N-Nucleon Effective Generators of the Poincaré Group
Derived from a Field Theory

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Abstract

It is shown that the ten generators of the Poincaré group acting in the Fock space of nucleons and mesons and based on standard Lagrangians can be blockdiagonalized by one and the same unitary transformation such that the space of a fixed number of nucleons is separated from the rest of the Fock space. The existence proof is carried through in a formal power series expansion in the coupling constant to all orders. In this manner one arrives at effective generators of the Poincaré group which act in the two subspaces separately.

I. INTRODUCTION

Low energy nuclear physics below the pion threshold is naturally formulated in terms of a fixed number of nucleonic degrees of freedom. In the overwhelming number of applications a nonrelativistic framework is used. This however is not sufficient if one investigates for instance electron scattering with high three-momentum transfers as one encounters in typical experiments performed nowadays. Also it is still an open question whether relativistic effects play a significant role when calculating the binding energy of nuclei. In three-nucleon scattering it has been found recently [1] that the total nd cross section evaluated with most modern NN forces and based on rigorous solutions of the 3N Faddeev equations deviate from the data above \( \approx 100 \text{ MeV nucleon lab} \) laboratory energy. That discrepancy reaches about 10% at 300 MeV and is very likely caused by the neglect of relativity. On all these grounds a relativistic generalization of the usual Schrödinger equation for \( N \) interacting nucleons is highly desirable.

In [2] Dirac proposed three forms of relativistic quantum mechanics for a given number of interacting particles. A realization thereof in the instant form was given by Bakamjian and Thomas [3]. That scheme however violates cluster separability [4]. Being less ambitious and searching just for relativistic correction terms to the generators of the Galilean group in leading orders Foldy and Krajcik have discussed [5] a \( \frac{1}{2} \) expansion of the ten generators of the Poincaré group. This scheme has been applied recently in a realistic context in the 3N system [6]. A way to treat the defect in the Bakamjian and Thomas scheme with respect to the cluster separability has been found by Sokolov [7]
and also worked out by Coester and Polyzou [8]. An extensive overview over the whole subject is given in [9].

There is however also another approach to the generators of the Poincaré group for a fixed number of particles. Relativistic field theory provides generators which act in the full Fock space. Thinking of applications for nuclear physics one considers interacting fields of nucleons and mesons. To arrive at generators which act in the space of a fixed number of $N$ nucleons one has to eliminate the mesonic degrees of freedom as well as the ones for antiparticles. A way to do this has been proposed in [10] and worked out in lowest order in the coupling constant for a field theory of “scalar nucleons” interacting with a scalar meson field. While this has been formulated in the instant form a corresponding derivation can also be performed in the light front form [11]. Numerical investigations based on those effective generators determined in leading order in the coupling constant have been carried through in [12], [13] and [14].

In this article we want to show that the derivation proposed in [10] can be carried through to arbitrary order in the coupling constant. Thus the effective generators of the Poincaré group in an $N$ nucleon subspace do exist at least in the sense of a formal series expansion. It will be interesting to investigate whether those generators are automatically also cluster separable. This is left to a future study.

In section II we formulate our way to derive the effective generators in the $N$ nucleon subspace out of a field theoretical model of interacting nucleon and meson fields. The existence proof is carried through in section III. We summarize briefly in section IV.

II. CONDITIONS FOR THE EFFECTIVE GENERATORS

We consider a field theory of interacting nucleons, antinucleons, and mesons given by a Lagrangian of the form

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$$

(1)

where $\mathcal{L}_0$ is the free part and the interacting part $\mathcal{L}_I$ is linear in the coupling constant $g$. We also assume that $\mathcal{L}_I$ is linear in creation and absorption operators for mesons as is the case for often used field theories. In a standard manner [15] one arrives at the ten generators of the Poincaré group for constant time slices (instant form). The Hamiltonian and the three boost operators carry interactions, whereas the total momentum and angular momentum operators are the free ones. The latter two leave the plane $t =$const invariant. Thus one has in obvious notation

$$H = H_0 + H_I$$

(2)

$$K_i = K_{0i} + K_{Ii}$$

(3)

$$P_i = P_{0i}$$

(4)

$$J_i = J_{0i}$$

(5)
where due to Eq. (1)

\[ H_I \sim g \]
\[ K_{II} \sim g \]

