Asymptotic Search for Ground States of $SU(2)$ Matrix Theory

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Abstract

We introduce a complete set of gauge-invariant variables and a generalized Born-Oppenheimer formulation to search for normalizable zero-energy asymptotic solutions of the Schrödinger equation of $SU(2)$ matrix theory. The asymptotic method gives only ground state candidates, which must be further tested for global stability. Our results include a set of such ground state candidates, including one state which is a singlet under spin(9).
1 Introduction

The $N = 16$ supersymmetric gauge quantum mechanics \cite{1, 2, 3}, including its action formulation by dimensional reduction, was first studied in 1984-85. The model was noted again in 1988-89 as a regularization \cite{4}, with continuous spectrum \cite{5}, of the $D = 11$ supermembrane. In early 1996 the model was identified \cite{6} as the dynamics of interacting $D_0$-branes, which led to further study, including a truncated version of the model \cite{7} and the identification by $D_0$-scattering \cite{8, 9} of the scale of $D_0$ physics with the scale of $D = 11$ supergravity.

Interest in the $N = 16$ model exploded in late 1996 when the large $n$ limit of the model, now christened Matrix theory, was proposed \cite{10} as a nonperturbative formulation of M-theory. Among the many papers since then, we mention only the extension \cite{11} of the conjecture to include finite $n$ and those papers with direct relevance to the ground state of the theory, in particular, the study of the Witten index of the theory \cite{12, 13, 14} and the ongoing study of the zero supercharge condition for SUSY ground states \cite{15, 16}.

Since the original work of Claudson and Halpern, however, the ground state wave function of the theory has remained elusive. One obstruction to the investigation of such dynamical questions, pointed out in the original paper, is that matrix theory has no conserved fermion number, which blocks the fermion sector analysis applicable to simpler supersymmetric quantum mechanical systems. As a consequence, one expects that any particular matrix theory eigenstate is spread over a considerable portion of the fermionic Hilbert space. The lore \cite{6, 10} is that the theory should have a unique normalizable zero-energy “threshold” bound state, which is a singlet under spin(9).

In this paper we develop an asymptotic method to search for zero-energy ground states of the $SU(2)$ matrix theory. The method has two basic ingredients,

- a complete set of gauge-invariant bosonic and fermionic variables
- a generalized Born-Oppenheimer formulation

which allow us to extend some of the ideas of Ref. \cite{7}. Moreover, there are strong parallels between our generalized Born-Oppenheimer formulation and the analysis of Ref. \cite{13}. The method yields only candidate ground states, which are gauge-invariant asymptotic solutions, near the flat directions of
the potential, of the zero-energy Schrödinger equation of the theory. The ground state candidates must be further checked for global stability at non-asymptotic values of the gauge-invariant distance $R$.

Our results include a set of such candidate ground states, including exactly one state which is a singlet under spin(9) and which, as it turns out, has bosonic angular momentum $l = 2$. The fermionic structure of the ground state candidates is relatively simple in the asymptotic domain, though one expects increasing complexity at higher order in $R^{-1}$.

**Matrix Theory**

We will follow the original notation [1] for the theory, beginning with the 16 supercharges $Q_\alpha$,

$$Q_\alpha = (\Gamma^m \Lambda_a)_\alpha \pi^m_a + igf_{abc}(\Sigma^{mn} \Lambda_a)_\alpha \phi^m_b \phi^n_c$$  \hspace{1cm} (1.1a)

$$[\phi^m_a, \pi^n_b] = i\hbar \delta_{ab} \delta_{mn}, \quad \{\Lambda_{aa}, \Lambda_{b\beta}\} = \delta_{ab} \delta_{\alpha\beta}$$  \hspace{1cm} (1.1b)

$$\{\Gamma^m, \Gamma^n\} = 2\delta_{mn}, \quad \Sigma^{mn} = -\frac{i}{4}[\Gamma^m, \Gamma^n]$$  \hspace{1cm} (1.1c)

$$a = 1 \ldots g, \quad m = 1 \ldots 9, \quad \alpha = 1 \ldots 16$$  \hspace{1cm} (1.1d)

where $\phi^m_a$ are the real bosonic variables and $f_{abc}$ are the Cartesian structure constants of any compact Lie algebra with dimension $g$. The gamma matrices $(\Gamma^m)_{a\beta}$ are real, symmetric and traceless and the fermions $\Lambda_{aa}$ are real. We will also need the generators $G_a$ of gauge transformations

$$G_a = f_{abc}(\phi^m_b \pi^n_c - \frac{i\hbar}{2} \Lambda_{aa} \Lambda_{\alpha\alpha})$$  \hspace{1cm} (1.2)

and the generators $J^{mn}$ of spin(9)

$$J^{mn} = \pi_a^{[m} \phi^n_a] - \frac{\hbar}{2} \Lambda_{aa}(\Sigma^{mn})_{\alpha\beta} \Lambda_{a\beta}$$  \hspace{1cm} (1.3)

where $[m \text{ } n]$ means antisymmetrization of indices.

The supercharges satisfy

$$\{Q_\alpha, Q_\beta\} = 2\delta_{\alpha\beta}H + 2g(\Gamma^m)_{a\beta} \phi^m_a G_a$$  \hspace{1cm} (1.4)
\[
H_B = \frac{1}{2} \pi_a \pi_a + V, \quad V = \frac{g^2}{4} f_{abc} \phi_b^m \phi_c^m f_{ade} \phi_d^m \phi_e^m \quad (1.5b)
\]

\[
H_F = -i g \frac{\hbar}{2} f_{abc} \Lambda_{\alpha\alpha} (\Gamma^m)_{\alpha\beta} \phi_b^m \Lambda_{\beta}\quad (1.5c)
\]

and the gauge-invariant states \(G_a \mid G.I.\) = 0 form the physical subspace of the theory.

2 Bosonic Preliminaries

In this section we sharpen our tools on some bosonic subproblems, allowing the Lorentz vector index to run over \(m = 1 \ldots d\) for generality, although \(d = 9\) for matrix theory.

2.1 Gauge-Invariant Bosonic Variables

For the gauge group \(SU(2)\), with \(f_{abc} = \epsilon_{abc}\), it is useful to define the real symmetric matrix \(\Phi\)

\[
\Phi_{ab} \equiv \phi_a^m \phi_b^m, \quad a, b = 1, 2, 3 \quad (2.1)
\]

and the solutions to its eigenvalue problem

\[
\Phi_{ab} \psi_b^i = \lambda_i^2 \psi_a^i, \quad i = 1, 2, 3 \quad (2.2a)
\]

\[
\psi_a^i \psi_a^j = \delta_{ij}, \quad \psi_a^i \psi_b^j = \delta_{ab} \quad (2.2b)
\]

\[
\lambda_3 \geq \lambda_2 \geq \lambda_1 \geq 0. \quad (2.2c)
\]

The eigenvectors \(\psi\) form a real orthogonal matrix and the eigenvalues \(\lambda\) are a complete set of rotation- and gauge-invariant bosonic variables for this case.

A complete set of \(3(d - 1)\) independent gauge-invariant bosonic variables includes the three eigenvalues \(\lambda\) and the \(3(d - 2)\) gauge-invariant angular variables

\[
\eta_i^m \equiv \phi_a^m \psi_a^i / \lambda_i, \quad \eta_i^m \eta_j^m = \delta_{ij}. \quad (2.3)
\]

In the first part of this paper, we focus primarily on the gauge- and rotation-invariant \(\lambda\)'s, returning to the \(\eta\)'s in Section 7. For the gauge group \(SU(3)\), there are more gauge-invariant variables, including \(d_{abc} \phi_a^m \phi_b^p \phi_c^p\).
On functions of \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \), the bosonic Hamiltonian takes the form

\[
H_B = -\frac{\hbar^2}{2} \Delta + V, \quad \Delta = \partial^m_a \partial^m_a
\]  

\[
\Delta f(\lambda) = \rho^{-1} \frac{\partial}{\partial \lambda_i} (\rho \frac{\partial}{\partial \lambda_i} f(\lambda))
\]

\[
V = \frac{g^2}{2} (\lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2)
\]

\[
\rho(\lambda) \equiv (\lambda_1 \lambda_2 \lambda_3)^{d-3}(\lambda_3^2 - \lambda_1^2)(\lambda_2^2 - \lambda_1^2)(\lambda_3^2 - \lambda_2^2) \geq 0
\]

where \( \partial^m_a = \partial/\partial \phi^m_a \) and \( H_B \) is hermitian in the inner product

\[
\int d^3 \lambda \rho(\lambda) f^*(\lambda) g(\lambda), \quad d^3 \lambda \equiv d\lambda_1 d\lambda_2 d\lambda_3.
\]

More generally, the full bosonic measure is

\[
(d\phi) = d^3 \lambda \rho(\lambda)(d\Omega)
\]

\[
\int (d\Omega) = 1
\]

\[
\int (d\Omega) f(\phi) = 0 \quad \text{when} \quad f(\phi) = -f(-\phi)
\]

where \( \Omega \) are \( 3(d-1) \) “angles” (which include the \( 3(d-2) \) gauge-invariant angles \( \eta \) in (2.3), plus three gauge degrees of freedom). Through Section 6 of this paper, the relations (2.6b) and (2.6c) are all we shall need to know about \( \Omega \).

### 2.2 Zero-Energy Hamilton-Jacobi Equation

It was emphasized by Claudson and Halpern that a SUSY ground state must satisfy the zero-energy Hamilton-Jacobi equation

\[
\psi \sim \exp[\pm \frac{S_0}{\hbar}]
\]

\[
\frac{1}{2} \mid \nabla S_0 \mid^2 = V
\]

in the extreme semiclassical limit, and this equation takes the \( d \)-independent form

\[
(\partial/\partial \lambda_i S_0(\lambda))(\partial/\partial \lambda_i S_0(\lambda)) = g^2 (\lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2)
\]
when we restrict ourselves to gauge- and rotation-invariant wave functions. These authors also gave an exact solution of eq. (2.7b) or (2.8),

\[ S_0(\lambda) = \sqrt{W} \]  

(2.9a)

\[ W = \frac{1}{6} g^2 \epsilon_{abc} \epsilon_{def} \phi_a^{m} \phi_b^{n} \phi_c^{p} \phi_d^{q} \phi_e^{r} \phi_f^{s} \]  

(2.9b)

\[ = g^2 \det \Phi \]  

(2.9c)

\[ = g^2 \lambda_1^2 \lambda_2^2 \lambda_3^2. \]  

(2.9d)

Here, we make a brief systematic study of the solutions of the zero-energy Hamilton-Jacobi equation (2.8).

For this investigation, it is convenient to introduce spherical coordinates

\[ \lambda_1 = r \sin \theta \cos \phi, \quad \lambda_2 = r \sin \theta \sin \phi, \quad \lambda_3 = r \cos \theta \]  

(2.10a)

\[ r^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \]  

(2.10b)

and to write the solution in the form

\[ S_0 = gr^3 F(\theta, \phi). \]  

(2.11)

Then the Hamilton-Jacobi equation (2.8) reduces to

\[ \sin^2 \theta (9F^2 + F_\phi^2) + F_\phi^2 = \sin^4 \theta (\cos^2 \theta + \sin^2 \theta \cos^2 \phi \sin^2 \phi) \]  

(2.12)

where the subscripts denote partial derivatives. The right side of (2.12) is proportional to \( V \), so the flat directions of \( V \) correspond to \( \theta = 0 \) in these variables.

For small \( \theta \), (2.12) admits two solutions which are non-singular as \( \theta \to 0 \),

\[ S_0 = \frac{gr^3 \theta^2}{2} \left\{ \sin 2(\phi - \phi_0) \right\} + O(\theta^3). \]  

(2.13)

The first solution, which we call the Claudson-Halpern (CH) branch, contains the CH solution \( S_0(\lambda) = \sqrt{W} \) when \( \phi_0 = 0 \), and the second solution is the solution \( S_0(\lambda) \approx \frac{V}{gr} \) studied later by Itoyama [17, 18]. The \( \phi_0 \neq 0 \) solutions of the CH branch are new.
Because the exponential decrease of $\exp[-\frac{S_0}{\hbar}]$ at large $r$ is lost near $\theta = 0$, none of these solutions is normalizable\(^1\), whether we choose the naive measure $d^3\lambda$ or the quantum measure $d^3\lambda\rho(\lambda)$. More precisely, we find non-normalizability in the flat directions

$$\int d^3\lambda \exp[-\frac{2S_0}{\hbar}] \propto \int_0^\theta d\theta$$

(2.14a)

$$\int d^3\lambda\rho(\lambda) \exp[-\frac{2S_0}{\hbar}] \propto \int_0^\theta d\theta$$

(2.14b)

for the Itoyama solution and the CH branch with $0 < \phi_0 < \frac{\pi}{4}$. (For small $\theta$ the range of the angle $\phi$ is $\frac{\pi}{4} \leq \phi \leq \frac{\pi}{2}$.) The CH solution itself, with $\phi_0 = 0$, has an extra multiplicative divergence in the $\phi$ integration.

It is clear that the Hamilton-Jacobi equation by itself is unable to choose among solutions or to answer the question of normalizability: any of the solutions might, in principle, be made normalizable by quantum corrections including the fermions, and

$$\psi \sim \exp[-\frac{1}{\hbar}(S_0 + \hbar\alpha \ln r)]$$

(2.15)

is normalizable in $d^3\lambda\rho(\lambda)$ when $\alpha > \frac{3}{2}$ for the Itoyama solution and for the CH branch with $0 < \phi_0 < \frac{\pi}{4}$. For the CH solution itself, normalizability requires that $\alpha > 3$. In what follows, our task is to study such quantum corrections in detail.

It is also important to note that our study of the zero-energy Hamilton-Jacobi equation is incomplete because the full equation (2.7b) allows other (gauge- but not rotation-invariant) solutions, which include the $\eta$ variables in (2.3) as well (see Section 7).

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\(^1\)We note in passing that the full bosonic Hamiltonian $H_B$ has exact gauge-invariant zero-energy solutions

$$\psi(W) = W^\gamma K_{2|\gamma|}(\frac{\sqrt{W}}{\hbar}) \sim \exp[-\sqrt{\frac{W}{\hbar}}], \quad \gamma = \frac{(4 - d)}{4}$$

and $K \to I$ where $K$ and $I$ are cylinder functions of imaginary argument. These solutions are quantum extensions of the Claudson-Halpern solution, but neither are normalizable.
2.3 Born-Oppenheimer Approximation

Our approach in this paper follows the line of the Born-Oppenheimer approximation [19], which we illustrate first on the gauge- and rotation-invariant sector of the bosonic Hamiltonian

\[ H_B \psi(\lambda) = E \psi(\lambda) \]  

\[ V = \frac{g^2}{2} [R^2(\lambda_1^2 + \lambda_2^2) + \lambda_1^2 \lambda_2^2] \]  

where we have set \( R = \lambda_3 \). Our goal is to study the asymptotic behavior at large \( R \) near the classical flat directions, \( \lambda_1 = \lambda_2 = 0 \), of \( V \). In the language of the Born-Oppenheimer approximation, we integrate out the “fast” variables \( \lambda_1, \lambda_2 \) to obtain an effective Hamiltonian for the asymptotic behavior in the “slow” variable \( R = \lambda_3 \).

