bmatrix}
\Delta Su_w/\rho & \rho p/r - \Delta Su_w/\rho \\
\rho p/r - \Delta Su_w/\rho & \rho^2
\end{bmatrix},
\]

We observe that because of \( \rho^2 > 0 \) the Killing vector \( \partial_x = r\partial_w + s\partial_v \) is always spacelike, while \( \partial_t = p\partial_w + q\partial_v \) becomes timelike if \( p \neq 0, \frac{q}{p} < -\text{sup} C_w \). We have thus obtained a stationary configuration in \( d \) dimensions from the null reduction with an extra Killing vector field. Any system of two commuting Killing vectors one of which is null can be viewed in this way as consisting of a spacelike and a timelike Killing vector. Note, however, that the converse is not correct in general.

To identify the physical fields we equate

\[
E_M^A = \begin{bmatrix}
E_m^a \\
0
\end{bmatrix} \begin{bmatrix}
\frac{1}{\sqrt{2}}(u_m - SC_m) \\
\frac{1}{\sqrt{2}}(u_m + SC_m)
\end{bmatrix} = \begin{bmatrix}
\rho^\chi e_{\mu}^a \\
0
\end{bmatrix} \begin{bmatrix}
\rho A_\mu \\
\rho
\end{bmatrix},
\]

where \( \mu = (m,t) \) and \( \alpha = 1, \ldots, d-1, 0 \). There is an arbitrariness specified by \( \chi \) in splitting the dilaton factor from the \( d \)-metric \( e_\mu^a \). If one takes \( \chi = -\frac{1}{d-2} \), the \( eR \) term takes the canonical form without any dilaton factor in the reduced action. Another common choice is \( \chi = -1/2 \) in \( d = 10 \), known as the “string metric”, in which the

\[\text{Note that } u_m, u_w \text{ and } S \text{ appear only through } Su_m \text{ or } Su_w \text{ in the } (d+1)\text{-metric. Since } \det E_M^A = Su_w \det E_m^a, \text{ we may assume } S > 0 \text{ and } u_w > 0.\]
dilaton factor disappears from the $F^2$ term. In our case yet another choice turns out to be more convenient, as we will see below.

From (12) we find the following expressions for the dilaton and the components of the KK vector field, respectively,

$$e^{2\phi} := \rho, \quad A_m = \frac{1}{\sqrt{2}}(u_m + SC_m)\rho^{-1}, \quad A_t = \frac{p}{r} - \frac{\Delta}{\rho^2} Su_w.$$  \hspace{1cm} (13)

In addition, adopting $\chi = 1$, we get the condition

$$e_t^0 = \frac{p}{r} - A_t.$$  \hspace{1cm} (14)

The BPS saturation for strictly stationary, asymptotically Minkowskian solutions is an immediate consequence of the relation (14). More precisely, we assume that $x$ can be globally separated from the $d + 1$-dimensional spacetime with a suitable identification of different local neighborhoods. We then require that the $d$-bein $e_{\mu}^\alpha$ in a local coordinate of the neighborhood of spatial infinity goes to $\pm \delta_{\mu}^\alpha$ as one approaches spatial infinity. In this case the total mass expressed in terms of the Komar integral can be rewritten as

$$M = \frac{1}{2\Omega_{d-2}} \int_{S_\infty} dS e^{\mu_1 \cdots \mu_{d-2}} e_{\mu_1 \cdots \mu_{d-2} \nu \sigma} \nabla^\nu \tau^\sigma$$

$$= \frac{1}{2\Omega_{d-2}} \int_{S_\infty} dS e N^\mu \xi^\nu \nabla_{\mu} \tau^\nu$$

$$= \frac{1}{2\Omega_{d-2}} \int_{S_\infty} dS \frac{1}{2} N^\mu \partial_{\mu} g_{tt}$$

$$= \frac{1}{2\Omega_{d-2}} \int_{S_\infty} dS N^\mu \left(\frac{p}{r} - A_t\right) F_{\mu t}$$

$$= \frac{|Q|}{2},$$  \hspace{1cm} (15)

which saturates the BPS bound $M \geq c|Q|$ with $c = 1/2$. $\tau^\mu = \delta_t^\mu$ is the timelike Killing vector and $Q$ is electric charge. All the fields are assumed to be $t$-independent in the neighborhood of infinity, in which the whole closed hypersurface $S_\infty$ “infinitely close to infinity” lies. $N^\mu$ and $\xi^\mu$ are the unit future and outward pointing vectors.