These ten operators fulfill the Lie algebra of the Poincaré group:

\[ [P_i, H] = 0 \] (7)
\[ [J_i, H] = 0 \] (8)
\[ [P_i, P_j] = 0 \] (9)
\[ [J_i, J_j] = i\epsilon_{ijk}J_k \] (10)
\[ [J_i, P_j] = i\epsilon_{ijk}P_k \] (11)
\[ [J_i, K_j] = i\epsilon_{ijk}K_k \] (12)
\[ [H, K_i] = -iP_i \] (13)
\[ [K_i, K_j] = -i\epsilon_{ijk}J_k \] (14)
\[ [P_i, K_j] = -i\delta_{ij}H \] (15)

Formally one can verify that using the equal time commutation relations of the underlying fields. Because \( \mathcal{L}_I \) is assumed to be linear in the creation and annihilation operators for mesons, \( H_I \) and \( K_{II} \) will be linear in these operators too. Hence the eigenstates of \( H \) will necessarily contain an infinite number of mesons in addition to the nucleons (and antinucleons). The behaviour of such an eigenstate under Lorentz transformation, however, is transparent. We regard the operator of four momentum

\[ P^\mu \equiv (H, P_1, P_2, P_3) \] (16)

and consider a Lorentz transformation \( T(\Lambda, a) \) defined by

\[ x^\mu T \rightarrow x'^\mu = \Lambda^\mu_{\nu}x^\nu + a^\mu \] (17)

Related to \( T \) is a unitary operator \( U(\Lambda, a) \) acting in the Hilbert space spanned by the eigenstates of \( H \). A consequence of the commutation relations (7)-(15) are the transformation properties of \( P^\mu \):

\[ P^\mu T \rightarrow P'^{\mu} = UP^\mu U^\dagger = \Lambda_\nu^{\mu}P^\nu \] (18)

Because of Eqs. (7) and (9) there exist simultaneous eigenstates related to the four components of the four-momentum operator, which fulfill

\[ P^{\mu}\Psi_p = p^{\mu}\Psi_p \] (19)

Applying \( U \) and using Eq. (18) one gets

\[ \Lambda_\nu^{\mu}P^\nu U\Psi_p = p^{\mu}U\Psi_p \] (20)
This can be rewritten into

\[ P^\nu U|\Psi_p\rangle = \Lambda^\nu_\mu P^\mu U|\Psi_p\rangle \]  

(21)

Thus up to a phase factor we get

\[ U|\Psi_p\rangle = |\Psi_{\Lambda p}\rangle \]  

(22)

and the “four dimensional Schrödinger equation” (19) reads in the new frame of reference

\[ P^\mu |\Psi_{\Lambda p}\rangle = (\Lambda p)^\mu |\Psi_{\Lambda p}\rangle \]  

(23)

Therefore the simultaneous eigenstates of \( P^\mu \) in the new frame are eigenstates in the old frame with Lorentz transformed eigenvalues of the overall four momentum.

We pose now the question if one can find a representation of the Poincaré algebra being restricted to a subspace of the Fock space with a fixed number of particles. We want to call those generators of the Poincaré group “effective”. If it is possible to find an effective representation of the Poincaré group one is able to formulate an effective Schrödinger equation in the subspace of a given number of particles, say \( N \) nucleons and no mesons, very much alike (19). The interesting point about that is that this equation would be easier to solve than Eq. (19) since the number of degrees of freedom is finite now. In addition, since we assume the Poincaré algebra to be fulfilled in that subspace, this effective Schrödinger equation inherits the nice transformation properties of Eq. (23).

A way to find effective generators is to unitarily transform the generators (2)-(5) by an operator \( U \) such that all ten generators are put into a blockdiagonal shape at the same time. One block would refer to the \( N \) nucleon subspace, the other block to the rest and the two blocks would not be coupled. Under a unitary transformation the commutation relations remain valid, of course. Let us denote the projection on the subspace of \( N \) nucleons by \( \eta \) and the projection on the rest by \( \Lambda \equiv 1 - \eta \). Then what we are looking for is a unitary transformation of the form

\[
\begin{align*}
H &\rightarrow \tilde{H} = \eta \tilde{H} \eta + \Lambda \tilde{H} \Lambda \\
K_i &\rightarrow \tilde{K}_i = \eta \tilde{K}_i \eta + \Lambda \tilde{K}_i \Lambda \\
P_i &\rightarrow \tilde{P}_i = \eta \tilde{P}_i \eta + \Lambda \tilde{P}_i \Lambda \\
J_i &\rightarrow \tilde{J}_i = \eta \tilde{J}_i \eta + \Lambda \tilde{J}_i \Lambda 
\end{align*}
\]

(24) (25) (26) (27)