Toward this end we first decompose the Hamiltonian and the measure as

\[ H_B = H_0 + H_1 \]  

\[ H_0 = -\frac{\hbar^2}{2} \left\{ \frac{\partial^2}{\partial \lambda_1^2} + \frac{(d - 3)}{\lambda_1} \frac{\partial}{\partial \lambda_1} + \frac{2 \lambda_1}{\lambda_1^2 - \lambda_2^2} \frac{\partial^2}{\partial \lambda_2^2} + \left\{ \frac{(d - 3)}{\lambda_2} + \frac{2 \lambda_2}{\lambda_2^2 - \lambda_1^2} \right\} \frac{\partial}{\partial \lambda_2} \right\} + \frac{g^2}{2} R^2 (\lambda_1^2 + \lambda_2^2) \]  

\[ H_1 = -\frac{\hbar^2}{2} \left\{ \frac{\partial^2}{\partial R^2} + \frac{(d + 1)}{R} \frac{\partial}{\partial R} + \frac{2 (\lambda_1^2 + \lambda_2^2)}{R^3} + \ldots \right\} \frac{\partial}{\partial R} - \frac{2}{R^2} (\lambda_1 \frac{\partial}{\partial \lambda_1} + \lambda_2 \frac{\partial}{\partial \lambda_2} + \ldots) \right\} + \frac{g^2}{2} \lambda_1^2 \lambda_2^2 \]  

\[ \rho = R^{d+1} \sigma \]  

\[ \sigma = (\lambda_1 \lambda_2)^{d-3} (\lambda_2^2 - \lambda_1^2) (1 - \lambda_1^2 / R^2) (1 - \lambda_2^2 / R^2) \]  

where we will see that \( H_0 \) is the dominant part of \( H_B \) at large \( R \) and the dots in \( H_1 \) indicate terms with higher inverse powers of \( R \).

The first term \( H_0 \) in (2.17b) describes a rotation- and gauge-invariant two-dimensional oscillator whose frequency is linear in \( R \). The nodeless eigenstate

\[ u_R(\lambda_1, \lambda_2) = C(R) R^{-\frac{(d-1)}{2}} \exp\left[-\frac{g R}{2\hbar}(\lambda_1^2 + \lambda_2^2)\right] \]
\[
E_0(R) = \hbar g R (d - 1) \tag{2.18b}
\]

\[
\int d^2 \lambda \sigma |u_R|^2 = 1 \tag{2.18c}
\]

is almost certainly the unique ground state of \(H_0\) (see Appendix G), where \(E_0(R)\) is the energy and \(d^2 \lambda = d\lambda_1 d\lambda_2\). The power of \(R\) in \(u_R\) guarantees that \(C(R)\) approaches a constant at large \(R\),

\[
|C(R)|^2 = \frac{(d - 3)!}{2^{d-1}} (1 + O(R^{-3})) \tag{2.19}
\]

because

\[
\lambda_1, \lambda_2 = O((\frac{\hbar}{g})^{\frac{1}{2}} R^{-\frac{1}{2}}) \tag{2.20}
\]

when averaged over \(|u_R|^2\). The orders of magnitude in (2.20) define the quantum neighborhood of the classical flat directions of the potential.

In this paper we compute only through \(O(R^{-2})\), and, for this purpose, \(C(R)\) may be treated as a constant. Similarly, the measure \(\sigma\) in (2.17e) can be replaced by its asymptotic form

\[
\sigma \to \sigma_\infty = (\lambda_1 \lambda_2)^{d-3} (\lambda_2^2 - \lambda_1^2) \tag{2.21}
\]

in all computations through \(O(R^{-2})\).

More generally, all the eigenfunctions of \(H_0\) can be written as the normalizing power of \(R\) in (2.18a) times functions of the scaled variables

\[
z_1 = \lambda_1 \left(\frac{g}{\hbar} R\right)^{\frac{1}{2}}, \quad z_2 = \lambda_2 \left(\frac{g}{\hbar} R\right)^{\frac{1}{2}} \tag{2.22}
\]

and the corresponding eigenvalues are all proportional to \(R\). At finite values of \(z_{1,2}\), it follows that, throughout the Hilbert space of \(H_0\), we may estimate the order of magnitude of \(\lambda_1\) or \(\lambda_2\) at large \(R\) as \(O(R^{-\frac{1}{2}})\) (as recorded in (2.20)), and the derivatives with respect to \(\lambda_1\) or \(\lambda_2\) as \(O(R^{\frac{3}{2}})\). Using these orders of magnitude, one sees that \(H_0\) is the dominant part of \(H_B\) (contains all terms of \(O(R)\)) in the gauge- and rotation-invariant sector and \(H_1 = O(R^{-2})\).

Another way to view the large \(R\) expansion of this paper, although we have chosen not to write things out in this way, is to use as independent variables \((z_1, z_2, \lambda_3 = R)\), and then formally expand in powers of \(R^{-1}\).
The conventional Born-Oppenheimer approximation is essentially first-order perturbation theory in $H_1$ around $u_R$. In variational language, we study a separable trial wave function of the form

$$\psi(\lambda) = u_R(\lambda_1, \lambda_2)\psi(R) \quad (2.23)$$

where $\psi(R)$ may be called the reduced wave function or state vector. Averaging over the fast variables, we obtain an effective Hamiltonian for the slow variable $R$,

$$H_{\text{eff}}(R)\psi(R) = E\psi(R)$$
$$H_{\text{eff}}(R) = \int d^2\lambda^* u_R^* H_B u_R$$
$$\int dR R^{d+1} |\psi(R)|^2 < \infty \quad (2.24a)$$

where the normalization condition on the reduced state vector is given in (2.24c). The effective Hamiltonian (2.24b) can be evaluated exactly but we confine ourselves in this paper to the leading terms (through $O(R^{-2})$) at large $R$.

Using the integrals given in Appendix F, we obtain the asymptotic form of the effective Hamiltonian

$$H_{\text{eff}}(R) = \frac{\hbar^2}{2} \left\{ \frac{d^2}{dR^2} + \frac{(d+1)}{R} \frac{d}{dR} + \frac{B}{R^2} + \ldots \right\} \quad (2.25a)$$

$$B = -\frac{(d-1)(d-9)}{4} \quad (2.25b)$$

whose linear potential is nothing but $E_0(R)$ in (2.18b). The coefficient of the first derivative term in (2.25a) could have been fixed in advance by hermiticity of $H_{\text{eff}}$ in the reduced measure $R^{d+1}$ of eq. (2.24c) and in fact the operator

$$\Delta_R = \frac{d^2}{dR^2} + \frac{(d+1)}{R} \frac{d}{dR} \quad (2.26)$$

is the natural Laplacian on this measure.

For this bosonic case, the positive potential growing linearly with $R$ gives $E > 0$ normalizable bound states which show exponential decrease

$$\psi(R) \sim \exp[-\frac{2}{3} \left( \frac{2g(d-1)}{\hbar} \right)^{\frac{1}{2}} R^\frac{3}{2}] \quad (2.27)$$
at large $R$. For the full matrix theory, we expect from Ref. [7] that the fermionic contributions will exactly cancel$^{2}$ the bosonic contribution $E_0(R) = 8\hbar R$, leaving an effective Hamiltonian of the form

$$H_{\text{eff}}(R) = -\frac{\hbar^2}{2} \left( \frac{d^2}{dR^2} + \frac{(d + 1)}{R} \frac{d}{dR} + \frac{B'}{R^2} + \ldots \right)$$

and such a Hamiltonian can have an asymptotic power-law behaved zero-energy normalizable bound state, provided that

$$B' < \frac{(d^2 - 4)}{4}.$$  

The problem here is that the Born-Oppenheimer approximation cannot be trusted to give the true value of the constant $B'$, even approximately, because (unlike molecular physics) matrix theory has no natural small parameters to control the approximation. In what follows, we develop an improved formalism which allows us to compute the necessary coefficient $B'$ exactly in matrix theory.

3 Generalized Perturbation Theory

In order to study the asymptotic behavior of the wavefunction, we need a procedure which combines the idea of the Born-Oppenheimer approximation with the techniques of perturbation theory. Here is such a general formalism for studying the equation

$$\mathcal{L} | \Psi \rangle = 0$$

in which the linear operator is $\mathcal{L} = H - E$ and $| \Psi \rangle$ is a vector in the Hilbert space of $H$.

We start by choosing a normalized state $| \cdot \rangle$ in the Hilbert space and its associated projection operators

$$P = P^2 = | \cdot \rangle \langle \cdot |, \quad Q = Q^2 = 1 - P$$

and the action of these projection operators on the state vector will be written as

$$| \Psi_P \rangle = P | \Psi \rangle, \quad | \Psi_Q \rangle = Q | \Psi \rangle.$$ 

$^{2}$Following Ref. [7], we expect that sectors with uncanceled linear $R$ terms are associated to excited states.
The original Schrödinger equation (3.1) is then broken down into two coupled equations. The first equation is

\[ P \mathcal{L} P | \Psi_P \rangle + P \mathcal{L} Q | \Psi_Q \rangle = 0 \]  

(3.4)
or, equivalently,

\[ \langle \cdot | \mathcal{L} | \cdot \rangle \langle \cdot | P | \Psi_P \rangle + \langle \cdot | \mathcal{L} Q | \Psi_Q \rangle = 0 \]  

(3.5)

and the second equation is

\[ Q \mathcal{L} P | \Psi_P \rangle + Q \mathcal{L} Q | \Psi_Q \rangle = 0 \]  

(3.6)

which can be formally solved as,

\[ | \Psi_Q \rangle = - (Q \mathcal{L} Q)^{-1} Q \mathcal{L} P | \Psi_P \rangle. \]  

(3.7)

If we substitute (3.7) into (3.4), we get the “reduced” Schrödinger equation:

\[ [P \mathcal{L} P - P \mathcal{L} Q (Q \mathcal{L} Q)^{-1} Q \mathcal{L} P] | \Psi_P \rangle = 0 \]  

(3.8)
or, equivalently,

\[ [\langle \cdot | \mathcal{L} | \cdot \rangle - \langle \cdot | \mathcal{L} Q (Q \mathcal{L} Q)^{-1} Q \mathcal{L} | \cdot \rangle] \langle \cdot | \Psi \rangle = 0. \]  

(3.9)

One can also write a variational principle for the exact solution of (3.6),

\[ J[\chi] = \langle \chi | Q \mathcal{L} Q | \chi \rangle + 2 \langle \chi | Q \mathcal{L} P | \Psi_P \rangle \]  

(3.10)

where \( J \) is stationary under variations of \( | \chi \rangle \) about \( | \Psi_Q \rangle \).

This formulation is exact and can be adapted to a number of different applications. For the familiar problem of non-degenerate perturbation theory, where \( \mathcal{L} = H_0 - E + V \), one chooses \( P \) to project onto a particular eigenstate of \( H_0 \) and then the introduction of power series expansions into eqs. (3.7) and (3.8) leads to familiar formulas.

We may illustrate this situation by choosing

\[ | \Psi \rangle = | p \rangle, \quad (H_0 + V) | p \rangle = E_p | p \rangle, \quad H_0 | p \rangle^0 = E_p^0 | p \rangle^0 \]  

(3.11a)

\[ | \cdot \rangle = | p \rangle^0, \quad P = | p \rangle^0 \langle p |, \quad | \Psi_P \rangle = | p \rangle^0 \langle p | p \rangle. \]  

(3.11b)
Equation (3.8) then becomes the energy equation,
\[ E_p = E_p^0 + V_{pp} + \sum_{m,n} V_{pm}[1 + Q(H_0 - E_p)^{-1}QVQ]_{mn}^{-1} \frac{V_{np}}{(E_p - E_n^0)} \] (3.12)
and equation (3.7) becomes the wavefunction equation,
\[ \langle m \neq p | p \rangle = \{\sum_n [1 + Q(H_0 - E_p)^{-1}QVQ]_{mn}^{-1} \frac{V_{np}}{(E_p - E_n^0)}\}^0 \langle p | p \rangle \] (3.13)
and both are easily iterated to any desired order of the perturbation \( V \). In this example, each choice of projector \( P \) is a choice to study a “nearby” exact state \( | \Psi \rangle \).

In the case of degenerate perturbation theory, one starts by choosing \( P \) as the projector into the degenerate subspace of interest and equation (3.8) becomes a matrix equation in that subspace.

For generalized Born-Oppenheimer problems, we proceed as follows. The original problem involves a number of coordinates, which we partition into two groups, called \( x \) (the “fast” variables) and \( y \) (the “slow” variables)
\[ | \Psi \rangle = | \Psi(x,y) \rangle \] (3.14)
and we choose a particular projector \( P \) to act only on the \( x \)-variables,
\[ | \cdot \rangle = | \psi_0(x) \rangle_R \] (3.15a)
\[ P = | \cdot \rangle \langle \cdot | = | \psi_0(x) \rangle_R \int dx' R \langle \psi_0(x') | . \] (3.15b)
In this class of applications, the partition into fast and slow variables and the choice of the projector state and its symmetry determines a preferred sector of the Hilbert space. In practice, our choice of projector state \( | \cdot \rangle \) below will be guided by the need to cancel the linear term in \( R \) in (2.25). The projected state is
\[ | \Psi_P \rangle = | \psi_0(x) \rangle_R | \psi(y) \rangle \] (3.16a)
\[ | \psi(y) \rangle = \langle \cdot | \Psi(x,y) \rangle = \int dx' R \langle \psi_0(x') | \Psi(x', y) \rangle \] (3.16b)
where \( | \psi(y) \rangle \) will be called the reduced state vector. Note that inner products with this projection operator involve integration over the fast variables.
x but not over the slow variables y; the symbol R stands for a subset of the y variables, and the subscript R is placed on the projector state $|\psi_0(x)\rangle_R$ to indicate that this vector in the Hilbert space of the x-variables may be parametrized by some of the y-variables.

In these applications equations (3.4) and (3.8) are reduced Schrodinger equations in the slow variables y, the fast variables x having being integrated out, and in particular eq. (3.9)

$$\{\langle \cdot | H - E | \cdot \rangle - \langle \cdot | HQ(Q(H - E)Q)^{-1}QH | \cdot \rangle\} |\psi(y)\rangle = 0 \quad (3.17)$$

is the effective Schrodinger equation for the reduced state vector $|\psi(y)\rangle$. The terms of (3.17) are in 1-1 correspondence with the terms of (3.4),

$$\int dx_R \langle \psi_0(x) | H - E | \psi_0(x) \rangle_R |\psi(y)\rangle + \int dx_R \langle \psi_0(x) | H | \Psi_Q(x,y) \rangle = 0 \quad (3.18)$$

and the first term would give the “first-order” Born-Oppenheimer approximation (that is, eqs. (2.24a,b)) if we were to ignore the second term. The second term can contribute in principle, however, to the effective Hamiltonian for $|\psi(y)\rangle$, and so we must proceed to solve the other equation (3.6)

$$QH |\psi_0(x)\rangle_R |\psi(y)\rangle + Q(H - E) |\Psi_Q(x,y)\rangle = 0 \quad (3.19)$$

for the state $|\Psi_Q(x,y)\rangle$. If we have some small quantity, such as $\frac{1}{R}$ at large $R$, solutions of the system may be carried out in practice to any desired order of the small quantity.

Application of this machinery to matrix theory requires that we first make some transformations from the original variables.

4 Canonical Transformations

We focus now on the fermionic variables $\Lambda_{\alpha\alpha}$ of matrix theory and carry out canonical transformations in order to introduce gauge-invariant fermions (Subsection 4.1) and to obtain a form of the Hamiltonian (Subsection 4.2) which is amenable to the computational method of the previous section.

In what follows we scale out $\hbar$ and the coupling constant g, according to the relations

$$Q_\alpha(\hbar, g; \phi) = (g\hbar^2)^{1/2} Q_\alpha(1,1; \phi') \quad (4.1a)$$

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\[ H(h, g; \phi) = (g \hbar^2)^{\frac{1}{2}} H(1, 1; \phi') \quad (4.1b) \]
\[ \phi = \left( \frac{\hbar}{g} \right)^{\frac{1}{2}} \phi' \quad (4.1c) \]
and it is really \( \phi' \) which appears below, although we drop the prime. At any point, the reader may reinstate these parameters with the substitution
\[ \phi \rightarrow \left( \frac{g}{\hbar} \right)^{\frac{1}{2}} \phi \quad (4.2) \]
and the rescalings of \( Q_\alpha \) and \( H \) above.