\footnote{Strictly speaking, one must have a precise notion of asymptotic flatness to describe the limiting procedure more rigorously. This would be obtained by generalizing the definition in $d = 4$ \cite{4}.}
normal to \( S_\infty \), respectively, which are orthogonal to each other. In the last line we used (14) and \( e_t^0 \to \pm 1 \). The fact that \( c = 1/2 \) for \( \chi = 1 \) can be confirmed by an explicit calculation using known BPS solutions such as extremal black holes. It should be emphasized that the formula (15) is derived from purely geometrical assumptions without any use of the field equations.

Because of the existence of extra isometries, \( p \)-brane solutions of usual type \( (p \geq 1) \) cannot be asymptotically Minkowskian in the \( d \)-dimensional sense. Therefore they are excluded from our discussion here (but can be easily discussed in parallel by considering mass/charge per unit world volume). Consequently magnetic charge is zero if \( d \geq 5 \). In \( d = 4 \) it is also zero in our case since the way we identified the KK vector field implicitly assumed that there exists a smooth section of the \( U(1) \) bundle (whose fiber is the orbit of \( \partial/\partial x \)-isometry) in the neighborhood of infinity.

The notion of BPS saturation is independent of how the dilaton is split from the \( d \)-metric (“frame”), but the constant factor \( c \) changes. For general \( \chi \) the relation (14) is replaced by

\[
e_t^0 = \rho^{1-\chi} \left( \frac{p}{r} - A_t \right). \tag{16}
\]

The expression of the total mass then depends also on the asymptotic behavior of the dilaton, which is not determined solely by geometrical constraints. The calculation using extremal black holes shows that the formula (15) must be replaced by

\[
M = \frac{1 + \chi}{4} |Q|. \tag{17}
\]

Let us now examine how the relation (17) is reconciled with the field equations in the case of the canonical metric \( \chi = -\frac{1}{d-2} \). Dimensional reduction of the Einstein–Hilbert action from \( d + 1 \) dimensions to \( d \) dimensions produces (up to a surface term) the standard result

\[
ER(E) = e \left[ R(e) - \frac{1}{4} \rho^{2\chi-1} F_{\mu\nu} F^{\mu\nu} - \frac{d-1}{d-2} \partial_\mu \ln \rho \partial^\mu \ln \rho \right]. \tag{18}
\]

Here \( E \equiv \det E_M^A, e \equiv \det e_\mu^\alpha, R \) denotes the Ricci scalar, \( F_{\mu\nu} := 2 \partial_{[\mu} A_{\nu]} \) is the KK field strength, and \( d \)-dimensional indices \( \mu, \nu \) are raised and lowered with the
metric $g_{\mu\nu} := \epsilon_{\mu}^{\alpha} \epsilon_{\nu\alpha}$. The dilaton field equation, Maxwell’s equations, and Einstein’s equation derived from the action are

\begin{align}
\nabla_\mu \partial_\mu \ln \rho - \frac{1}{4} \rho^{\frac{d-2}{d-4}} F^2 &= 0, \\
\nabla_\mu \left( \rho^{\frac{d-2}{d-4}} F^{\mu\nu} \right) &= 0, \\
R_{\mu\nu} - \frac{1}{2} \rho^{\frac{d-2}{d-4}} F_{\mu\sigma} F^{\nu\sigma} + \frac{1}{4(d-2)} g_{\mu\nu} \rho^{\frac{d-2}{d-4}} F^2 - \frac{d-1}{d-2} \partial_\mu \ln \rho \partial_\nu \ln \rho &= 0,
\end{align}

where $\nabla_\mu$ denotes the covariant derivative associated with the Levi–Civita connection derived from $g_{\mu\nu}$.