While \( H \) and \( K_i \) (in the instant form) couple the \( \eta \) and \( \Lambda \) spaces, this is by assumption no longer the case for \( \tilde{H} \) and \( \tilde{K}_i \) and the operators \( \eta \tilde{H} \eta, \eta \tilde{K}_i \eta, \eta \tilde{P}_i \eta \) and \( \eta \tilde{J}_i \eta \) are effective generators of the Poincaré group. Now one can look for eigenstates of \( \tilde{P}^\mu \) whose \( \Lambda \)-components are zero. Lorentz transformations on those states are generated by the effective operators and we may write down the effective Schrödinger equation

\[ \eta \tilde{P}^\mu \eta |\psi\rangle = p^\mu \eta |\psi\rangle \]  

(28)
In [10] such a path has been initiated and will be worked out more stringently now. In [16] Okubo proposed a way to transform an arbitrary hermitian operator

$$O = \begin{pmatrix} \eta O \eta & \eta O \Lambda \\ O \Lambda & \Lambda O \Lambda \end{pmatrix}$$  \hspace{1cm} (29)

into a block diagonal form by means of a unitary transformation $U$:

$$O \rightarrow \tilde{O} = UOU^\dagger = \eta \tilde{O} \eta + \Lambda \tilde{O} \Lambda$$  \hspace{1cm} (30)

We follow Okubo for the choice of the unitary operator $U$

$$U = \begin{pmatrix} \eta \Lambda \eta & \eta \Lambda \Lambda \\ \Lambda \eta & \Lambda \Lambda \end{pmatrix} = \begin{pmatrix} (1 + A^\dagger A)^{-\frac{1}{2}} \eta & (1 + A^\dagger A)^{-\frac{1}{2}} A^\dagger \\ -(1 + AA^\dagger)^{-\frac{1}{2}} A & (1 + AA^\dagger)^{-\frac{1}{2}} \Lambda \end{pmatrix}$$  \hspace{1cm} (31)

where $A$ has the form

$$A = \Lambda A \eta$$  \hspace{1cm} (32)

Unitary transformations within the subspaces $\eta$ and $\Lambda$ are put to 1. Using the forms (29) and (31) the requirement for blockdiagonalization is

$$\Lambda \left( [O, A] + O - AO A \right) \eta = 0$$  \hspace{1cm} (33)

Since it is `a priori not obvious that it will be possible to blockdiagonalize each generator using the same $U$ we label $A$ with the generator to be blockdiagonalized. Noting Eqs. (4) and (5) telling that $P_i$ and $J_i$ do not connect the $\eta$ and $\Lambda$ spaces the conditions for the ten operators $A_H, A_K, A_P, \text{and} A_J$ turn out to be

$$\Lambda \left( [H_0, A_H] + H_I A_H + H_I - A_H H_I A_H \right) \eta = 0$$  \hspace{1cm} (34)

$$\Lambda \left( [K_0, A_K] + K_I A_K + K_I - A_K K_I A_K \right) \eta = 0$$  \hspace{1cm} (35)

$$\Lambda [P_i, A_P] \eta = 0$$  \hspace{1cm} (36)

$$\Lambda [J_i, A_J] \eta = 0$$  \hspace{1cm} (37)

Here we made use of the assumption that $L_I$ and hence $H_I$ and $K_I$ are linear in the meson operators such that $\eta H_I \eta = 0 = \eta K_I \eta$. If one and the same $A$ can be found that fulfills the set of conditions (34)-(37) the existence of ten effective generators of the Poincaré group in the separate subspaces $\eta$ and $\Lambda$ is proven.

### III. PROOF OF THE EXISTENCE OF $A$

The conditions (34)-(37) are nonlinear in the $A$'s. One can linearize them by searching for $A$ in the form of a Taylor expansion in the coupling constant
\[ A = \sum_{\nu=1}^{\infty} A_{\nu} g^\nu \]  

(38)

The term of order \( g^0 \) is absent since for a free theory (\( g \to 0 \)) the generators are already blockdiagonal and \( U = 1 \) is achieved with \( A = 0 \). It is easy to rewrite the set under the assumption (38) by equating equal powers of \( g \). Keeping in mind that we assumed \( \eta H_I \eta = 0 = \eta K_I \eta \) we get the result:

\[
\begin{align*}
[H_0, A_{\eta}] &= -H_I \eta \\
[H_0, A_{A_{\eta}}] &= -\Lambda H_I A_{\eta} \\
[H_0, A_{\eta_{n+1}}] &= -\Lambda H_I A_{\eta_n} + \sum_{\nu=1}^{n-1} A_{\eta_{\nu}} H_I A_{\eta_{n-\nu}} & n \geq 2 \\
[K_{0i}, A_{K_{\eta}}] &= -K_I \eta \\
[K_{0i}, A_{K_{\eta_2}}] &= -\Lambda K_I A_{K_{\eta}} \\
[K_{0i}, A_{K_{\eta_{n+1}}}] &= -\Lambda K_I A_{K_{\eta_n}} + \sum_{\nu=1}^{n-1} A_{K_{\nu}} K_I A_{K_{n-\nu}} & n \geq 2 \\
[P_{i}, A_{\eta_{n}}] &= 0 & n \geq 1 \\
[J_{i}, A_{\eta_{n}}] &= 0 & n \geq 1
\end{align*}
\]

(39)-(46)

Let us introduce short hand notations. An arbitrary Fock state in the \( \Lambda \) space describing a certain number of noninteracting particles with momenta \( p \) is simply denoted by \( |\Lambda\rangle \). Its energy, the eigenvalue to \( H_0 \), is denoted by \( E_{\Lambda} \). The projection operator into the \( \Lambda \) space is

\[
\Lambda \equiv \int d^3p_{\Lambda} |\Lambda\rangle\langle\Lambda|
\]

(47)

where \( d^3p_{\Lambda} \) stands for all momentum integrations. Similarly we denote an arbitrary state in the \( \eta \) space by \( |\eta\rangle \), its energy by \( E_{\eta} \) and the projection operator into the \( \eta \) space by

\[
\eta \equiv \int d^3p_{\eta} |\eta\rangle\langle\eta|
\]

(48)

Then it is very convenient to use the following notation

\[
\int d^3p_{\Lambda} d^3p_{\eta} \frac{1}{E_{\Lambda} - E_{\eta}} |\Lambda\rangle\langle\Lambda|B|\eta\rangle\langle\eta| = \frac{1}{E_{\Lambda} - E_{\eta}} \Lambda B \eta
\]

(49)

where \( B \) is an arbitrary operator.

The set (39)-(41) can now be solved recursively. Note that according to Eq. (32) \( A \) connects the \( \Lambda \) and the \( \eta \) spaces. Using now the notation (49) one finds the following explicit expressions for \( A_{\eta_{\nu}} \):
\begin{align*}
A_{H_1} &= -\frac{1}{E_\Lambda - E_\eta} \Lambda H_1 \eta \\
A_{H_2} &= -\frac{1}{E_\Lambda - E_\eta} \Lambda H_1 A_{H_1} \eta \\
A_{H_{n+1}} &= -\frac{1}{E_\Lambda - E_\eta} \Lambda H_1 A_{H_n} \eta + \frac{1}{E_\Lambda - E_\eta} \Lambda A_{H_n} H_1 A_{H_{n-\nu}} \eta \quad n \geq 2 \tag{52}
\end{align*}

With the help of the Lie algebra (7)-(15) we will now show by induction that $A_{H_n}$ given by Eqs. (50)-(52) satisfies also Eqs. (42)-(46) and therefore one and the same $A$ blockdiagonalizes all ten generators.

First we look at Eq. (45). Because of

\[
[P_i, \Lambda] = [P_i, \eta] = 0 \tag{53}
\]

one has the following identity for any operator $B(H)$

\[
[P_i, \frac{1}{E_\Lambda - E_\eta} \Lambda B(H) \eta] = \frac{1}{E_\Lambda - E_\eta} \Lambda [P_i, B(H)] \eta \tag{54}
\]

Further, since $P_i$ commutes with $H_0$ and $H$, it also commutes with $H_I \equiv H - H_0$. Consequently we get

\[
[P_i, A_{H_1}] = -[P_i, \frac{1}{E_\Lambda - E_\eta} \Lambda H_1 \eta] \\
= -\frac{1}{E_\Lambda - E_\eta} \Lambda [P_i, H_I] \eta = 0 \tag{55}
\]

By induction this carries over to $A_{H_2}$ and $A_{H_n}$ with $n \geq 3$ and we can write

\[
A_{P_i} = A_H \tag{56}
\]

The proof of (46) using $A_H$ is very similar. Eqs. (53)-(55) remain valid replacing $P_i$ by $J_i$ and $J_i$ commutes with $H$ and $H_0$. Consequently we can also put

\[
A_{J_i} = A_H \tag{57}
\]