### 4.1 Gauge-Invariant Fermions

Our first step involves the introduction of gauge-invariant fermions, using the eigenvectors \( \psi_i^a \) which were introduced in (2.2). The gauge-invariant fermions are defined as
\[ \Lambda'_{i\alpha} \equiv \psi^i_a \Lambda_{a\alpha}, \quad i = 1, 2, 3, \quad \alpha = 1 \ldots 16 \quad (4.3) \]
and these preserve the anti-commutation relations
\[ \{ \Lambda'_{i\alpha}, \Lambda'_{j\beta} \} = \delta_{ij} \delta_{\alpha\beta}. \quad (4.4) \]
Moreover, the gauge-invariant fermions allow us to write the Yukawa term in the Hamiltonian (1.5) as
\[ H_F = -\frac{i}{2} \epsilon_{ijk} \Lambda'_{i\alpha} (\Gamma_j)_{\alpha\beta} \Lambda'_{k\beta} \lambda_j \equiv -\frac{i}{2} \epsilon_{ijk} (\Lambda'_i \Gamma_j \Lambda'_k) \lambda_j. \quad (4.5) \]
The real symmetric, traceless and gauge-invariant matrices \( \Gamma_i \) in (4.5) are defined by
\[ \Gamma_i \equiv \frac{\Gamma^m \tilde{\phi}^m_a \psi^i_a}{\lambda_i} = \Gamma^m \eta^m_i, \quad i = 1, 2, 3 \quad (4.6a) \]
\[ \{ \Gamma_i, \Gamma_j \} = 2 \delta_{ij} \quad (4.6b) \]
and they preserve the Clifford algebra as shown. (The 21 gauge-invariant angles \( \eta^m_i \) are defined in (2.3).) In what follows, \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) are the components of \( \Gamma_i \).

The eigenvectors \( \psi^i_a \) are functions of the bosonic variables \( \phi^m_a \), so the gauge-invariant fermions \( \Lambda' \) are coordinate-dependent and do not commute.
with the bosonic derivatives $\pi$. We rectify this situation by making an additional canonical transformation to obtain independent bosonic momenta $\pi'$:

$$\pi'^m_a = \pi^m_a + F^m_a \tag{4.7a}$$

$$F^m_a = i \frac{1}{2} (\Lambda'_i (T^m_a)_{ij} \Lambda'_j), \quad (T^m_a)_{ij} = \psi^i_b \delta^m_a \psi^j_b \tag{4.7b}$$

$$[\pi'^m_a, \Lambda'_{ia}] = 0 \tag{4.7c}$$

where $\pi'$ and $\phi$ remain canonical. This allows us to specify that

$$\pi'^m_a | \Lambda' \rangle = 0 \tag{4.8a}$$

$$\pi'^m_a [f(\phi) | \Lambda' \rangle] = -i (\partial^m_a f(\phi)) | \Lambda' \rangle \tag{4.8b}$$

where $| \Lambda' \rangle$ is any state formed with the gauge-invariant fermions. In what follows, we describe this situation by writing

$$\pi'^n_m = -i \partial^m_n \Lambda', \quad \partial^m_n \Lambda' = 0. \tag{4.9}$$

Further details of this transformation are given in Appendix A, which notes that the matrices $T^m_a$ are divergence-free flat connections.

Appendix A also shows that the gauge generators (1.2) become purely bosonic

$$G_a = \epsilon_{abc} (\phi^m_b \pi'^m_c - i \frac{1}{2} (\Lambda_b \Lambda_c)) = \epsilon_{abc} \phi^m_b \pi'^m_c \tag{4.10}$$

when written in terms of the independent canonical momenta $\pi'$. This result confirms that $G_a$ commutes with $\Lambda'$ and tells us that states formed with the $\Lambda'$ fermions

$$G_a f(\lambda, \eta) | \Lambda' \rangle = 0 \tag{4.11}$$

are gauge invariant, as expected, when the bosonic coefficient $f$ is separately gauge invariant.

The rotation generators (1.3) also maintain a simple form

$$J^{mn} = \pi'[m \phi'[n] - \frac{1}{2} (\Lambda'_i \Sigma^{mn} \Lambda'_j) \tag{4.12}$$

when expressed in terms of the independent momenta $\pi'$. This result shows that, because $\psi'_{\lambda}$ is rotation-invariant, the gauge-invariant fermions remain spinors under spin(9).
Other applications of the gauge-invariant fermions include the following: The supercharges $Q_\alpha$ and the Hamiltonian $H$ can be written entirely in terms of gauge-invariant quantities. This gauge-invariant formulation of $SU(2)$ matrix theory is given in Appendix B and continued in Appendix G. Moreover, the complete diagonalization of the Yukawa term

$$H_F = -\sum_{k,\nu} \mu_k a^+_k a_{k\nu}$$

(4.13)

is discussed in Appendix C.

### 4.2 Further Transformations

For our consideration below of large $R = \lambda_3$ asymptotic behavior, it is convenient to make another transformation to simplify the leading term in $H_F$, 

$$i\Lambda'_1 (\Gamma_3)_{\alpha\beta} \Lambda'_2 \lambda_3 = i(\Lambda'_1 \Gamma_3 \Lambda'_2)R.$$  

(4.14)

We further define

$$\begin{align*}
\Lambda''_1 &= \Gamma_3 \Lambda'_1, \\
\Lambda''_2 &= \Lambda'_2, \\
\Lambda''_3 &= \Lambda'_3
\end{align*}$$

(4.15a)

$$\{\Lambda''_{i\alpha}, \Lambda''_{j\beta}\} = \delta_{ij} \delta_{\alpha\beta}$$

(4.15b)

along with another canonical transformation,

$$\pi''_{am} = \pi''_{am} + G^m_a, \quad [\pi''_{am}, \Lambda''_{i\alpha}] = 0$$

(4.16)

for which $\pi''$ and $\phi$ remain canonical. See Appendix A for further details.

The final form of the gauge generators is

$$G_a = \epsilon_{abc} \phi^m_b \pi''_{cm}$$

(4.17)

because $\Lambda''$ are also gauge-invariant fermions. The rotation generators are now

$$J^{mn} = \pi''_{[m} a^*_{n]} - \frac{1}{2} (\Lambda'' \Sigma^{mn} \Lambda'')$$

(4.18)

and it follows that the $\Lambda''$ fermions remain spinors under spin(9).

The final form of the Hamiltonian that results from our canonical transformations is the following (we now drop all primes for simplicity):

$$H = H_B + H_F + H_S$$

(4.19)
where

\[ H_B = \frac{1}{2} \pi_a^m \pi_a^m + V, \] (4.20a)

\[ H_F = i(\Lambda_1 \Lambda_2) R + i(\Lambda_2 \Gamma_1 \Lambda_3) \lambda_1 + i(\Lambda_3 \Gamma_2 \Lambda_1) \lambda_2, \] (4.20b)

\[ H_S = -(F_a^m + G_a^m) \pi_a^m + \frac{1}{2} F_a^m F_a^m + \frac{1}{2} G_a^m G_a^m + F_a^m G_a^m. \] (4.20c)

Here

\[ \pi_a^m = -i \partial_a^m = -i \partial / \partial \phi_a^m, \quad \partial_a^m \Lambda_{ia} = 0 \] (4.21a)

\[ F_a^m = i(\Lambda_1 \Gamma_3 \Lambda_2)(T_a^m)_{12} + i(\Lambda_1 \Gamma_3 \Lambda_3)(T_a^m)_{13} + i(\Lambda_2 \Lambda_3)(T_a^m)_{23} \] (4.21b)

\[ G_a^m = \frac{i}{2}(\Lambda_1 \Gamma_3 \partial_a^m \Gamma_3 \Lambda_1) \] (4.21c)

and the connection \( T \) is defined in (4.7). Note that \( H_F \) in (4.20b) is the original Yukawa term, now written in terms of the gauge-invariant fermions, and the shift term \( H_S \), which is quartic in the gauge-invariant fermions, is the result of our canonical transformations.

Our third and final step is to introduce gauge-invariant fermion creation and annihilation operators for \( \Lambda_1 \) and \( \Lambda_2 \):

\[ \Lambda_{1a} = \frac{(a_a + a_a^+)}{\sqrt{2}}, \quad \Lambda_{2a} = \frac{(a_a - a_a^+)}{i \sqrt{2}}. \] (4.22a)

\[ \{a_\alpha, a_\beta^+\} = \delta_{\alpha \beta} \] (4.22b)

This gives the first term in \( H_F \) as

\[ i(\Lambda_1 \Lambda_2) R = R(\sum_a a_a^+ a_a - 8) \] (4.23)

and the gauge-invariant empty state \( |0\rangle \), defined by

\[ a_\alpha |0\rangle = 0 \] (4.24)

gives the lowest value \(-8R\) for this operator.

The final form of the rotation generators is

\[ J^{mn} = \pi_a^{[m} \phi_a^{n]} - a_a^+ (\Sigma^{mn})_{\alpha \beta} a_\beta - \frac{1}{2} (\Lambda_3 \Sigma^{mn} \Lambda_3) \] (4.25)

so that the state \( |0\rangle \) is invariant under rotations of the \( \Lambda_1, \Lambda_2 \) fermions.
5 The First Computation

The Hamiltonian (4.19) acts in the Hilbert space of the following 75 variables:

- 27 bosonic variables $\phi^a_n$, which we have packaged into 3 gauge-and rotational-invariant “lengths” – $\lambda_1, \lambda_2, \lambda_3$ – and 24 remaining “angles” $\Omega$.

- 48 fermionic operators, where 32 have been packaged into the gauge-invariant annihilation and creation operators $a_\alpha$ and $a_\alpha^+$ and another 16 gauge-invariant fermions $\Lambda_3\alpha$.

We begin the computation by choosing a partition into the fast variables, “$x$” variables: $\lambda_1, \lambda_2, a_\alpha, a_\alpha^+, \Omega$ \hspace{1cm} (5.1)

and the slow variables, “$y$” variables: $\lambda_3 = R, \Lambda_3\alpha$ \hspace{1cm} (5.2)

although we will discuss a slightly different partition in Section 7.

Next, we must choose a particular projection operator $P$ and its associated projector state $| \cdot \rangle$. Our choice is

\[
| \cdot \rangle = | \psi_0(x) \rangle_R = u_R(\lambda_1, \lambda_2) | 0 \rangle \hspace{1cm} (5.3a)
\]

\[
| \Psi_P \rangle = | \cdot \rangle | \psi(R, \Lambda_3) \rangle = | \psi_0(x) \rangle_R | \psi(R, \Lambda_3) \rangle \hspace{1cm} (5.3b)
\]

where $u_R(\lambda_1, \lambda_2)$ is defined in equation (2.18) and $| 0 \rangle$ is the empty fermion state for $\Lambda_1$ and $\Lambda_2$ defined in (4.24). This state $| \cdot \rangle$ is the gauge-invariant analogue of the approximate ground state introduced in Ref. [7], and, as discussed by these authors, it will guarantee the desired cancellation of the term linear in $R$ in the effective Hamiltonian (2.25).

This leaves us to study the reduced state vector $| \psi(R, \Lambda_3) \rangle$ in the “$y$” variables,

\[
| \psi(R, \Lambda_3) \rangle = \langle \cdot | \Psi(x, y) \rangle = \langle \cdot | \Psi_P \rangle = \int d^2 \lambda (d\Omega) \sigma u_R^*(\lambda_1, \lambda_2) \langle 0 | \psi(x, y) \rangle \hspace{1cm} (5.4)
\]

where $\sigma$ is defined in (2.17e) and the normalization integral is

\[
\int dRR^{10} \langle \psi(R, \Lambda_3) | \psi(R, \Lambda_3) \rangle < \infty. \hspace{1cm} (5.5)
\]
Consistent with earlier notation, we define
\[
\langle \cdot \mid A \mid \cdot \rangle \equiv \int d^2 \lambda (d\Omega) \sigma u_R^* (\lambda_1, \lambda_2) \langle 0 \mid A \mid 0 \rangle u_R (\lambda_1, \lambda_2)
\] (5.6)
where \( A \) is any operator which may depend upon both the \( x \) and \( y \) variables. The result of this partial average is an operator that depends only upon the \( y \) variables and their derivatives.

We must now go through all the terms in the Hamiltonian (4.19) and answer the following questions for each operator:

1. What fermionic selection rules apply with respect to the number operator \( N_F = \sum \alpha a_\alpha^+ a_\alpha \)?

2. What is the order of magnitude of the operators in powers of \( R \)? Here, it is important to remember that \( \lambda_1 \) and \( \lambda_2 \) are of order \( R^{-1/2} \) at large \( R \).

The details of this assessment are given in Appendix D. The results are given below, phrased in the language of “matrix elements,” \( PHP, PHQ, QHP \) and \( QHQ \), as these appear in the basic equations (3.4) and (3.6). For reference, the first of these equations reads:
\[
P(H - E)P \mid \Psi_P \rangle + PHQ \mid \Psi_Q \rangle = 0
\] (5.7)
or, equivalently,
\[
\langle \cdot \mid H - E \mid \cdot \rangle \mid \psi(R, \Lambda_3) \rangle + \int d^2 \lambda (d\Omega) \sigma u_R^* (\lambda_1, \lambda_2) \langle 0 \mid H \mid \Psi_Q(x, y) \rangle = 0
\] (5.8)
and we begin by evaluating the first term of this equation.

Generically, \( H \) is dominated by terms of \( O(R) \). However, for the “diagonal matrix element” \( PHP \), these leading terms cancel, as anticipated in the discussion of Section 2.3, and we are left with the terms through \( O(R^{-2}) \):
\[
\langle \cdot \mid H - E \mid \cdot \rangle = -\frac{1}{2} \frac{d^2}{dR^2} - \frac{5}{R} \frac{d}{dR} + \frac{12}{R^2} - E + \ldots
\] (5.9)
which will act on the reduced state vector \( \mid \psi(R, \Lambda_3) \rangle \). Here, the dots indicate higher order terms in \( \frac{1}{R} \). If we were to stop here, the asymptotic effective Hamiltonian would be
\[
H_{eff}^{(1)} = \frac{1}{2} \frac{d^2}{dR^2} - \frac{5}{R} \frac{d}{dR} + \frac{12}{R^2}
\] (5.10)
which defines the full first-order Born-Oppenheimer approximation, now including the fermions. Comparing with the earlier bosonic result (2.25) for $H_{\text{eff}}$, we see that the fermionic contributions have cancelled the term linear in $R$ and added the term $+\frac{1}{R^2}$, which comes from the $(F^2+G^2)$ terms in (4.20c). But we cannot stop here because there are other terms of order $\frac{1}{R^2}$ to be found from the second term of (5.8) and to evaluate this term we need $|\Psi_Q(x,y)\rangle$.

To solve for $|\Psi_Q(x,y)\rangle$ we turn to the other basic equation (3.6) which we write as follows:

$$Q(H - E)Q |\Psi_Q\rangle + QHP |\Psi_P\rangle = 0$$ (5.11)

or, equivalently,

$$Q(H - E) |\Psi_Q(x,y)\rangle + QH[u_R(\lambda_1, \lambda_2) | 0 \rangle | \psi(R, \Lambda_3)\rangle] = 0.$$ (5.12)

The formal solution of this equation is

$$|\Psi_Q\rangle = -(Q(H - E)Q)^{-1}QHP |\Psi_P\rangle.$$ (5.13)

For the term $QHP$ in (5.11) (an “off-diagonal matrix element”) the leading contribution is of order $R^{-\frac{3}{2}}$ and comes only from the second and third terms of $H_F$ in (4.20b):

$$QHP |\Psi_P\rangle = QH[u_R(\lambda_1, \lambda_2) | 0 \rangle | \psi(R, \Lambda_3)\rangle]$$

$$\simeq -\lambda_1(a^+\Gamma_1\Lambda_3) + i\lambda_2(\Lambda_3\Gamma_2\Gamma_3a^+)\sqrt{2}u_R(\lambda_1, \lambda_2) | 0 \rangle | \psi(R, \Lambda_3)\rangle.$$ (5.14a)

The projection operator $Q$ does not appear in this last expression since we can write $Q = 1 - P$ and $P$ annihilates (5.14b) because $\langle 0 | a^+ | 0 \rangle = 0$. This state has $N_F = 1$ and so the first term of (5.11) also has $N_F = 1$.