To proceed, we assume that the $d$-dimensional spacetime is parameterized by a single global coordinate system and that the timelike Killing vector $\tau \equiv \tau^M \partial_M = \partial_t$ is orthogonal to a hypersurface parameterized by $x^m$, $m = 1, \ldots, d - 1$. Since $R_{\mu\nu} \tau^\nu = -\nabla^\nu \nabla_\nu \tau_\mu$ for any Killing vector $\tau$ (see e.g. [5]), we get

\begin{align}
M &= \frac{1}{2 \Omega_{d-2}} \int_\Sigma dV \epsilon_{\mu_1 \ldots \mu_{d-1}} \epsilon_{\nu_1 \ldots \nu_{d-1}} \nabla_\nu \nabla^{\nu} \tau^\sigma \\
&= -\frac{1}{2 \Omega_{d-2}} \int_\Sigma dV \epsilon_{x^1 \ldots x^{d-1}} \epsilon_{x^1 \ldots x^{d-1}} R_t^t \\
&= -\frac{1}{2 \Omega_{d-2}} \int_\Sigma dV \epsilon_{\mu} \frac{1}{2} \rho^{\frac{d-2}{d-4}} \left( F_{tm} F^{tm} - \frac{1}{2(d-2)} F^2 \right) \\
&= -\frac{1}{2 \Omega_{d-2}} \int_\Sigma dV \nabla_\mu \left[ \epsilon \rho^{\frac{d-2}{d-4}} \left( \frac{1}{2} \left( 1 - \frac{1}{d-2} \right) A_t F^{tm} - \frac{1}{2(d-2)} A_m F^{tm} \right) \right],
\end{align}

where $\Sigma := \text{Int} S_\infty$. The second term becomes a surface integral, which can be interpreted as the net current flow through $S_\infty$ provided that the $A_m \rho^{\frac{d-2}{d-4}}$ factor varies sufficiently slowly near infinity. So let us assume here that it drops out. The first term can be also written as a surface integral. At first sight $A_t$ can be shifted by an arbitrary integration constant. This ambiguity can be fixed by taking account of the missing source term in RHS of Maxwell’s equation (20). Indeed, it is the equation only for the vacuum region outside the locations of point charge, without which we would get $Q = 0$. So suppose for simplicity that we have a single delta-function singularity at $p$. Then it picks up the value of $A_t$ at $p$ when the third equality in (22)
is written into a surface integral. This enables us to obtain the expression

\[ M = \frac{1}{4} (A_t(p) - A_t(\infty)) \left( 1 - \frac{1}{d-2} \right) Q \]

\[ = \frac{1}{4} \left( \rho^{\frac{d-1}{2}} e_t^0(\infty) - \rho^{\frac{d-1}{2}} e_t^0(p) \right) \left( 1 - \frac{1}{d-2} \right) Q, \]  

(23)

which does not have the ambiguity any more.

For asymptotically Minkowskian solutions \( e_t^0(\infty) = \pm 1 \). If \( \rho(\infty) = 1 \) and \( \rho^{\frac{d-1}{2}} \) blows up at least faster than \( e_t^0 \) at \( p \) (as is the case for extremal black holes), (23) becomes

\[ M = \frac{1}{4} \left( 1 - \frac{1}{d-2} \right) |Q|, \]  

which is in agreement with (17) with \( \chi = -\frac{1}{d-2} \).

As a final comment we note that the mass expressed as a Komar integral (15) or (17) coincides with the ADM mass in \( d = 4 \) [6] but in general they are different. For extremal black holes they differ by a factor \( 2(1 - \frac{1}{d-2}) \).

We have shown that the BPS saturation for strictly stationary, asymptotically Minkowskian solutions of a null reduction is a purely geometrical consequence. It would be interesting to investigate whether the relation between null reduction and BPS saturation extends to general non-stationary cases. A comparison with the supersymmetric string waves of [7], which only assume a covariantly constant null (Killing) vector, may give us a hint.

Let us conclude with a speculation how the hyperbolic Kac–Moody algebra \( E_{10} \) might give rise to a duality symmetry of M theory. As was shown in [8, 9], to obtain an \( E_{10} \) hidden symmetry in the dimensional reduction of \( d = 11 \) supergravity, the final step from \( d = 2 \) to \( d = 1 \) must be a null reduction. The order in which the dimensional reductions are performed should not affect the final symmetry group. Hence suppose in particular that we first perform a null reduction from \( d = 11 \) to \( d = 10 \) and then reduce the other nine spatial dimensions. In this case all the solutions that satisfy the conditions we assumed in this paper are BPS saturated in \( d = 10 \). Upon compactification we could also find corresponding BPS solutions in lower dimensions (e.g. by taking a periodic array of solutions). So this would mean that \( E_{10} \) is a purely stringy symmetry in the sense that its action may be defined only
on BPS solutions.

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