On the $M_V(\text{peak})$ Versus Orbital Period Relation for Dwarf Nova Outbursts

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We present computations for the accretion disk limit cycle model in an attempt to explain the empirical relation for dwarf nova outbursts between the peak visual absolute magnitude and orbital period found by Warner. For longer period systems one sees intrinsically brighter outbursts. This is accounted for in the limit cycle model by the scaling with radius of the critical surface density $\Sigma_{\text{max}}$ which triggers the dwarf nova outbursts. During the storage phase of the instability the accretion disk mass must be less than some maximum value, a value which scales with radius and therefore orbital period. When the instability is triggered and the accumulated mass is redistributed into a quasi steady state disk in outburst, the resultant peak optical flux from the disk is a measure of the total mass which was stored in quiescence. We compute light curves for a range in outer disk radius (or equivalently, orbital period), and find that our peak values of $M_V$ are within $< 1$ mag of the observed relation $M_V(\text{peak}) = 5.64 - 0.259P_{\text{orbital}}(h)$ for $2 \lesssim P_{\text{orbital}}(h) \lesssim 8$.

Subject Headings: accretion, accretion disks – instabilities – cataclysmic variables – stars: individual (SS Cygni)
1. INTRODUCTION

Dwarf novae are a subclass of the cataclysmic variables—interacting binary stars with orbital periods of several hours in which a Roche lobe filling K or M dwarf secondary transfers matter at a rate $\dot{M}$ into an accretion disk surrounding a white dwarf (WD) primary. Dwarf novae are characterized observationally by having outbursts of several magnitudes which recur on time scales of days to decades and which can last from $\sim 1$ day to several weeks (Warner 1995). The accretion disk limit cycle model has been developed and refined to account for the outbursts (Meyer & Meyer-Hofmeister 1981; for recent reviews see Cannizzo 1993a and Osaki 1996). In this model, material is stored up at large radii in a relatively inviscid accretion disk as neutral gas, and then dumped onto the central star as ionized gas when a certain critical surface density is achieved somewhere in the disk.

Warner (1987) presented a thorough systematic study of the dwarf novae. He noted several interesting correlations between various attributes associated with the outbursts. Of particular interest is his finding of a relation between the peak absolute magnitude of dwarf nova outbursts and their orbital periods: $M_V(\text{peak}) = 5.64 - 0.259P_{\text{orbital}}(\text{h})$. In deriving this relation, Warner made corrections for distance and inclination for $\sim 20$ systems that were well enough studied to have reliable values for $M_V(\text{peak})$ and orbital period $P_{\text{orbital}}$. The distance determinations were made primarily on the basis of infrared fluxes of the secondary stars, as explained in Warner (1987).

In this work we quantify the theoretical relation between $M_V(\text{peak})$ and $P_{\text{orbital}}$ by running computations of our time dependent code which describes the accretion disk limit cycle model for a range in values of $r_{\text{outer}}$, the outer radius of the accretion disk. We compare our results to Warner’s empirical relation.
2. BACKGROUND

The physical basis for the limit cycle lies within the vertical structure of the accretion disk. In particular, the change in the functional form of the opacity at \( \sim 10^4 \) K coincident with the transition from neutral to ionized gas leads to a hysteresis relation between the effective temperature \( T_{\text{eff}} \) and surface density \( \Sigma \) in vertical structure computations carried out at a given annulus. This hysteresis leads to local maxima \( \Sigma_{\text{max}} \) and minima \( \Sigma_{\text{min}} \) in \( \Sigma \) when the locus of solutions is plotted as \( T_{\text{eff}} \) vs. \( \Sigma \). The resultant series of steady state solutions resembles an "S". Instability analyses of the equilibrium vertical structure solutions and their linearly perturbed states show that the upper and lower branches of the "S" curve, those portions where \( d \log T_{\text{eff}} / d \log \Sigma > 0 \), are viscously and thermally stable and can accommodate physically attainable states during quiescence and outburst. The "middle" part of the "S" where \( d \log T_{\text{eff}} / d \log \Sigma < 0 \) is unstable and physically unattainable. During quiescence material accumulates at large radii in the disk because the viscosity is so low that there is little inward transport of the gas. The disk is far from steady state. When \( \Sigma > \Sigma_{\text{max}} \) somewhere in the disk, a heating instability is triggered which begins an outburst; and when \( \Sigma < \Sigma_{\text{min}} \), a cooling transition front is started which controls the decay of the outburst from maximum light. As the disk goes from quiescence to outburst, the matter in the disk is redistributed from large to small radii, and the local \( \dot{M}(r) \) profile becomes close to steady state, except for a strong outflow at large radii. The local maximum in \( \Sigma \) determined from the vertical structure computations is

\[
\Sigma_{\text{max}} = 11.4 \, \text{g cm}^{-2} \, r_{10}^{1.05} \, m_1^{-0.35} \, \alpha_{\text{cold}}^{-0.86},
\]  