The leading terms in $QHQ$ (the “energy denominator”) will have the generic $O(R)$ behavior of $H$. The $O(R^{-\frac{3}{2}})$ estimate holds for $PHQ$ in equation (5.7), the same as for $QHP$. Then we can see that the formal expression

$$(PHQ)(Q(H - E)Q)^{-1}(QHP) \sim O(R^{-\frac{3}{2}}(R - E + O(R^{-\frac{3}{2}}))^{-1}R^{-\frac{3}{2}})$$ (5.15)

(substitute (5.13) into (5.7)) will contribute a term of order $\frac{1}{R^2}$ to the (second term of the) reduced wave equation (5.7) and we must determine its numerical
coefficient. For this purpose we need compute only the terms of $O(R)$ in $QHQ$.

From the details in Appendix D we find that four terms in $H$ contribute to $QHQ$ at order $R$ and these include differential operators as well as multiplicative operators in the bosonic variables. This makes the explicit inversion of the operator $Q(H - E)Q$ a difficult problem, so we shall go back to equation (5.11) and solve it, at large $R$, as an inhomogeneous differential equation for $|\Psi_Q\rangle$. This procedure is closely related to an early technique [20, 21] in atomic physics.

To solve equation (5.11) make the asymptotic ansatz,

$$|\Psi_Q(x, y)\rangle = \frac{-f_1(\lambda_1, \lambda_2)(a^+\Gamma_1\Lambda_3) + if_2(\lambda_1, \lambda_2)(\Lambda_3\Gamma_2\Gamma_3a^+)}{\sqrt{2}} \times u_R(\lambda_1, \lambda_2) |0\rangle |\psi(R, \Lambda_3)\rangle \quad (5.16)$$

which is modeled after equation (5.14), with the insertion of two unknown functions $f_1$ and $f_2$. The ansatz is again annihilated by $P$ because $\langle 0 | a^+ | 0 \rangle = 0$. Calculating the action of $Q(H - E)Q$ on this $|\Psi_Q\rangle$, we find (see Appendix D) that the asymptotic form of $Q(H - E) |\Psi_Q(x, y)\rangle$ has the same form as (5.14), involving the same fermion bilinears $(a^+\Gamma_1\Lambda_3)$ and $(\Lambda_3\Gamma_2\Gamma_3a^+)$. Setting the total coefficients of these fermion bilinears to zero in (5.11) gives the coupled inhomogeneous differential equations

$$[-\frac{1}{2}(\Delta_1 + \Delta_2) + RD + \frac{3}{\lambda_1^2} + U + R - E]f_1 - Zf_2 = -\lambda_1 \quad (5.17a)$$
$$[-\frac{1}{2}(\Delta_1 + \Delta_2) + RD + \frac{3}{\lambda_2^2} + U + R - E]f_2 - Zf_1 = -\lambda_2 \quad (5.17b)$$

$$\Delta_1 \equiv (\frac{\partial}{\partial \lambda_1})^2 + \left[\frac{6}{\lambda_1} + 2\frac{\lambda_1}{(\lambda_1^2 - \lambda_2^2)}\right]\frac{\partial}{\partial \lambda_1} \quad (5.17c)$$
$$\Delta_2 \equiv (\frac{\partial}{\partial \lambda_2})^2 + \left[\frac{6}{\lambda_2} + 2\frac{\lambda_2}{(\lambda_2^2 - \lambda_1^2)}\right]\frac{\partial}{\partial \lambda_2} \quad (5.17d)$$
$$D \equiv \lambda_1\frac{\partial}{\partial \lambda_1} + \lambda_2\frac{\partial}{\partial \lambda_2} \quad (5.17e)$$
$$U \equiv \frac{\lambda_1^2 + \lambda_2^2}{(\lambda_1^2 - \lambda_2^2)^2} \quad (5.17f)$$
\[ Z \equiv \frac{2\lambda_1 \lambda_2}{(\lambda_1^2 - \lambda_2^2)^2} \]  
\( (5.17g) \)

for the unknown functions \( f_1 \) and \( f_2 \). As planned, we have kept only terms of \( O(R) \) multiplying \( f_1 \) and \( f_2 \) on the left of these equations, and the inhomogeneous terms on the right come from the \( \lambda_1 \) and \( \lambda_2 \) factors in (5.14b).

In fact, we have found a simple exact particular solution of these equations:

\[ f_1 = -\frac{\lambda_1}{(2R - E)}, \quad f_2 = -\frac{\lambda_2}{(2R - E)}. \]

\( (5.18) \)

(Barring such luck, we would have carried out numerical computations, using, for example, the variational principle mentioned earlier.)

The general solution to (5.17) can also include any solution to the homogeneous version of the equations, in addition to this particular solution. Because of the non-vanishing linear terms in \( R \) in these equations (which represent \( Q(H - E)Q \)), any solutions of the homogeneous equations will decay exponentially at large \( R \), as in equation (2.27), and can thus be consistently ignored compared to the asymptotic power-law behavior expected for the reduced state vector \( \left| \psi(R, \Lambda_3) \right> \).

Now that we know \( \left| \Psi_Q(x,y) \right> \) in (5.16), we can compute the large \( R \) contribution to the second term of (5.7):

\[ \langle \cdot | HQ | \Psi_Q \rangle = \langle \cdot | \frac{\lambda_1 (a \Gamma_1 \Lambda_3) + i \lambda_2 (\Lambda_3 \Gamma_2 \Gamma_3 a)}{\sqrt{2}} | \Psi_Q(x,y) \rangle \]

\( (5.19a) \)

\[ = -\frac{1}{(2(2R - E))} \langle \cdot | (\Lambda_3 (\lambda_1^2 + \lambda_2^2 + 2\lambda_1 \lambda_2 \Theta) \Lambda_3) | \cdot \rangle \]

\[ \times | \psi(R, \Lambda_3) \rangle \]

\( (5.19b) \)

\[ = -\frac{1}{(2(2R - E))} \int d^2 \lambda (d\Omega) | u_R(\lambda_1, \lambda_2) |^2 \]

\[ \times (\Lambda_3 (\lambda_1^2 + \lambda_2^2 + 2\lambda_1 \lambda_2 \Theta) \Lambda_3) | \psi(R, \Lambda_3) \rangle. \]

\( (5.19c) \)

The gauge-invariant matrix \( \Theta \) in (5.19) is defined as

\[ \Theta \equiv -i \Gamma_1 \Gamma_2 \Gamma_3 \]

\( (5.20) \)

and when we take the average over angles,

\[ \int (d\Omega) \Theta = 0 \]

\( (5.21) \)
because $\Theta$ is odd under reflection of all the $\phi$ variables (see (E.18)). From Appendix F we find that the average value of $(\lambda_1^2 + \lambda_2^2)$ is $\frac{8}{\pi}$; and from the anticommutation relations we have $(\Lambda_3 \Lambda_3) = 8$, so that our result is independent of any representation we might choose for the $\Lambda_3$ variables. Then the result of this “second order” calculation,

$$\langle \cdot | HQ | \Psi_Q \rangle = -\frac{16}{R^2} | \psi(R, \Lambda_3) \rangle \quad (5.22)$$

is exact through order $\frac{1}{R^2}$.

6 The First Set of Candidate Ground States

Adding the result in eq. (5.22) to the “first order” terms in eq. (5.10), we find the asymptotic form of eq. (5.7), exact through order $\frac{1}{R^2}$:

$$H_{\text{eff}} | \psi(R, \Lambda_3) \rangle = E | \psi(R, \Lambda_3) \rangle \quad (6.1a)$$

$$H_{\text{eff}} = -\frac{1}{2} \frac{d^2}{dR^2} - \frac{5}{R} \frac{d}{dR} - \frac{4}{R^2}. \quad (6.1b)$$

This reduced Schrödinger equation has two solutions at $E = 0$: $R^{-1}$ or $R^{-8}$ times any state $| \Lambda_3 \rangle$ formed with the gauge-invariant fermions $\Lambda_3$. The second solution

$$| \psi(R, \Lambda_3) \rangle \simeq R^{-8} | \Lambda_3 \rangle \quad (6.2)$$

allows the normalization integral (5.5) to converge at large $R$. (In the language of eq. (2.28), we have found that $B'' = 8 < \frac{\pi}{4}$. The result (6.2) is our first asymptotic set of ground state candidates, which must be tested further for global stability at non-asymptotic values of $R$.

These solutions also confirm [5] a continuous spectrum for $E > 0$. With $E = \frac{k^2}{2}$, the effective Hamiltonian (6.1) yields plane-wave normalizable solutions which behave as

$$| \psi(R, \Lambda_3) \rangle_{\pm} \simeq R^{-5} e^{\pm ikR} | \Lambda_3 \rangle \quad (6.3)$$

at large $R$.

We can also follow the computation backward to reconstruct the asymptotic form of the candidate ground states $| \Psi \rangle$ near the flat directions of the
potential \( V \). Using eqs. (5.3), (5.16) and (6.2), we find the asymptotic forms

\[
| \Psi_P \rangle = R^{-8} u_R(\lambda_1, \lambda_2) | 0 \rangle | \Lambda_3 \rangle
\]

(6.4a)

\[
| \Psi_Q \rangle = \frac{1}{(2\sqrt{2})} | \lambda_1(a^+ \Gamma_1 \Lambda_3) - i\lambda_2(\Lambda_3 \Gamma_2 \Gamma_3 a^+) \rangle | \Psi_P \rangle
\]

(6.4b)

where \( | 0 \rangle \) is the ground state of the gauge-invariant fermions \( \Lambda_1, \Lambda_2 \) and \( u_R \) is given in eq. (2.18).

Adding these results, we obtain the full asymptotic form of the candidate ground states, up to an overall normalization constant,

\[
| \Psi \rangle \simeq \{1 + \frac{R^{-\frac{3}{2}}}{(2\sqrt{2})} [z_1(a^+ \Gamma_1 \Lambda_3) - iz_2(\Lambda_3 \Gamma_2 \Gamma_3 a^+)]\}
\]

\[
\times R^{-4} \exp\left(-\frac{(z_1^2 + z_2^2)}{2}\right) | 0 \rangle | \Lambda_3 \rangle
\]

(6.5a)

\[
z_1 = \lambda_1 R^{\frac{1}{2}}, \quad z_2 = \lambda_2 R^{\frac{1}{2}}
\]

(6.5b)

where the scaled variables \( z_1 \) and \( z_2 \) are those defined earlier in eq. (2.22).

In this form of the candidate ground states, the variables \( z_{1,2} \) are finite and only \( R \) is large.

This result can also be written through this order in \( \frac{1}{R} \) as

\[
| \Psi \rangle \simeq \exp\left(-\frac{S}{\hbar}\right) | 0 \rangle | \Lambda_3 \rangle
\]

(6.6a)

\[
S = \frac{V}{gr} + \left\{ \frac{H_F}{2gr} + 4\hbar \ln r \right\}
\]

(6.6b)

\[
r = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{\frac{1}{2}} = (\phi_a^m \phi_a^m)^{\frac{1}{2}} = R + O(R^{-2})
\]

(6.6c)

\[
\lambda_{1,2} = O((\frac{\hbar}{g})^{\frac{1}{2}} R^{-\frac{1}{2}}) = O((\frac{\hbar}{g})^{\frac{1}{2}} r^{-\frac{1}{2}})
\]

(6.6d)

where \( V \) is the bosonic potential, \( H_F \) is the Yukawa term and we have reinstated \( \hbar \) and \( g \) following the rule (4.2). The range of validity in (6.6d) (finite \( z_{1,2} \) at large \( r \)) defines the quantum neighborhood of the classical flat direction \( \lambda_1 = \lambda_2 = 0 \). This form of the result shows that these candidate ground states are quantum extensions of Itoyama’s solution of the zero-energy Hamilton-Jacobi equation, now made normalizable by the quantum correction \( 4\hbar \ln r \) (this corresponds to \( \alpha = 4 > \frac{3}{2} \) in the discussion of Section 2.2).
This set of candidate ground states does not include a singlet under spin(9). To see this explicitly, we note that the bosonic prefactor \( \exp(-\frac{\xi}{\hbar}) \) in (6.6) is rotation invariant while the rotation generators (4.25) give

\[
J^{mn} |0\rangle |\Lambda_3\rangle = |0\rangle (\frac{1}{2} (\Lambda_3 \Sigma^{mn} \Lambda_3)) |\Lambda_3\rangle
\]

or

\[
\frac{1}{2} J^{mn} J^{mn} |0\rangle |\Lambda_3\rangle = 18 |0\rangle |\Lambda_3\rangle
\]

on the fermion states. The evaluation of the Casimir operator in (6.7b) follows from Fierz transformations and properties of the \( \Gamma \) matrices. This shows that all three irreducible representations of spin(9) in \(|\Lambda_3\rangle\) (and hence in the candidate ground states)

\[
|\Lambda_3\rangle = |256\rangle = |44\rangle \oplus |84\rangle \oplus |128\rangle
\]

have the same value of the Casimir. These irreps correspond respectively to the spin(9) irreps of the 11-dimensional supergraviton:

1. a symmetric, traceless second rank tensor \( (g_{mn}) \)
2. a totally antisymmetric third rank tensor \( (H_{mnp}) \)
3. a “gravitino” or Rarita-Schwinger irrep \( (B_\alpha^m) \)

where the first two irreps are bosonic and the last is fermionic.

We also remark that the set (6.6) of 256 candidate ground states forms a “zero-index unit” whose presence cannot violate the index theorem [13] for \( SU(2) \) matrix theory. In this connection, it is clear that there are strong parallels between our generalized Born-Oppenheimer formulation and the computational method of Ref. [13]. It is difficult to make a quantitative comparison, however, because we are computing different quantities.

### 7 A More General Set of Candidates

Having obtained our first set (6.6) of ground state candidates, we are now in a position to obtain a more general set of candidates.

One crucial observation is that the angular integration \( (d\Omega) \) played a very limited role in the computation of Section 5: Because our projector state \( |\cdot\rangle \) was independent of the angular variables \( \Omega \), we needed only \( \int (d\Omega) = 1 \) in
every stage except for eq. (5.19), where \( (d\Omega)\Theta = 0 \) eliminated the term proportional to the operator \( \Theta \). This opens the possibility of broadening our perspective by partitioning the variables \( \Omega \) into fast and slow variables, while maintaining the requirement that no linear terms in \( R \) should appear in the effective Hamiltonian. In what follows, we ignore the 3 gauge degrees of freedom in \( \Omega \), keeping only the 21 gauge-invariant angular variables \( \eta \) which we give again here for reference,

\[
\eta^m_i = \frac{\phi^m_a \psi^i_a}{\lambda_i}, \quad \eta^m_i \eta^m_j = \delta_{ij}. \tag{7.1}
\]

Recall that these 21 variables plus the 3 \( \lambda \)'s are a complete set of \( 27 - 3 = 24 \) gauge-invariant bosonic variables for \( SU(2) \).