The proof that $A_{H_n}$ solves the set (42)-(44) is the difficult one. We need the following relations. From

\[
[H, K_i] = -i P_i \tag{58}
\]

and the linear dependence of $H$ and $K_i$ on $g$ (see Eqs. (2), (3), and (6)) we find easily by equating operators related to equal powers in $g$

\[
[H_0, K_{0i}] = -i P_i \tag{59}
\]

\[
[H_0, K_{Ii}] = [K_{0i}, H_I] \tag{60}
\]

\[
[H_I, K_{Ii}] = 0 \tag{61}
\]
Further one has

$$[K_{0i}, C(H_0)] = iP_i \frac{\partial}{\partial H_0} C(H_0)$$

(62)

where $C$ depends on $H_0$ only. Because of the free state kinematics it is also easily seen that

$$[K_{0i}, \Lambda] = [K_{0i}, \eta] = 0$$

(63)

Using all that one verifies that

$$[K_{0i}, A_{H1}] = [K_{0i}, -\frac{1}{E_\Lambda - E_\eta} \Lambda B(H_0)\eta]$$

$$= [K_{0i}, -\frac{1}{H_0 - E_\eta} \Lambda B(H_0)\eta]$$

$$= [K_{0i}, -\frac{1}{H_0 - E_\eta}] \Lambda B \eta + \frac{1}{E_\Lambda - E_\eta} \Lambda [K_{0i}, B] \eta + \Lambda B [K_{0i}, \frac{1}{E_\Lambda - H_0}] \eta$$

(64)

$$= \frac{1}{E_\Lambda - E_\eta} \Lambda [K_{0i}, B] \eta$$

(65)

This enables us now to show that $A_{H_1}$ solves Eq. (42):

$$[K_{0i}, A_{H1}] = [K_{0i}, -\frac{1}{E_\Lambda - E_\eta} \Lambda H_1 \eta]$$

$$= -\frac{1}{E_\Lambda - E_\eta} \Lambda [K_{0i}, H_1] \eta$$

$$= -\frac{1}{E_\Lambda - E_\eta} \Lambda [H_0, K_{H_1}] \eta$$

$$= -\frac{1}{E_\Lambda - E_\eta} \Lambda (E_\Lambda - E_\eta) K_{H_1} \eta = -\Lambda K_{H_1} \eta$$

(66)

Next let us verify that $A_{H_2}$ from Eq. (51) solves Eq. (43):

$$[K_{0i}, A_{H2}] = -[K_{0i}, -\frac{1}{E_\Lambda - E_\eta} \Lambda H_1 A_{H_1} \eta]$$

$$= -\frac{1}{E_\Lambda - E_\eta} \Lambda [K_{0i}, H_1 A_{H_1}] \eta$$

$$= -\frac{1}{E_\Lambda - E_\eta} \Lambda [H_0, K_{H_1}] A_{H_1} \eta$$

$$- \frac{1}{E_\Lambda - E_\eta} \Lambda H_1 [K_{0i}, A_{H_1}] \eta$$

(67)

Using Eq. (66) and applying $H_0$ this can be rewritten as
\[ [K_{0i}, A_{n+1}] = -\frac{1}{E_\Lambda - E_\eta} \Lambda K_{Ii}(E_\Lambda - H_0) A_{n\eta} \]
\[ + \frac{1}{E_\Lambda - E_\eta} \Lambda H_I K_{Ii} \eta \]  
(68)

Further using Eq. (61) we get
\[ [K_{0i}, A_{n}] = -\frac{1}{E_\Lambda - E_\eta} \Lambda K_{Ii}(E_\Lambda - H_0 - A_{nI} - H_I) \eta \]
\[ = -\Lambda K_{Ii} A_{nI} - \frac{1}{E_\Lambda - E_\eta} \Lambda K_{Ii}(-[H_0, A_{nI}] - H_I) \eta \]
\[ = -\Lambda K_{Ii} A_{nI} \]  
(69)

The last step follows from Eq. (39) and proves \( A_{nI} = A_{nI} \).

Due to the structure of the set (42)-(44) it turns out that the proof for \( A_{nI} \) and \( A_{nI} \) should also be treated separately. Since the algebra is rather lengthy and the steps used for the general case \( A_{nI}, n \geq 5 \) include all the ones for the simpler cases \( n = 3 \) and \( n = 4 \) we leave the verification for those simpler cases to the reader.

We embark now in the proof that \( A_{nI}, n \geq 4 \) is a solution of Eq. (44) provided every \( A_{nI}, \nu \leq n \) is a solution of the set (42)-(44). From (52) we get
\[ [K_{0I}, A_{n+1}] = \frac{1}{E_\Lambda - E_\eta} \Lambda \left[ K_{0I}, \left( -H_I A_{nI} + \sum_{\nu=1}^{n-1} A_{nI} H_I A_{nI-\nu} \right) \right] \eta \]  
(70)

Using Eq. (60) this can be rewritten as
\[ [K_{0I}, A_{n+1}] \]
\[ = \frac{1}{E_\Lambda - E_\eta} \Lambda \left( -[H_0, K_{Ii}] A_{nI} - H_I [K_{0I}, A_{nI}] \right. \]
\[ + \sum_{\nu=1}^{n-1} [K_{0I}, A_{nI}] H_I A_{nI-\nu} \]
\[ + \sum_{\nu=1}^{n-1} A_{nI} [H_0, K_{Ii}] A_{nI-\nu} \]
\[ + \sum_{\nu=1}^{n-1} A_{nI} H_I [K_{0I}, A_{nI-\nu}] \right) \eta \]  
(71)

The first term in Eq. (71) is changed as
\[ -\frac{1}{E_\Lambda - E_\eta} \Lambda [H_0, K_{Ii}] A_{nI} \eta = -\Lambda K_{Ii} A_{nI} \eta - \frac{1}{E_\Lambda - E_\eta} \Lambda K_{Ii} (E_\eta - H_0) A_{nI} \eta \]  
(72)
Inserting this into Eq. (71) and using the assumption that \( A_{n, \nu}, \nu \leq n \) solves the set (42)-(44) we get

\[
[K_{0i}, A_{n+1}] = -\Lambda K_{Ii} A_{n} + \frac{1}{E_{\Lambda} - E_{I}} \Lambda \left( -K_{Ii} (E_{\eta} - H_{0}) A_{n} + H_{I} \Lambda K_{Ii} A_{n-1} - \sum_{\nu=3}^{n-1} A_{n, \nu} K_{Ii} A_{n-\nu-1} \right) \]

\[
+ \left[ -K_{Ii} \eta' H_{I} A_{n-1} - K_{Ii} A_{n} H_{I} A_{n-2} \right] + \sum_{\nu=1}^{n-2} \left( -K_{Ii} A_{n-\nu-1} + \sum_{\nu'=1}^{\nu-2} A_{n, \nu'} K_{Ii} A_{n-\nu-1-\nu'} \right) H_{I} A_{n}\nu \]

\[
+ \sum_{\nu=1}^{n-1} A_{n, \nu} \left[ H_{0} + K_{Ii} \right] A_{n-\nu} \left[ -K_{Ii} \eta' H_{I} A_{n-1} - K_{Ii} \Lambda K_{Ii} \right] \eta
\]

\[
\quad + \frac{1}{E_{\Lambda} - E_{I}} \Lambda \left( K_{Ii} \Lambda' H_{I} A_{n-1} - K_{Ii} \Lambda' H_{I} A_{n-2} \right) \eta = 0
\]

(73)

The square brackets are inserted to group the terms together resulting from the commutators with \( K_{0i} \). Moreover we added a zero at the end and used the identity

\[
A_{n} H_{I} \Lambda K_{Ii} = A_{n} H_{I} K_{Ii}
\]

(74)

which is valid because \( A_{n} \) has an \( \eta \)-projector on the right. Eq. (73) can be simplified by means of the two identities

\[
-K_{Ii} \Lambda H_{I} + H_{I} \Lambda K_{Ii} - K_{Ii} \eta H_{I} = -K_{Ii} H_{I} + H_{I} \Lambda K_{Ii} = -H_{I} \eta K_{Ii}
\]

(75)

and

\[
\frac{1}{E_{\Lambda} - E_{I}} \Lambda \left( K_{Ii} (H_{0} - E_{\eta}) A_{n} + K_{Ii} \Lambda' H_{I} A_{n-1} - \sum_{\nu=3}^{n-1} A_{n, \nu} H_{I} A_{n-\nu} \right. \\
\left. -K_{Ii} A_{n} H_{I} A_{n-2} \right) \eta = 0
\]

(76)

The second one is just a consequence of Eq. (41). Using Eqs. (75) and (76) we find

\[
[K_{0i}, A_{n+1}] = -\Lambda K_{Ii} A_{n} + \frac{1}{E_{\Lambda} - E_{I}} \Lambda \left( -H_{I} \eta' K_{Ii} A_{n-1} - H_{I} \sum_{\nu=1}^{n-2} A_{n, \nu} K_{Ii} A_{n-1-\nu} \right.
\]

\[
+ \frac{1}{E_{\Lambda} - E_{I}} \Lambda \left( \eta - K_{Ii} \Lambda' H_{I} A_{n-1} \right.
\]