More precisely, we begin our second computation by choosing the partition

\[
\text{fast (x) variables: } \lambda_1, \lambda_2, \Lambda_1, \Lambda_2, \eta_1, \eta_2 \tag{7.2a}
\]

\[
\text{slow (y) variables: } \lambda_3, \Lambda_3, \eta_3 \tag{7.2b}
\]

because, as demonstrated below, this will allow us to avoid \( R \) terms in the effective Hamiltonian for the reduced state vector \( |\psi(R, \Lambda_3, \eta_3)\rangle \). Moreover, we choose the same \( \eta_1, \eta_2 \)-independent projector state

\[
| \cdot \rangle = u_R(\lambda_1, \lambda_2) | 0 \rangle \tag{7.3}
\]

used in the first computation, but now we must specify the decomposition of the \( \eta \)-measure in order to integrate out the fast variables \( \eta_1 \) and \( \eta_2 \). The full gauge-invariant measure can be written

\[
(d\phi) = d^3 \lambda \rho(\lambda)(d^3 \eta) \tag{7.4a}
\]

\[
(d^2 \eta) = (d^2 \eta)(d\eta_3) \tag{7.4b}
\]

\[
(d^2 \eta) = \left( \prod_{i=1,2} \left( \prod_{m=1}^9 d\eta^m_i \right) \delta(\eta^n_i \eta^n_i - 1) \right) \delta(\eta^n_3 \eta^n_3 - 1) \tag{7.4c}
\]

\[
(d\eta_3) = \left( \prod_{m=1}^9 d\eta^m_3 \right) \delta(\eta^n_3 \eta^n_3 - 1) \tag{7.4d}
\]

and, in what follows, we will need only the following two properties of \( (d^2 \eta) \),

\[
\int (d^2 \eta) = 1 \tag{7.5a}
\]

\[
\int (d^2 \eta) \Theta = 0. \tag{7.5b}
\]
It is straightforward to see that \( f(d^2 \eta) \) is independent of \( \eta_3 \) and (7.5a) is a convenient convention. The property in (7.5b) follows because the matrix \( \Theta \)

\[
\Theta = -i \Gamma_1 \Gamma_2 \Gamma_3 = -i \Gamma^m \Gamma^n \eta_1^m \eta_2^n \eta_3^p
\]

is odd in each of the three \( \eta \)'s while \( (d^2 \eta) \) is even in \( \eta_1 \) and/or \( \eta_2 \).

Relative to our first computation, we now have the replacement

\[
(d\Omega) \rightarrow (d^2 \eta)
\]

in all averages over fast variables. This includes, for example, the new form of eq. (5.6)

\[
\langle \cdot \mid A \mid \cdot \rangle = \int d^2 \lambda (d^2 \eta) \sigma u_R^* (\lambda_1, \lambda_2) \langle 0 \mid A \mid 0 \rangle u_R (\lambda_1, \lambda_2).
\]

Correspondingly, the integration over \( \eta_3 \) appears only in the new normalization condition

\[
\int dRR^{10} (d\eta_3) \langle \psi (R, \Lambda_3, \eta_3) \mid \psi (R, \Lambda_3, \eta_3) \rangle < \infty
\]

for the reduced state vector.

The second computation (see Appendix D) then proceeds exactly as did the first computation, using the same ansatz (5.16) for \( | \Psi_Q \rangle \) now with

\[
| \psi (R, \Lambda_3) \rangle \rightarrow | \psi (R, \Lambda_3, \eta_3) \rangle
\]

for the reduced state vector. The same contributions are obtained (now by \( f(d^2 \eta) = 1 \)) from each term, including the elimination of the \( \Theta \) term (now by \( f(d^2 \eta) \Theta = 0 \)) in the new version of eq. (5.19). There is, however, one new contribution to PHP from the action of the Laplacian \( \Delta \) on the angular variables \( \eta_3 \) of the reduced state vector \( | \psi (R, \Lambda_3, \eta_3) \rangle \). We sketch here only the asymptotic results that we need for this computation, referring the reader to Appendix G for the full structure of the Laplacian on general gauge-invariant functions \( f(\lambda, \eta) \).

To study the new contribution of the Laplacian, we begin with the identity

\[
(\partial^m_a \eta_i^a)(\partial^m_a \lambda_j) = 0
\]

which is, in fact, equivalent to eq. (E.14). It follows that the Laplacian is separable in the form

\[
\Delta = \Delta_\lambda + \Delta_\eta
\]
where $\Delta_{\lambda}$, which contains the $\lambda$ derivatives, is defined in (2.4b) and $\Delta_\eta$ contains only derivatives with respect to the $\eta$ variables. With the chain rule and the asymptotic identity
\[
\partial_m \eta_n^3 = \frac{1}{R} (\delta^{mn} - \eta_3^m \eta_3^n) \psi_3^3 + O(R^{-\frac{5}{2}})
\]  
(7.13)
we can easily compute the extra asymptotic contribution to $PHP$ as
\[
\langle \cdot | -\frac{1}{2} \Delta_\eta | \cdot \rangle | \psi(R, \Lambda_3, \eta_3) \rangle \quad (7.14a)
\]
\[
= \int d^2 \lambda (d^2 \eta) \sigma u_R^*(\lambda_1, \lambda_2) ( -\frac{1}{2} \Delta_\eta ) u_R(\lambda_1, \lambda_2) | \psi(R, \Lambda_3, \eta_3) \rangle \quad (7.14b)
\]
\[
= ( \frac{L_3^2}{2R^2} + O(R^{-3}) ) | \psi(R, \Lambda_3, \eta_3) \rangle. \quad (7.14c)
\]
Derivatives with respect to $\eta_1$ and $\eta_2$ do not contribute in this computation because there is no dependence on these variables in $u_R(\lambda_1, \lambda_2)$ or the reduced state vector $| \psi(R, \Lambda_3, \eta_3) \rangle$. Moreover, we have organized the result into the angular momentum operators of $\eta_3$,
\[
L_{mn}^3 = -i \delta_3 [m \partial_3 n], \quad L_3^2 = \frac{1}{2} L_{mn}^3 L_{mn}^3
\]  
(7.15)
where $\partial_3^m = \partial / \partial \eta_3^m$. In this result, the $\eta$ derivatives may be taken as the naive derivative
\[
\partial_3^m \eta_3^n = \delta^{mn}
\]  
(7.16)
or the constrained derivative
\[
\partial_3^m \eta_3^n = \delta^{mn} - \eta_3^m \eta_3^n
\]  
(7.17)
which respects the constraint $\eta_3^m \eta_3^n = 1$: Both give the same operators $L_{mn}^3$, which generate a bosonic $SO(9)$.

Adding this extra term then, we find the new asymptotic effective Hamiltonian
\[
H_{eff} | \psi(R, \Lambda_3, \eta_3) \rangle = E | \psi(R, \Lambda_3, \eta_3) \rangle \quad (7.18a)
\]
\[
H_{eff} = -\frac{1}{2} \frac{d^2}{dR^2} - \frac{5}{R} \frac{d}{dR} + \frac{L_3^2}{2R^2} - 8
\]  
(7.18b)
which is exact through $O(R^{-2})$. This reduced system becomes a simple radial wave equation when we introduce the spherical harmonics $Y_l(\eta_3)$ of $SO(9)$

$$L_3^2Y_l(\eta_3) = l(l + 7)Y_l(\eta_3), \quad l = 0, 1, 2 \ldots$$

(7.19)

with “magnetic” degeneracy

$$\text{deg}(l) = \frac{(2l + 7)(l + 6)!}{l!7!}. \quad (7.20)$$

Using the spherical harmonics, we can immediately write down the normalizable asymptotic solutions

$$|\psi(R, \Lambda_3, \eta_3)\rangle \asymp R^{-l-8}Y_l(\eta_3)|\Lambda_3\rangle \quad (7.21a)$$

$$|\Lambda_3\rangle = |256\rangle = |44\rangle \oplus |84\rangle \oplus |128\rangle \quad (7.21b)$$

for the reduced state vector.

This set of solutions contains our first solution (6.2) as the special case with $l = 0$, and the set contains exactly one state

$$|\psi(R, \Lambda_3, \eta_3)\rangle_{l=2} \simeq R^{-10}Y_2^{mn}(\eta_3)|44; mn\rangle$$

(7.22)

which is a singlet under spin(9). Here we have used an explicit form of the 44-dimensional

$$Y_2^{mn}(\eta_3) = \eta_3^m \eta_3^n - \frac{1}{9} \delta^{mn} \quad (7.23)$$

to perform the invariant sum over $Y_2$ times the 44-dimensional irrep in $|\Lambda_3\rangle$.

The new effective Hamiltonian (7.18) also exhibits plane-wave normalizable solutions

$$|\psi(R, \Lambda_3, \eta_3)\rangle \pm \simeq \frac{Y_l(\eta_3)e^{\pm ikR}}{R^5} |\Lambda_3\rangle \quad (7.24)$$

and hence a continuous spectrum for $E = \frac{k^2}{2} > 0$. The earlier result (6.3) is included in (7.24) when $l = 0$.

Finally, we may follow the new computation backward to obtain the full asymptotic form of our general set of candidate SUSY ground states. One
obtains the generalization of (6.5),

\[ |\Psi\rangle \simeq \left\{ 1 + \frac{R^\frac{3}{2}}{(2\sqrt{2})} \left[ z_1(a^+\Gamma_1\Lambda_3) - iz_2(\Lambda_3\Gamma_2\Gamma_3a^+) \right] \right\} \times R^{-l-4}Y_l(\eta_3) \exp\left( -\frac{(z_1^2 + z_2^2)}{2} \right) |0\rangle |\Lambda_3\rangle \]  

(7.25a)

\[ z_1 = \lambda_1 R^\frac{3}{2}, \quad z_2 = \lambda_2 R^\frac{3}{2} \]  

(7.25b)

and the generalization of (6.6),

\[ |\Psi\rangle \simeq \exp\left( -\frac{S}{\hbar} \right) Y_l(\eta_3) |0\rangle |\Lambda_3\rangle \]  

(7.26a)

\[ S = \frac{V}{gr} + \frac{H_F}{2gr} + (l + 4)\hbar \ln r \]  

(7.26b)

\[ r = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{\frac{1}{2}} = (\phi^m\phi^m)^{\frac{1}{2}} \]  

(7.26c)

\[ \lambda_{1,2} = O((\frac{\hbar}{g})^{\frac{1}{2}}r^{-\frac{1}{2}}) \]  

(7.26d)

\[ |\Psi\rangle_{l=2} \simeq \exp\left( -\frac{S}{\hbar} \right) Y_2^{mn}(\eta_3) |0\rangle |44; mn\rangle \]  

(7.26e)

\[ J^{mn} |\Psi\rangle_{l=2} = 0 \]  

(7.26f)

where we have recorded in (7.26e) the unique candidate which is a singlet under spin(9). The extreme semiclassical limit of the \( l \neq 0 \) solutions in (7.26) are gauge- but not rotation-invariant solutions of the zero-energy Hamilton-Jacobi equation.

The apparent simplicity of the ground state candidates (7.26a,b) suggests that there may be a more elegant path to this result.

Each of these candidates is a normalizable zero-energy asymptotic solution of the Schrodinger equation of SU(2) matrix theory, but each must be tested further for stability at non-asymptotic values of \( R \). The high-\( l \) solutions are particularly suspect because they are associated to the growing centrifugal barrier \( \frac{L^2}{2R^2} \). Since the singlet state has \( l = 2 \), this leaves the states with \( l \leq 2 \) as the most auspicious candidates.

Appendix G also outlines a strategy for a proof of a conjecture which, if true, would tell us that our projector state in (7.3) is the only state in the Hilbert space of the fast variables (7.2a) that avoids linear terms in \( R \) in the effective Hamiltonian. In this case, our set of candidate ground states would
be a complete list for the partition (7.2) of the variables of SU(2) matrix theory.

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Appendix A: Canonical Transformations

In further detail, the shift in (4.7) is

\[ \pi'_m^a = \pi_m^a + F_m^a \tag{A.1a} \]

\[ F_m^a = \frac{1}{2i} \Lambda_{ba}(S_m^a)_{bc} \Lambda_{ca} = \frac{i}{2} \Lambda'_{i\alpha}(T_m^a)_{ij} \Lambda'_{j\alpha} \tag{A.1b} \]

\[ (S_m^a)_{bc} = \psi_b^i \delta_m^a \psi_i^c, \quad (T_m^a)_{ij} = \psi_i^b \delta_m^a \psi_j^b \tag{A.1c} \]

where \( \Lambda' \) are the gauge-invariant fermions (4.3) and \( \psi_i^a \) are the eigenvectors of \( \Phi \) introduced in (2.2). Further properties of \( T_m^a \) are found in Appendix E. The quantities \( S_m^a \) and \( T_m^a \) are real antisymmetric matrices, which we call connections.

Using the orthonormality and completeness of \( \psi_i^a \) in (2.2), one verifies that \( \phi \) and \( \pi' \) are canonical variables which are independent of \( \Lambda' \):

\[ [\pi'^m_a, \Lambda'_{i\alpha}] = 0 \tag{A.2a} \]

\[ [\phi^m_a, \pi'^m_b] = i \delta_{mn} \delta_{ab} \tag{A.2b} \]

\[ [\pi'^m_a, \pi'^m_b] = 0 \tag{A.2c} \]

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and this tells us that
\[ \pi'^m_a = -i \partial'^m_a = -i \frac{\partial}{\partial \phi'^m_a}, \quad \partial'^m_a \Lambda'_{i\alpha} = 0 \] (A.3)

in coordinate representation. These derivatives could be written more precisely as \( (\partial'^m_a)_{\Lambda'} \) to show that they act on the bosons as usual, but at fixed \( \Lambda' \).

The statement (A.2c) is equivalent to the fact that \( S'^m_a \) and \( T'^m_a \) are flat connections
\[ \partial'^m_a S'^m_b - \partial'^m_b S'^m_a + [S'^m_a, S'^m_b] = 0 \] (A.4a)
\[ \partial'^m_a T'^m_b - \partial'^m_b T'^m_a + [T'^m_a, T'^m_b] = 0 \] (A.4b)
which follows directly from the properties (2.2) of \( \psi'^i_a \). Moreover, we find that both connections are divergence-free
\[ \partial'^m_a S'^m_a = \partial'^m_a T'^m_a = 0. \] (A.5)

To see this for \( T'^m_a \), follow the steps
\[ \partial'^m_a (T'^m_a)_{ij} = \frac{1}{2} \partial'^m_a [\psi'^i_b \partial'^m_a \psi'^j_b - (i \leftrightarrow j)] \] (A.6a)
\[ = \frac{1}{2} [\psi'^i_b \Delta \psi'^j_b - (i \leftrightarrow j)] \] (A.6b)
\[ = 0 \] (A.6c)
where we used (E.5) in the last step, and similarly for \( S'^m_a \). It also follows that
\[ \partial'^m_a F'^m_b - \partial'^m_b F'^m_a + i[F'^m_a, F'^m_b] = 0, \quad \partial'^m_a F'^m_a = 0 \] (A.7)
so the current \( F'^m_a \) in eq. (A.1b) is also a flat divergenceless connection.

The canonical transformation (4.7) or (A.1) can also be understood in terms of a unitary (but not gauge-invariant) transformation \( K(\phi, \Lambda'(\phi)) \)
\[ \pi'^m_a = K^{-1} \pi'^m_a K \] (A.8a)
\[ \Lambda'_{i\alpha}(\phi) = K^{-1} \Lambda'_{i\alpha}(\phi_0) K \] (A.8b)
\[ \Lambda'_{i\alpha}(\phi_0) = \psi'^i_a(\phi_0) \Lambda_{i\alpha} \] (A.8c)
\[ \partial'^m_a K = iKF'^m_a, \quad K(\phi_0) = 1 \] (A.8d)
\[ K = P \exp[i \int_{\phi_0}^{\phi'} d\phi' \cdot F(\phi', \Lambda'(\phi'))] \] (A.8e)
where \( \phi_0 \) is a reference point of \( \phi \) and \( \Lambda_{a \alpha} \) are the original constant but not gauge-invariant fermions. The path-ordered operator \( K \) is well defined because the current \( F^m_a \) is a (divergenceless) flat connection.