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Next we exchange the orders of summation

\[
\sum_{\nu=1}^{n-1} \sum_{\nu'=-2}^{n-3} A_{H_{\nu}} K_{I_{1}} A_{H_{n-1-\nu}} H_{1} A_{H_{n-\nu}} + \sum_{\nu=1}^{n-1} A_{H_{\nu}} [H_{0}, K_{I_{1}}] A_{H_{n-\nu}} \\
+ \sum_{\nu=1}^{n-3} A_{H_{\nu}} H_{1} (-K_{I_{1}} A_{H_{n-1-\nu}} + \sum_{\nu'=-1}^{n-2} A_{H_{\nu}} K_{I_{1}} A_{H_{n-1-\nu'}}) \\
\]

\[-A_{H_{n-2}} H_{1} K_{I_{1}} A_{H_{1}} - A_{H_{n-1}} H_{1} K_{I_{1}}) \eta \]

(77)

Using Eq. (78) we can group together some terms from Eq. (77) taking Eq. (61) and the set (39)-(41) into account:

\[
\sum_{\nu=1}^{n-1} \sum_{\nu'=-2}^{n-3} A_{H_{\nu}} K_{I_{1}} A_{H_{n-1-\nu}} H_{1} A_{H_{n-\nu}} \\
- \sum_{\nu=1}^{n-3} A_{H_{\nu}} K_{I_{1}} A_{H_{n-1-\nu}} H_{1} A_{H_{n-\nu}} - \sum_{\nu=1}^{n-3} A_{H_{\nu}} K_{I_{1}} A_{H_{n-1-\nu}} H_{1} A_{H_{n-\nu}} \\
- A_{H_{n-2}} H_{1} K_{I_{1}} A_{H_{1}} - A_{H_{n-1}} H_{1} K_{I_{1}} \\
= \sum_{\nu=1}^{n-3} \sum_{\nu'=1}^{n-2} A_{H_{\nu}} K_{I_{1}} A_{H_{n-1-\nu}} H_{1} A_{H_{n-\nu}} - \sum_{\nu=1}^{n-3} A_{H_{\nu}} K_{I_{1}} H_{1} A_{H_{n-\nu}} \\
- A_{H_{n-2}} K_{I_{1}} H_{1} A_{H_{1}} - A_{H_{n-1}} K_{I_{1}} H_{1} \\
= + \sum_{\nu'=1}^{n-3} A_{H_{\nu}} K_{I_{1}} [H_{0}, A_{H_{n-\nu'}}] \\
+ A_{H_{n-2}} K_{I_{1}} [H_{0}, A_{H_{n-\nu'}}] + A_{H_{n-1}} K_{I_{1}} [H_{0}, A_{H_{n-\nu'}}] \\
= \sum_{\nu'=1}^{n-1} A_{H_{\nu}} K_{I_{1}} [H_{0}, A_{H_{n-\nu'}}] \\
\]

(79)
Similarly the expression (79) can be grouped together with two more terms from Eq. (77)

\[
\sum_{\nu=1}^{n-3} A_{\eta_\nu} H_{I} \sum_{\nu'=1}^{n-\nu-2} A_{\eta_\nu'} K_{I_i} A_{\eta_{n-\nu}-\nu'} - \Lambda H_I \eta' K_{I_i} A_{\eta_{n-1}} - \Lambda H_I \sum_{\nu'=1}^{n-2} A_{\eta_\nu'} K_{I_i} A_{\eta_{n-1}-\nu'} \\
\sum_{\nu'=2}^{n-2} (-\Lambda H_I A_{\eta_\nu'} + \sum_{\nu=1}^{n-\nu'-1} A_{\eta_\nu} H_{I} A_{\eta_{n-\nu'}}) K_{I_i} A_{\eta_{n-1}-\nu'} - \Lambda H_I \eta' K_{I_i} \Lambda A_{\eta_{n-1}} \\
-\Lambda H_I A_{\eta_\nu} K_{I_i} A_{\eta_{n-2}} \\
= \sum_{\nu'=2}^{n-2} [H_0, A_{\eta_\nu}] K_{I_i} A_{\eta_{n-1}-\nu'} + [H_0, A_{\eta_1}] K_{I_i} A_{\eta_{n-1}} + [H_0, A_{\eta_2}] K_{I_i} A_{\eta_{n-2}} \\
= \sum_{\nu'=3}^{n-1} [H_0, A_{\eta_\nu}] K_{I_i} A_{\eta_{n-\nu'}} + [H_0, A_{\eta_1}] K_{I_i} A_{\eta_{n-1}} + [H_0, A_{\eta_2}] K_{I_i} A_{\eta_{n-2}} \\
= \sum_{\nu'=1}^{n-1} [H_0, A_{\eta_\nu}] K_{I_i} A_{\eta_{n-\nu'}} \tag{81}
\]