The gauge-invariant states \( | \Lambda' \rangle \) satisfy

\[
\pi^m_a | \Lambda'(\phi) \rangle = (-i \partial^m_a + F^m_a) | \Lambda'(\phi) \rangle = 0 \quad (A.9)
\]

and so may be written in terms of the original fermions as

\[
| \{ \Lambda'_{i \alpha}(\phi) \} \rangle = K^{-1} | \{ \Lambda'_{i \alpha}(\phi_0) = \psi^i_a(\phi_0) \Lambda_{a \alpha} \} \rangle \quad (A.10)
\]

although neither factor on the right is separately gauge invariant.

To see the cancellation (4.10) of fermionic terms in the gauge generator \( G_a \), use eq. (E.2) for \( \partial_m \psi^i_b \) to verify the intermediate steps

\[
\epsilon_{abc} \phi^m_a \partial^m_n \psi^i_b = \epsilon_{abd} \psi^i_b \quad (A.11a)
\]

\[
\epsilon_{abc} \phi^m_a (S^m_c)_{df} = \epsilon_{adf} \quad (A.11b)
\]

where \( S^m_c \) is the flat connection in (A.1). The result in (A.11a) says that \( \psi^i_a \) transforms in the adjoint of the gauge group.

Similarly, the form (4.12) of the rotation generators follows with the steps

\[
\phi^{|m|}_a \partial^{|m|}_a \psi^j_b = 0 \quad (A.12a)
\]

\[
\phi^{|m|}_a (T^m_a)_{ij} = \psi^i_b \psi^{|m|}_a \partial^{|m|}_a \psi^j_b = 0 \quad (A.12b)
\]

where (A.12a) says that \( \psi^i_a \) are singlets under \( \text{spin}(9) \).

When we make the substitution (A.1) in the Hamiltonian, we encounter

\[
\frac{1}{2} \pi^m_a \pi^m_a = \frac{1}{2} \pi^m_a \pi^m_a - F^m_a \pi^m_a + \frac{1}{2} F^m_a F^m_a - \frac{1}{2} [\pi^m_a, F^m_a] \quad (A.13)
\]

where the \( F^2 \) terms from the shift are quartic in the gauge-invariant fermions. The last term in (A.13) vanishes, however, because the flat connection \( T^m_a \) has zero divergence.

The explicit form of the shift term in the second canonical transformation (4.16) is

\[
G^m_a = \frac{1}{2i} (\Lambda'_1 (\Gamma_3 \partial^m_a \Gamma_3) \Lambda'_1) = \frac{i}{2} (\Lambda''_1 (\Gamma_3 \partial^m_a \Gamma_3) \Lambda''_1) \quad (A.14)
\]

where \( \Lambda''_1 \) are the final gauge-invariant fermions. Further details of the derivatives of the matrices \( \Gamma_i \) can be found in Appendix E. Here again we find that
\( \pi'' \) and \( \phi \) are canonical variables independent of \( \Lambda'' \). Moreover, as above, the statement \([\pi''_a, \pi''_b] = 0\) is equivalent to the fact that \( \Gamma_3 \partial_3 \Gamma_3 \) is a flat connection, and using eq. (E.13) we find that this connection is also divergence free.

The final form (4.17) of the gauge generators \( G_a \) is obtained because the shift term \( G^m_a \) does not contribute to \( G_a \). To see this, use eq. (E.11) to verify explicitly that \( \Gamma_i \) is gauge invariant

\[
\epsilon_{abc} \phi^m_b \partial^m_c \Gamma_i = 0 \tag{A.15}
\]

and hence that \( \epsilon_{abc} \phi^m_b G^m_c = 0 \).

We found the following identities helpful

\[
\Gamma_3 \phi^{[m}_a \partial^n_a \Gamma_3 = -\Gamma_3 \Gamma^{[m} \eta^{n]}_3 \tag{A.16a}
\]

\[
\Gamma_3 \Sigma^{mn} \Gamma_3 + i \Gamma_3 \Gamma^{[m} \eta^{n]}_3 = \Sigma^{mn} \tag{A.16b}
\]

in obtaining the form (4.18) of the rotation generators. Here \( \eta^m_i \) are the gauge-invariant angular variables defined in (2.3).

**Appendix B: Gauge-Invariant Formulation**

Given the gauge-invariant fermions \( \Lambda' \) of Section 4.1 and the additional gauge-invariant (but not canonical) coordinates and momenta

\[
\begin{align*}
\phi^m_i &\equiv \psi^i_a \varphi^m_a, \\
\pi''_i &\equiv \psi^i_a \pi''_a = -i \hbar \psi^i_a \partial^m_a = -i \hbar D^m_i \\
\left[ \pi''_i, \Lambda'_{j\alpha} \right] &= 0
\end{align*} \tag{B.1a}
\]

\[
\left( \pi''^m_i \right. \text{ is the independent momentum in (4.7)) we can rewrite the supercharges and the Hamiltonian of } SU(2) \text{ matrix theory entirely in terms of gauge-invariant quantities.}
\]

Using the derivative formulas of Appendix E, the results are

\[
\begin{align*}
Q_\alpha &= (\Gamma^m \Lambda'_j)_{\alpha} \pi''_i - i \hbar \sum_{i \neq j} (\lambda_i \Gamma_i \Lambda'_j)_{\alpha} \frac{(\Lambda'_i \Lambda'_j)}{\lambda^2_i - \lambda^2_j} \\
&\quad + \frac{g}{2} \epsilon_{ijk} (\lambda_i \Gamma_i \lambda_j \Gamma_j \Lambda'_k)_{\alpha} \tag{B.2a}
\end{align*}
\]

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\[ H = \frac{1}{2} \pi_i^m \pi_i^m + \frac{g^2}{2} \left( \lambda_i^2 \lambda_j^2 + \lambda_i^2 \lambda_j^3 + \lambda_i^3 \lambda_j^3 \right) \]
\[ + \frac{i}{2} \sum_{i \neq j} \frac{1}{\lambda_i^2 - \lambda_j^2} \left\{ \phi_j^m \pi_j^m + \left( [\Lambda_i', \Lambda_j'] \phi_i^m \pi_j^m \right) \right\} \]
\[ - \frac{\hbar^2}{4} \sum_{i \neq j} (\Lambda_i' \Lambda_j')^2 \frac{\lambda_i^2 + \lambda_j^2}{(\lambda_i^2 - \lambda_j^2)^2} + \frac{i}{2} \epsilon_{ijk} (\Lambda_i' \lambda_k \Lambda_j') \] (B.2b)

where \((\Lambda_i'BA'_j) = \Lambda_{ia}' B_{\alpha\beta} \Lambda_{ja}'\) and \(([\Lambda_i', \Lambda_j']) = \Lambda_{ia}' \Lambda_{j\alpha}' - \Lambda_{ja}' \Lambda_{i\alpha}'\). As expected, \(Q\) and \(H\) are respectively cubic and quartic in the gauge-invariant fermions, and the quartic term in the Hamiltonian is just the \(F^2\) term of the shift. The last term in the Hamiltonian is the Yukawa term \(H_F\), whose diagonalization is discussed in Appendix C.

Using chain rules, the gauge-invariant momenta \(\pi_i^m\) can be evaluated explicitly when operating on general gauge-invariant bosonic functions \(f(\lambda, \eta)\), where the \(\eta\) variables are given in (2.3). The results for \(\pi_i^m\) and the bosonic Laplacian on \(f(\lambda, \eta)\),

\[-\hbar^2 \Delta = \pi_i^m \pi_i^m + i \hbar \sum_{i \neq j} \frac{1}{\lambda_i^2 - \lambda_j^2} \phi_j^m \pi_j^m \] (B.3a)
\[-\hbar^2 (\Delta_{\lambda} + \Delta_{\eta}) \] (B.3b)

are given explicitly in Appendix G. Here \(\Delta_{\lambda}\), which contains the \(\lambda\) derivatives, is given in (2.4b) and \(\Delta_{\eta}\) contains the \(\eta\) derivatives.

We also mention some alternate gauge-invariant forms for the supercharges,

\[ Q_\alpha = (\Gamma^m (\pi_a^m + i \Theta \partial_a^m \sqrt{W}) \Lambda_a)_{\alpha} \] (B.4a)
\[ = (\Gamma^m \Lambda'_a)_{\alpha} \pi_i^m - i \hbar \sum_{i \neq j} (\lambda_i \Gamma_i \Lambda'_j)_{\alpha} \frac{(\Lambda_i' \Lambda_j')}{\lambda_i^2 - \lambda_j^2} \]
\[ + i (\Gamma^m \Theta \Lambda'_a)_{\alpha} D_i^m \sqrt{W} \] (B.4b)

Here, \(W = g^2 \det \Phi\) is the Claudson-Halpern variable and \(\Theta\) is the gauge-invariant matrix

\[ \Theta = -i \Gamma_1 \Gamma_2 \Gamma_3 \] (B.5)

which satisfies \(\Theta^2 = 1\). Still another form is

\[ Q_\alpha = (\Gamma^m \phi_i^m \Lambda'_j)_{\beta} C_{\beta \alpha} - i \hbar \sum_{i \neq j} (\lambda_i \Gamma_i \Lambda'_j)_{\alpha} \frac{(\Lambda_i' \Lambda_j')}{\lambda_i^2 - \lambda_j^2} \] (B.6a)

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\[
C_{\alpha\beta} = [-i\hbar\lambda; \frac{\partial}{\partial \lambda_i}] + i\Sigma^{mn} M^{mn}_i - i\Theta \sqrt{W}]_{\alpha\beta} \quad (B.6b)
\]
\[
M^{mn}_i = \phi_{[m}^{i} \pi_{n]} \quad (B.6c)
\]
where the second term in \(C_i\), which is of “spin-orbit” form, contains all the \(\eta\) derivatives in the supercharge. See Appendix G for further details.

**Appendix C: Diagonalization of the Yukawa Term**

In this Appendix we discuss the exact diagonalization of the Yukawa term \(H_F\) in the Hamiltonian, keeping \(\bar{h} = g = 1\).

We begin with the expression (4.5) for \(H_F\) in terms of the gauge-invariant fermions \(\Lambda'\),
\[
H_F = -\frac{i}{2} \epsilon_{abc} \Lambda_a \Gamma^m \phi_b^m \Lambda_c = \frac{1}{2} \Lambda'_{ia} M_{ia,j\beta} \Lambda'_{j\beta} \quad (C.1a)
\]
\[
i, j = 1, 2, 3, \quad \alpha, \beta = 1 \ldots 16 \quad (C.1b)
\]
\[
M = -i \begin{pmatrix}
0 & \lambda_3 \Gamma_3 & -\lambda_2 \Gamma_2 \\
-\lambda_3 \Gamma_3 & 0 & \lambda_1 \Gamma_1 \\
\lambda_2 \Gamma_2 & -\lambda_1 \Gamma_1 & 0
\end{pmatrix} \quad (C.1c)
\]

where we have noted that \(\epsilon_{abc} \psi_A^b \psi_c^d = \epsilon_{ijk} \psi_A^k\) because the eigenvector \(\psi\) is a group element in the adjoint of \(SU(2)\). The gauge-invariant matrix \(M\) is hermitian and imaginary, which means that its eigenvalues \(\mu\) are real and occur in \(\pm\) pairs: if \(U\) is one of the 48 eigenvectors of \(M\) with eigenvalue \(\mu\), then \(U^*\) is also an eigenvector, with eigenvalue \(-\mu\).

The matrix \(M\) also satisfies
\[
(M^2)_{ij} = (r^2 - 2\lambda^2) \delta_{ij} + \lambda_1 \Gamma_1 \lambda_2 \Gamma_j 
\]
\[
[M(M^2 - r^2)]_{ij} = 2\lambda_1 \lambda_2 \lambda_3 \Theta \delta_{ij} \quad (C.2b)
\]
where we have defined \(r^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2\) and
\[
\Theta = -i \Gamma_1 \Gamma_2 \Gamma_3. \quad (C.3)
\]

The gauge-invariant matrix \(\Theta\) (which occurs throughout this paper) is hermitian and squares to one. Then (C.2b) gives us a sixth-order algebraic equation for the eigenvalues of \(M\),
\[
[\mu(\mu^2 - r^2)]^2 = 4W \quad (C.4)
\]
where \( W = (\lambda_1 \lambda_2 \lambda_3)^2 = \det(\Phi) \) is the Claudson-Halpern variable. The solutions of this algebraic equation are six real numbers, in three \( \pm \) pairs, so that each eigenvalue is 8-fold degenerate.

Furthermore, \( \Theta \) commutes with all of the \( \Gamma_i \), and hence with the matrix \( M \), so we can label the eigenvectors of \( M \) by their \( \Theta \) eigenvalues \( \pm 1 \). From equation (C.4) above, we find that the three eigenvalues \( \mu_k \), \( k = 1, 2, 3 \), corresponding to the \( +1 \) eigenvalue of \( \Theta \) satisfy

\[
\begin{align*}
 r & \leq \mu_3 \leq \frac{2r}{\sqrt{3}}, \\
 -r & \leq \mu_2 \leq -\frac{r}{\sqrt{3}}, \\
 -\frac{r}{\sqrt{3}} & \leq \mu_1 \leq 0
\end{align*}
\]

(C.5a)

\[
\mu_1 + \mu_2 + \mu_3 = 0 \quad \text{(C.5b)}
\]

and the roots \(-\mu_k \) correspond to the \( -1 \) eigenvalue of \( \Theta \).

The origin of the linear relation (C.5b) is as follows: The algebraic equation (C.4) gives the eigenvalues as functions of the gauge-invariant \( \lambda \)’s, but in fact only two combinations out of three occur, so that \( \mu_k = \mu_k(r, W) \). We also note for use below that the positive eigenvalue \( \mu_3 \) in (C.5a) behaves as

\[
\mu_3 = R + \frac{(\lambda_1 + \lambda_2)^2}{2R} + \ldots = R + O(R^{-2})
\]

(C.6)

for large \( R = \lambda_3 \) and \( \lambda_1, \lambda_2 = O(R^{-\frac{1}{2}}) \).

We are now ready to be more explicit about the eigenfunctions of \( M \), which may be labelled as

\[
\begin{align*}
 M_{ia,j\beta} U_{j\beta}^{k\nu} & = \mu_k U_{ia}^{k\nu}, \quad \Theta_{\alpha\beta} U_{ia}^{k\nu} = +U_{ia}^{k\nu} \\
 M_{ia,j\beta} U_{i\beta}^{k\nu*} & = -\mu_k U_{ia}^{k\nu*}, \quad \Theta_{\alpha\beta} U_{ia}^{k\nu*} = -U_{ia}^{k\nu*}
\end{align*}
\]

(C.7a)

\[
\begin{align*}
 k & = 1, 2, 3; \quad \nu = 1 \ldots 8.
\end{align*}
\]

(C.7c)

These eigenvectors \( U \) and \( U^* \) form a complete orthonormal set

\[
\begin{align*}
 U_{ia}^{k\nu} U_{ia}^{k'\nu'} & = \delta_{kk'}\delta_{\nu\nu'}, \quad U_{ia}^{k\nu} U_{ia}^{k'\nu'} = 0 \\
 U_{ia}^{k\nu} U_{ja}^{k\nu} & = \delta_{ij} \left( \frac{1 - \Theta}{2} \right)_{\alpha\beta}, \quad U_{ia}^{k\nu} U_{ja}^{k\nu} = \delta_{ij} \left( \frac{1 + \Theta}{2} \right)_{\alpha\beta}
\end{align*}
\]

(C.8a)

so we can use them to define gauge-invariant creation and annihilation operators

\[
\Lambda_{ia}' = \sum_{k,\nu} (U_{ia}^{k\nu} a_{k\nu} + U_{ia}^{k\nu*} a_{k\nu}^*)
\]

(C.9a)
\[ \{a_{k\nu}, a^+_{k'\nu'}\} = \delta_{kk'} \delta_{\nu\nu'} \quad (C.9b) \]

and we emphasize the pivotal role of the matrix \( \Theta \) in the separation into creation and annihilation terms.

With this expansion, the original Yukawa term is completely diagonalized

\[ H_F = -\sum_{k,\nu} \mu_k a_{k\nu}^+ a_{k\nu} \quad (C.10) \]

and this is the main result of this Appendix. Defining \( | \tilde{0} \rangle \) by \( a_{k\nu} | \tilde{0} \rangle = 0 \) as usual, we find that the state with the lowest fermionic energy \( H_F \implies E^F_0 \) is

\[ (\prod_{\nu=1}^8 a_{3\nu}^+) | \tilde{0} \rangle : E^F_0 = -8\mu_3 \quad (C.11a) \]

\[ E^F_0 = -8R + O(R^{-2}) \quad (C.11b) \]

and we note that the asymptotic form of this energy is the negative of the bosonic energy \( E_0(R) \) in (2.18).

In this case, one can also make a canonical transformation to independent canonical momenta \( \tilde{\pi}_m^a \) which commute with the fermion creation and annihilation operators,

\[ \tilde{\pi}_m^a = \pi_0^m + \frac{1}{2} \Lambda'_m^{ia}(R_a^m)_{ia,\beta}^\prime \Lambda'_j^{\beta} \quad (C.12a) \]

\[ (R_a^m)_{ia,\beta} = i \sum_{k,\nu} (\partial_a^{-m} U_{ik\nu} U^{ik\nu} + \partial_a^{m} U_{ik\nu} U^{ik\nu}) \quad (C.12b) \]

\[ [\tilde{\pi}_m^a, a_{k\nu}] = [\tilde{\pi}_m^a, a_{k\nu}^+] = 0 \quad (C.12c) \]

where \( R_a^m \) is again a flat connection.

We have used this transformation and the decomposition

\[ U_{ia}^{k\nu} = u_i^k(\lambda)(\Gamma_i)_{\alpha\beta} \chi_{\beta}^\nu, \quad U_{ia}^{k\nu'} = u_i^k(\lambda)(\Gamma_i)^{\alpha\beta} \chi_{\beta}^{\nu'} \quad (C.13a) \]

\[ \sum_i u_i^k(\lambda)u_i^{k'}(\lambda) = \delta^{kk'} \quad (C.13b) \]

\[ \Theta \chi_{\nu} = +\chi_{\nu}, \quad \Theta \chi^{\nu} = -\chi^{\nu} \quad (C.13c) \]
to study the rotational properties of the states in this fermionic Hilbert space. The explicit form of the functions $u^k_i(\lambda)$ is easily obtained, but is not needed here. The rotation generators take the form

$$J^{mn} = \tilde{\sigma}^m_{\alpha a} \phi^a_{\alpha} + \frac{i}{2} \sum_{k,\nu,\nu'} [a_{k\nu}, a_{k\nu'}^+ ] \chi^{\nu^*} (D^{mn} + i \Sigma^{mn} ) \chi^{\nu'}$$  \hspace{1cm} (C.14a)

$$D^{mn} = \phi^{[m}_{\alpha a} \phi^{n]}_{\alpha}$$  \hspace{1cm} (C.14b)

in this case, and the following list collects the states which are singlets ($J^{mn} = 0$) under spin(9):

$$| \tilde{0} \rangle e^{-\frac{3}{2} \omega}, \quad A_k | \tilde{0} \rangle e^{-\frac{\pi}{2} \omega},$$

$$A_k A_{k'} | \tilde{0} \rangle e^{\frac{\pi}{2} \omega}, \quad A_1 A_2 A_3 | \tilde{0} \rangle e^{\frac{3}{2} \omega}.$$  \hspace{1cm} (C.15a)

Here we have defined

$$A_k \equiv \prod_{\nu=1}^{8} a_{k\nu}^+, \quad \omega^m_a \equiv \chi^{\nu^*} \partial^m_a \chi^{\nu} = \partial^m_a \omega$$  \hspace{1cm} (C.16)

and the last relation follows because $\omega^m_a$ is a flat connection. The “lowest” state (C.11a) appears in this list, and, owing to (C.11b), this set of states may provide an alternative description of the spin(9) singlet ground state candidate obtained in Section 7.

**Appendix D: Assessment of Terms in the Hamiltonian**

Here we examine individual terms, or groups of terms, in the transformed Hamiltonian (4.19) and note for each:

1. its selection rule with respect to the fermion number operator

$$N_F = \Sigma a^+_\alpha a_\alpha ;$$  \hspace{1cm} (D.1)

2. its order of magnitude in powers of $R$, using $\lambda_3 = R$ and the fact that $\lambda_1$ and $\lambda_2$ are of order $R^{-\frac{3}{2}}$ at large $R$;

3. its contribution to the asymptotic computation, keeping only terms through $O(R^{-2})$ in the effective Hamiltonian.
The details below are given for the “first computation” of the text, and comments are added at the end which discuss the changes needed for the second computation (which allows $\eta_3$ dependence in the reduced state vector).

In this discussion, we will use the shorthand $PHP, QHQ, PHQ$ and $QHP$ for the various terms in the basic equations, where $PHP$ refers to $\langle \cdot | H | \cdot \rangle$ in (5.8), $QHQ$ refers to $QH | \Psi_Q \rangle$ with the ansatz (5.16) for $| \Psi_Q \rangle$, etc. In this language it will be helpful to state in advance the large $R$ systematics

\[
\begin{align*}
H & = O(R) \quad \text{(D.2a)} \\
PHP & = O(R^{-2}) \quad \text{(D.2b)} \\
QHP, PHQ & = O(R^{-\frac{1}{2}}) \quad \text{(D.2c)} \\
QHQ & = O(R) \quad \text{(D.2d)}
\end{align*}
\]

which we will verify below. These orders of magnitude (and the fact that $P | \Psi_Q \rangle = 0$) tell us that

\[
Q(H - E) | \Psi_Q \rangle = (H - E) | \Psi_Q \rangle + O(R^{-\frac{1}{2}}) \quad \text{(D.3)}
\]

and since (as explained in the text) we are only interested in the order $R$ contributions to these terms, the asymptotic results given here for $QHQ$ come entirely from the first term of (D.3).

Finally, it will be useful to note that the shift terms $F^m_a$ in (4.20c) and their squares can be written as

\[
\begin{align*}
F^m_a & = (a^+ \Gamma_3 a)(T^m_a)_{12} + i(\Lambda_1 \Gamma_3 \Lambda_3)(T^m_a)_{13} + i(\Lambda_2 \Lambda_3)(T^m_a)_{23} \quad \text{(D.4a)} \\
F^m_a F^m_a & = (a^+ \Gamma_3 a)^2 U_{12} - (\Lambda_1 \Gamma_3 \Lambda_3)^2 U_{13} - (\Lambda_2 \Lambda_3)^2 U_{23} \quad \text{(D.4b)}
\end{align*}
\]

where $U_{ij}$ is defined in (E.6) and we have used (E.10) to verify that there are no cross terms in (D.4b).

1. $H_B$ and the first term of $H_F$:

\[
H_0 + H_1 + \lambda_3 (N_F - 8) \quad \text{(D.5)}
\]

where $H_B = H_0 + H_1$ is the bosonic Hamiltonian in (4.19). The decomposition of $H_B$ is given in (2.17), now written in terms of independent bosonic derivatives. This group of terms is $O(R)$ and diagonal in $N_F$, but the terms of order $R$ cancel in $HP$ because

\[
(H_0 - 8R)u_R = 0. \quad \text{(D.6)}
\]
The $O(R^{-2})$ contributions of these terms to $PHP$ are the derivative terms $(\frac{d^2}{dR^2}, \frac{d}{dR})$ in (5.10), as in the bosonic computation of Section 2.3. The contribution of these terms to $QHP$ and $PHQ$ are negligible in this computation.

For $QHQ$, it is important to note first that $H_1 = O(R^{-2})$, so these terms can be ignored in the present computation. We find that the remaining terms contribute the $O(R)$ terms which are the first four terms on the left of each of (5.17a,b), plus the $R$ terms and half of the $U$ terms: The term $\frac{1}{2}U$ comes from the operation of $\Delta$ on each $\Gamma_i$ in $|\Psi_Q\rangle$ (using (E.12)), while the $R$ term follows from the $\lambda_3N_F$ term of (D.5) and the fact that $|\Psi_Q\rangle$ has $N_F = 1$. The $RD$ terms come from $\Delta$ acting as one derivative on the $f$’s and one derivative on $u_R$. Other “cross derivatives” vanish by virtue of (E.14).

2. The second and third terms of $H_F$:

$$i(\lambda_2 \Gamma_1 \lambda_3) \lambda_1 + i(\lambda_3 \Gamma_2 \lambda_3) \lambda_2 \quad \text{(D.7)}$$

This operator changes $N_F$ by $+1$ or $-1$ and is of order $R^{-\frac{3}{2}}$. It gives the entire asymptotic contribution to $QHP$ (and to $PHQ$) for our calculation and is written out in equation (5.14).

3. The first term in $-F^m a^m$:

$$i(\lambda_2 \Gamma_1 \lambda_3) \lambda_1$$ \quad \text{(D.8)}

This is diagonal in $N_F$ and of order $R$. However, it is zero when acting on $P$, owing to (E.9). Its only significant contribution is in $QHQ$, where it acts upon the matrices $\Gamma_1$ and $\Gamma_2$, according to (E.15). This term exchanges the fermion bilinears

$$(T_a^m)_{12} \partial_a [(a^+ \Gamma_2 a) \left\{ \begin{array}{c} (a^+ \Gamma_1 \lambda_3) \\ (\lambda_3 \Gamma_2 \lambda_3 a^+) \end{array} \right\} |0\rangle]$$

$$= Z \left\{ \begin{array}{c} (\lambda_2 \Gamma_2 \lambda_3 a^+) \\ -(a^+ \Gamma_1 \lambda_3) \end{array} \right\} |0\rangle \quad \text{(D.9)}$$

to leading order in $R$ and produces the “mixing” terms in eqs. (5.17) proportional to $Z$.

4. The second and third terms in $-F^m a^m$:

These terms (see (D.4a)) raise or lower $N_F$ by one and are zero acting on
5. The term $-G^m_{a}G^m_{a}$:
This gives zero in PHP by (E.14) and is too small to contribute elsewhere.

6. The first term in $\frac{1}{2}F^m_{a}F^m_{a}$:
This term (see (D.4b)), is diagonal in $N_F$ but zero when acting on $P$. A useful fact here is

$$ (a^+\Gamma_3 a)^2 a^+_a | 0 \rangle = a^+_a | 0 \rangle $$
\hspace{1cm} (D.10)

and the asymptotic contribution $\frac{1}{2}U_{12} | \Psi_Q \rangle$ is obtained for this term in $QH | \Psi_Q \rangle$. This gives the remaining half of the $U$ terms in (5.17).

7. The second and third terms in $\frac{1}{2}F^m_{a}F^m_{a}$:
These contribute to PHP as

$$ \langle 0 | \frac{1}{2}F^m_{a}F^m_{a} | 0 \rangle = 2(U_{13} + U_{23}) = \frac{4}{R^2} + \ldots $$
\hspace{1cm} (D.11)

and hence make a contribution of $\frac{4}{R^2}$ to (5.9).

8. The term $\frac{1}{2}G^m_{a}G^m_{a}$:
This contributes to PHP as follows. Using (E.16) and (E.12) we compute

$$ \langle 0 | \frac{1}{2}G^m_{a}G^m_{a} | 0 \rangle = \frac{1}{16} \text{Trace}[(\partial^m a^m)\Gamma_3(\partial^m a^m)\Gamma_3] = -\frac{1}{16} \text{Trace}[(\Gamma_3\Delta\Gamma_3)] $$
\hspace{1cm}
$$ = \frac{6}{\Lambda_3^2} + U_{13} + U_{23} = \frac{8}{R^2} + \ldots $$
\hspace{1cm} (D.12)

and hence this group of terms contributes $\frac{4}{R^2}$ to equation (5.9).

9. The term $F^m_{a}G^m_{a}$.
Using (E.15), this term is negligible in this calculation.

For the second computation, we must allow for the fact that the reduced state vector $| \psi(R, \Lambda_3, \eta_3) \rangle$ is a function also of the gauge-invariant angular variable $\eta_3$. This means that we must reexamine those terms above which involve derivatives with respect to $\eta_3$, namely $\Delta$ and the shift terms $F_\pi$ and $G_\pi$. The result for $\Delta$ is discussed in Section 7, and, because derivatives of $\eta_3$
are at least one power of $R^{-1}$ smaller than the terms we have kept, we find no new contributions from the shift terms.

**Appendix E: Derivatives**

We list here a number of useful formulas for the differentiation of the bosonic variables introduced in the text. The notation is

$$
\partial_a^m = \frac{\partial}{\partial \phi_a^m}, \quad \Delta = \partial_a^m \partial_a^m
$$

and we adopt here the generalization

$$
m, n = 1 \ldots d; \quad a, b, c = 1 \ldots g; \quad i, j, k = 1 \ldots g \quad (g \leq d)
$$

although only $d = 9$ and $g = 3$ apply for $SU(2)$ matrix theory.

Using the familiar method of matrix-perturbation theory, one derives the following two basic formulas for differentiation of $\lambda_i$ and $\psi_a^i$, defined in (2.2),

$$
\partial_a^m \lambda_i = \psi_a^i \psi_b^i \phi_b^m \frac{\partial}{\partial \lambda_i} \quad (E.1)
$$

$$
\partial_a^m \psi_b^i = \sum_{j \neq i} \psi_b^j \phi_c^m \psi_c^i + \psi_c^i \psi_a^j \psi_b^i \frac{\partial}{\partial \lambda_i} \quad (E.2)
$$

All that follows is derived by repeated application of these relations and the prior definitions.