Again we used the set (39)-(41) several times. Inserting all that into Eq. (77) that expression simplifies greatly and leads to the desired result:

\[
[K_{0\nu}, A_{\eta_{n+1}}] = -\Lambda K_{I_i} A_{\eta_n} \\
+ \frac{1}{E_\Lambda - E_\eta} \Lambda \sum_{\nu=1}^{n-1} \left[ A_{\eta_\nu} K_{I_i} [H_0, A_{\eta_{n-\nu}}] + A_{\eta_\nu} [H_0, K_{I_i}] A_{\eta_{n-\nu}} \right] \eta \\
- \Lambda K_{I_i} A_{\eta_n} \\
+ \frac{1}{E_\Lambda - E_\eta} \Lambda \sum_{\nu=1}^{n-1} A_{\eta_\nu} \eta' K_{I_i} A_{\eta'_{n-\nu}} (E_{\eta'} - E_\eta + E_{\eta'} - E_{\eta'} + E_\Lambda - E_{\eta'}) \eta \\
= -\Lambda K_{I_i} A_{\eta_n} + \sum_{\nu=1}^{n-1} A_{\eta_\nu} K_{I_i} A_{\eta_{n-\nu}} \tag{82}
\]

This is our final result stating that \( A_{\eta_{n+1}} \) solves Eq. (44) for \( n \geq 4 \) provided that \( A_{\eta_\nu}, \nu = 1, 2, 3, 4 \) is a solution of Eqs. (42)-(44). This however is the induction assumption and has been shown before.

We conclude that

\[
A_{K_{\eta_n}} = A_{\eta_n} \quad n \geq 1 \tag{83}
\]

and hence

\[
A_{K_{\eta}} = A_{\eta} \tag{84}
\]

Together with Eqs. (56) and (57) we arrive at the important result that \( A_{\eta} \) solves the set of all Eqs. (34)-(37) and all indices on the \( A \)’s can be omitted. In practice one will
use the recursion relations (50)-(52) for calculating $A$ since they are easier to solve than the ones resulting from Eqs. (42)-(44) and further the conditions (45) and (46) are not specific enough.

**IV. SUMMARY**

The ten generators of the Poincaré group acting in the full Fock space of nucleons, antinucleons, and mesons and having the form (2)-(5) according to often used Lagrange densities can be blockdiagonalized at the same time by a single unitary transformation. Thereby the blocks are defined by two projection operators which span the Fock space one referring to a fixed number of nucleons and the other to the rest of the space. As a consequence the resulting unitarily transformed generators act in these two spaces separately and specifically we gained effective generators in the space of $N$ nucleons which are true representations of the Poincaré algebra. This result has been proven using a (formal) power series expansion in the coupling constant. We see the importance of that result in the existence proof. Clearly in practice this series has to be truncated as is usually done in evaluating $NN$ forces in low orders of meson exchanges.

Results of numerical studies in [12] - [14] are promising. They show for instance that the relativistic energy-momentum relation of a two body state is rather well fulfilled if one solves the Schrödinger equation using the effective Hamiltonian in frames where the total momentum of the two-body system is different from zero. They also show that contributions to the relativistic Hamiltonian which remain undetermined in the scheme of an $\frac{1}{g^2}$ expansion of the Poincaré generators [5] can now be determined for any given field theory and are different from zero. In fact in the numerical examples studied [12] they are as important as those enforced by the Poincaré algebra. Thus that scheme discussed in this article provides interesting structural insight.

In addition one can pose now various questions. One of the effective generators, the Hamiltonian in the space of $N$ nucleons, will contain $NN$ but also many body forces. Will they fulfill the cluster separability? Since the effective generators are constructed in a power series expansion in the coupling constant one encounters in all orders $g^n$ with $n \geq 4$ meson exchange diagrams together with vertex corrections for instance. The question then arises whether the Poincaré algebra for the ten effective generators, which is fulfilled in each order in $g$, requires all the terms of a certain order in $g$ or whether subgroups fulfill the algebra separately. In the first case the Poincaré algebra would impose conditions on the acceptable vertex corrections which would play the role of strong formfactors. Further investigations of that type are planned.
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