When $f(\lambda)$ is any function of the $\lambda_i$, we have

$$
\Delta f(\lambda) = \sum_i \left\{ \frac{\partial^2}{\partial \lambda_i^2} \right\} f(\lambda) + \frac{(d - g)}{\lambda_i} \sum_{j \neq i} \left\{ \frac{2\lambda_i}{(\lambda_i^2 - \lambda_j^2)} \right\} \frac{\partial}{\partial \lambda_i} f(\lambda) \quad (E.3)
$$

$$
(\partial_a^m \psi_b^i)(\partial_a^m f(\lambda)) = 0 \quad (E.4)
$$

$$
\Delta \psi_a^i = \psi_a^i \left\{ - \sum_{j \neq i} U_{ij} \right\} \quad (E.5)
$$

$$
U_{ij} \equiv \frac{(\lambda_i^2 + \lambda_j^2)}{(\lambda_i^2 - \lambda_j^2)^2} \quad (E.6)
$$
The flat matrix connections $T$ were introduced in (4.7) and Appendix A:
\[
(T^m_a)_{ij} = (\psi^b_i \delta^m_a \psi^j_b) = (1 - \delta_{ij}) \phi^m_a \frac{\psi^b_i \psi^j_b}{(\lambda^2_j - \lambda^2_i)} \quad \text{(E.7)}
\]
\[
\partial^m_a (T^m_a)_{ij} = 0 \quad \text{(E.8)}
\]
\[
(T^m_a)_{ij} (\partial^m_a f(\lambda)) = 0 \quad \text{(E.9)}
\]
\[
(T^m_a)_{ij} (T^m_a)_{kl} = (1 - \delta_{ij}) (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) U_{ij}. \quad \text{(E.10)}
\]
The gauge-invariant matrices $\Gamma_i$ are defined in (4.6). They are real, symmetric, traceless, anti-commuting and satisfy $(\Gamma_i)^2 = 1$:
\[
\partial^m_a \Gamma_i = \frac{\Gamma^n}{\lambda^i} \left\{ \delta_{nm} \psi^j_a - \phi^m_a \phi^m_c \psi^j_b \psi^j_c \frac{\psi^j_a}{\lambda^2_i} + \phi^m_a \phi^m_c \sum_{j \neq i} \psi^j_b \left( \frac{\psi^j_c \psi^j_d + \psi^j_i \psi^j_d}{\lambda^2_j - \lambda^2_i} \right) \right\} \quad \text{(E.11)}
\]
\[
\Delta \Gamma_i = \Gamma_i \left\{ - \frac{(d - g)}{\lambda^i} - \sum_{j \neq i} U_{ij} \right\} \quad \text{(E.12)}
\]
\[
\partial^m_a (\Gamma_i \partial^m_a \Gamma_i) = 0 \quad \text{(no sum on $i$)} \quad \text{(E.13)}
\]
\[
(\partial^m_a \Gamma_i)(\partial^m_a f(\lambda)) = 0 \quad \text{(E.14)}
\]
\[
(T^m_a)_{ij} (\partial^m_a \Gamma_k) = (1 - \delta_{ij}) \frac{2 \lambda^k}{(\lambda^2_i - \lambda^2_j)^2} \left\{ \delta_{kj} \lambda^i \Gamma_i - \delta_{ki} \lambda^j \Gamma_j \right\} \quad \text{(E.15)}
\]
\[
(\partial^m_a \Gamma_i)_{\alpha\beta}(\partial^m_a \Gamma_j)_{\gamma\delta} = \delta_{ij} \left\{ \frac{[(\Gamma^m)^{\alpha\beta}(\Gamma^m)_{\gamma\delta} - \sum_k (\Gamma_k)^{\alpha\beta}(\Gamma_k)_{\gamma\delta}]}{\lambda^2_i} + \sum_{k \neq i} U_{ik} (\Gamma_k)^{\alpha\beta}(\Gamma_k)_{\gamma\delta} \right\} - (1 - \delta_{ij}) U_{ij} (\Gamma_j)^{\alpha\beta}(\Gamma_i)_{\gamma\delta} \quad \text{(E.16)}
\]
\[
\Delta (\Gamma_i \Gamma_j) = (1 - \delta_{ij}) \Gamma_i \Gamma_j \left\{ - \frac{(d - g)}{\lambda^2_i} - \sum_{k \neq i} U_{ik} - \frac{(d - g)}{\lambda^2_i} \right\} - \sum_{k \neq j} U_{jk} + 2 U_{ij} \quad \text{(E.17)}
\]
In the case of $SU(2)$, the special gauge-invariant matrix
\[
\Theta = -i \Gamma_1 \Gamma_2 \Gamma_3 = \frac{i}{6} \epsilon_{abc} \Gamma^m \Gamma^n \Gamma^p \phi^m_a \phi^p_b \phi^p_c \frac{\phi^m_a \phi^p_b \phi^p_c}{\lambda_1 \lambda_2 \lambda_3} \quad \text{(E.18)}
\]
is imaginary, antisymmetric, traceless, has square equal to the unit matrix, and commutes with the matrices $\Gamma_i$:

\[
\phi_0^m \phi_0^m \Theta = 0 \quad (E.19)
\]
\[
\Delta \Theta = \Theta \{ -(d - 3) \sum_i 1 / \lambda_i \} \quad (E.20)
\]
\[
\partial_a (\Theta \partial_a \Theta) = 0. \quad (E.21)
\]

**Appendix F: Integrals**

When we average over the fast variables $\lambda_1$ and $\lambda_2$ with the Gaussian function \((2.18a)\), the following class of two-dimensional integrals occur:

\[
\int_0^\infty ds \int_0^s dt (s^2 - t^2)(st)^M (s^2 + t^2)^N \exp(-s^2 - t^2) = \frac{(N + M + 1)!}{(M + 1)2^{M+2}}. \quad (F.1)
\]

This formula gives us useful averages for our asymptotic calculation. Using the notation

\[
\langle f(\lambda_1, \lambda_2) \rangle = \int d^2 \lambda \sigma_\infty | u_R |^2 \frac{f(\lambda_1, \lambda_2)}{\int d^2 \lambda \sigma_\infty | u_R |^2} \quad (F.2)
\]

(see eq. (2.21) and set $\hbar = g = 1$) we find, for general values of $d$,

\[
\langle \lambda_1^2 + \lambda_2^2 \rangle = \frac{(d - 1)}{R} \quad (F.3)
\]
\[
\langle (\lambda_1^2 + \lambda_2^2)^2 \rangle = \frac{d(d - 1)}{R^2} \quad (F.4)
\]
\[
\langle \lambda_1 \lambda_2 \rangle = \frac{(d - 2)}{2R} \quad (F.5)
\]
\[
\langle \lambda_1^2 \lambda_2^2 \rangle = \frac{(d - 1)(d - 2)}{4R^2} \quad (F.6)
\]

where $d = 9$ for matrix theory.

**Appendix G: Gauge-Invariant Angular Variables**
Here we will express the bosonic Laplacian in terms of the complete set \((\lambda_i, \eta^m)\) of gauge-invariant variables, regarding the \(\lambda\)’s and \(\eta\)’s respectively as radial and angular variables. The result can be arranged in several ways and the one shown below has particular advantages for our work.

We start, as in Appendix B, with the gauge-invariant bosonic variables

\[
\phi^m_i \equiv \psi_a^i \phi^m_a \tag{G.1}
\]

and the Lie derivatives (which do not act on the gauge-invariant fermions)

\[
i \pi^m_i = D^m_i \equiv \psi_a^i \frac{\partial}{\partial \phi^m_a} \tag{G.2}
\]
so that the Laplacian can be written as

\[
\Delta = D^m_i D^m_i + (\partial^m_i \psi^j_a) D^m_j = D^m_i D^m_i - \sum_{i \neq j} y_{ij} \phi^m_j D^m_j. \tag{G.3}
\]

Here we have made use of (E.2) and

\[
y_{ij} \equiv \frac{1}{(\lambda_i^2 - \lambda_j^2)}. \tag{G.4}
\]

The formula

\[
D^m_i \phi^n_j = \delta_{ij} \delta_{mn} + \sum_{k \neq i} y_{ik} \phi^m_k \phi^n_k - (1 - \delta_{ij}) y_{ij} \phi^m_j \phi^n_i \tag{G.5}
\]
also follows from results in Appendix E and will be used below.

Next, we write the orbital angular momentum operator \(M^{mn}\) in terms of these new derivatives,

\[
M^{mn} \equiv -i [\phi^m_a \partial_a^m - \phi^n_a \partial_a^n] = \sum_i M^{mn}_i \tag{G.6a}
\]

\[
M^{mn}_i \equiv -i [\phi^m_i D^m_i - \phi^n_i D^m_i] \tag{G.6b}
\]
and note that the operators \(M^{mn}_i\) are hermitian in the measure \((d\phi)\), although their algebra is not simple. Now calculate the trace of the square of each \(M_i\):

\[
M_i^2 = -\sum_{m < n} [\phi^m_i D^n_i - \phi^n_i D^m_i]^2 \tag{G.7a}
\]

\[
= -\lambda_i^2 (D^m_i)^2 + (\phi^m_i D^m_i)^2 + [d - 2 + \sum_{k \neq i} y_{ik} \lambda_k^2] \phi^m_i D^m_i. \tag{G.7b}
\]
Combining this result with equation (G.3) for the Laplacian, we find

$$\Delta = \frac{1}{\lambda_i^2} \{ (\phi_i^m D_i^m)^2 - M_{i}^2 + [d-2 + \sum_{j \neq i} y_{ij}(\lambda_i^2 + \lambda_j^2)]\phi_i^m D_i^m \} \quad \text{(G.8)}$$

and then noting that

$$\phi_i^m D_i^m = \lambda_i \frac{\partial}{\partial \lambda_i} \quad \text{(no sum on } i) \quad \text{(G.9)}$$

we can simplify this formula to the nice form

$$\Delta = \Delta_\lambda + \Delta_\eta \quad \text{(G.10a)}$$
$$\Delta_\eta = -\sum_i M_{i}^2 \lambda_i^2 \quad \text{(G.10b)}$$

Here $\Delta_\lambda$, which contains the $\lambda$ derivatives, was given earlier in eq. (E.3) and we will see that $\Delta_\eta$, which is negative semidefinite, contains only derivatives with respect to the angular variables $\eta$

$$\eta_i^m \equiv \frac{\phi_i^m}{\lambda_i} \quad \text{(G.11)}$$

which complement the radial variables $\lambda$.

For any function $f(\lambda, \eta)$, the chain rule gives

$$D_i^m f = \eta_i^m \frac{\partial}{\partial \lambda_i} f + (D_i^m \eta_j^j) \partial_j^m f \quad \text{(G.12a)}$$
$$D_i^m \eta_j^j = \frac{\delta_{ij}}{\lambda_i} [\delta_{mn} - \sum_k \eta_k^m \eta_k^n + \sum_k \eta_{ik} \lambda_i^2 \eta_k^m \eta_k^n]$$
$$- (1 - \delta_{ij}) y_{ij} \lambda_i \eta_j^m \eta_i^n \quad \text{(G.12b)}$$

where we have defined the $\eta$ derivative

$$\partial_i^m \equiv \frac{\partial}{\partial \eta_i^m}, \quad \partial_i^m \eta_j^j = \delta_{mn} \delta_{ij} \quad \text{(G.13)}$$

and (G.12b) is closely related to (E.11).

Using (G.12), we re-express the operators $M_{i}^{mn}$ in terms of the variables $\lambda$ and $\eta$. As expected, all $\frac{\partial}{\partial \lambda_i}$ terms cancel out and we find that
\[ M_{i}^{mn} = L_{i}^{mn} + i \sum_{j \neq i} \eta_{i}^{[m} \eta_{j}^{n]} \{ x_{ij}(\eta_{i} \partial_{j}) + x_{ji}(\eta_{j} \partial_{i}) \} \quad (G.14a) \]

where \( L_{i}^{mn} \) is the naive angular momentum operator for the \( \eta \) variables and

\[ x_{ij} \equiv \frac{\lambda_{i}^{2}}{(\lambda_{i}^{2} - \lambda_{j}^{2})}, \quad (\eta_{i} \partial_{j}) \equiv \eta_{i}^{m} \partial_{j}^{m}. \]  

Using the naive \( \eta \) derivative in \((G.13)\), it is not difficult to check that the operators \( M_{i} \) in \((G.14)\) respect the \( \eta \) constraints

\[ M_{mn}^{i} (\eta_{p}^{i} \eta_{p}^{j}) = 0 \]  

and it follows directly that the operators \( M_{i}^{mn} \) are hermitian in the gauge-invariant measure \( d^{3}\lambda (d^{3}\eta) \) (see \((7.4)\)). Taken with \((G.14)\), the form of the Laplacian in \((G.10)\) is the central result of this appendix.

For the discussion below, we will also need the form of the operators \( M_{i} \)

\[ M_{3}^{mn} = L_{3}^{mn} + i \sum_{j=1,2} \eta_{3}^{[m} \eta_{j}^{n]} (\eta_{3} \partial_{j}) + \ldots 
\]

\[ M_{1}^{mn} = -i \{ \eta_{1}^{[m} \partial_{1}^{n]} - \eta_{1}^{[m} \eta_{2}^{n]} [x_{21}(\eta_{2} \partial_{1}) + x_{12}(\eta_{1} \partial_{2})] \}
+ i \frac{\lambda_{1}^{2}}{R^{2}} \eta_{1}^{[m} \eta_{3}^{n]} (\eta_{3} \partial_{1}) + \ldots \] 

\[ M_{2}^{mn} = -i \{ \eta_{2}^{[m} \partial_{2}^{n]} - \eta_{2}^{[m} \eta_{1}^{n]} [x_{12}(\eta_{1} \partial_{2}) + x_{21}(\eta_{2} \partial_{1})] \}
+ i \frac{\lambda_{2}^{2}}{R^{2}} \eta_{2}^{[m} \eta_{3}^{n]} (\eta_{3} \partial_{2}) + \ldots \]  

in the asymptotic region, \( R = \lambda_{3} \gg \lambda_{1}, \lambda_{2} = O(R^{-\frac{1}{2}}) \). The extra term \((7.14)\) of the second computation in the text

\[ M_{3}^{mn} f(\eta_{3}) = L_{3}^{mn} f(\eta_{3}) + O(R^{-1}) \]  

\[ -\frac{1}{2} \Delta f(\eta_{3}) = (\frac{L_{3}^{2}}{2R^{2}} + O(R^{-3})) f(\eta_{3}) \]  

follows immediately from the asymptotic form of \( M_{3} \) in \((G.17)\), the \( M_{1,2} \) terms failing to contribute at this order.

In what follows, we will use the results above to outline a strategy for proving the following conjecture:
a) the eigenvalues $\epsilon$ of the bosonic operator $H_0$ in (2.17) satisfy
\[ \epsilon \geq (d-1)R = 8R \] (G.19)
and $u_R$ in (2.18) is the only state which realizes the minimum.

b) the eigenvalues of the bosonic operator
\[ H'_0 = H_0 + \frac{M^2_1}{2\lambda^2_1} + \frac{M^2_2}{2\lambda^2_2} \] (G.20)
also satisfy $\epsilon \geq (d-1)R = 8R$ and $u_R$ is the only state which realizes the minimum. Here $M_{1,2}$ are given by their leading (first four) terms in (G.17b,c).

The operator $H_0$ is the dominant part (that is, it contains all terms of order $R$) of the bosonic Hamiltonian in the gauge- and rotation-invariant sector; it contains only the fast derivatives $\frac{\partial}{\partial \lambda_{1,2}}$ with the slow variable $R = \lambda_3$ as a parameter. The operator $H'_0$ is the dominant part (in the same sense) of the full bosonic Hamiltonian $H_B$ at large $R$, including the gauge-invariant angular excitations; it involves only the fast derivatives $\frac{\partial}{\partial \lambda_{1,2}}$ and $\frac{\partial}{\partial \eta_{1,2}}$, with the slow variables $R = \lambda_3$ and $\eta_3$ as parameters.

If true, this conjecture implies that our $\eta_1, \eta_2$-independent projector state $|\cdot\rangle$ in (7.3) is the only state whose associated effective Hamiltonian (including $-8R$ from the fermions) has no linear term in $R$.

There is strong evidence for (a), though we have not tried to prove it: It is straightforward to find a large class of radial eigenfunctions $u_{m,n}(\lambda_1, \lambda_2)$ of $H_0$ (or $H'_0$) with
\[ \epsilon = R(d - 1 + 2(m + n)), \quad m, n = 0, 1, 2 \ldots \] (G.21)
where $u_{0,0} = u_R$. Assuming (a), we can prove (b) as follows. The positive semi-definite operators $M^2_i/\lambda^2_i$, $i = 1, 2$ can only give additional positive semi-definite contributions to $\epsilon$, beyond $(d - 1)R$. So to prove (b), we only need to show that there are no non-constant solutions to the differential equations
\[ M_i^{mn}(\eta_1, \eta_2) = 0; \quad i = 1, 2 \quad \forall mn \] (G.22)
where the $M$’s are given by their leading terms at large $R$. We have explicitly checked that this is true.
References