where \( r_{10} \) is the radius in units of \( 10^{10} \) cm, \( m_1 \) is the WD primary mass in solar units \( M_1 / M_\odot \), and \( \alpha_{\text{cold}} \) is the value of the Shakura & Sunyaev (1973) viscosity parameter on the lower (or neutral) stable branch of the \( S- \) curve (Cannizzo & Wheeler 1984). (To avoid confusion, we use the subscript “max” to refer to properties of the accretion disk associated
with $\Sigma_{\text{max}}$, and “peak” to refer to either the peak flux or peak accretion rate during an outburst.) The values for $\Sigma_{\text{max}}$ found by other workers are similar, reflecting slight differences in the equations of state, opacities, or treatments of the boundary conditions. The largest uncertainties may turn out to be due to the treatment of convection in the vertical structure. Within the confines of standard mixing length theory which is used in the vertical structure computations, convection may in fact be less important in dwarf novae accretion disks than was previously thought (Gu & Vishniac 1998). Nevertheless, even if convection is set to zero, one still obtains the hysteresis between $T_{\text{eff}}$ and $\Sigma$ (see Fig. 4 of Ludwig et al 1994).

Cannizzo (1993b, hereafter C93b) commented briefly on Warner’s finding of the empirical relation between the peak flux level in a dwarf nova outburst and the orbital period: $M_V(\text{peak}) = 5.64 - 0.259P_{\text{orbital}}(h)$. C93b used the following argument to derive a scaling law for the rate of accretion onto the WD during the peak of a dwarf nova outburst. One can express the mass of the disk at the end of the quiescence interval during which time material accumulates in the disk as $fM_{\text{max}}$, where $M_{\text{max}} = \int 2\pi r dr \Sigma_{\text{max}}$ is the “maximum mass” that the disk could have reached in quiescence if the disk were filled up to the level $\Sigma_{\text{max}}$ at every radius. Once the outburst has begun and progressed for a short time, the surface density profile adjusts from one which was non-steady state in quiescence to one which is in quasi-steady state in outburst. This basically involves a sloshing around of the gas from large to small radii, so that $\Sigma_{\text{quiescence}} \propto r$ and $\Sigma_{\text{outburst}} \propto r^{-1}$. If this time interval is short, one may equate the mass of the disk at the end of the quiescence interval to the mass of the disk at the beginning of the outburst. Therefore if one integrates a Shakura-Sunyaev scaling for $\Sigma_{\text{outburst}} = \Sigma(r, \alpha, \dot{M})$ over the disk as was done for the quiescent state by taking $M_{\text{outburst}} = \int 2\pi r dr \Sigma_{\text{outburst}}$, sets this equal to $fM_{\text{max}}$, and then inverts this expression to obtain $\dot{M}$, one derives an approximation for the peak $\dot{M}$ value in
the disk during outburst

\[ \dot{M}_{\text{peak}} = 1.1 \times 10^{-8} \, M_\odot \, \text{yr}^{-1} \left( \frac{\alpha_{\text{hot}}}{0.1} \right)^{1.14} \left( \frac{\alpha_{\text{cold}}}{0.02} \right)^{-1.23} \left( \frac{r_{\text{outer}}}{4 \times 10^{10} \, \text{cm}} \right)^{2.57} \left( \frac{f}{0.4} \right)^{1.43}, \quad (2) \]

(C93b), where \( \alpha_{\text{hot}} \) is the alpha value on the upper (or ionized) stable branch, and \( r_{\text{outer}} \) is the outer radius of the accretion disk (which is set by the orbital period). The values of the parameters entering into eqn. (2) have been scaled to the values which C93b found to be relevant for SS Cygni, a dwarf nova with \( P_{\text{orbital}} = 6.6 \, \text{h} \). From Kepler’s law \( P_{\text{orbital}}^2 \propto a^3 \), where \( a \) is the orbital separation. So if \( r_{\text{outer}} \propto a \) we basically have \( \dot{M}_{\text{peak}} \) scaling as \( P_{\text{orbital}}^{1.7} \), assuming the the other parameters in eqn. (2) do not vary with orbital period. If the visual flux were to scale linearly with \( \dot{M} \) in the disk, then this would imply \( M_V(\text{peak}) \propto -0.68 \log P_{\text{orbital}} \), a different functionality than that observed. One additional consideration which comes into play for dwarf novae at increasingly longer orbital periods is that \( r_{\text{outer}} \) varies nonlinearly with \( a \) in the regime where the secondary star transitions from being less massive than the primary star to being of comparable mass.

Warner (1995, see his Fig. 3.10) over-plotted eqn. (2) with the data taken from Warner (1987) to show that the analytical expression does a reasonable job of characterizing the observations. The analytical expression for \( \dot{M}_{\text{peak}} \) derived above is only useful in comparing with observations, however, if the variables appearing in eqn. (2) do not vary appreciably with radius. Of particular concern is the variable \( f \) which gives \( M_{\text{outburst}}/M_{\text{max}} \) at the time of burst onset. One might expect for \( f \) to vary significantly with orbital period \( P_{\text{orbital}} \) or secondary mass transfer rate \( \dot{M}_T \), in which case the dependence on \( r_{\text{outer}} \) which appears in eqn. (2) would be misleading. Our aim in this work is to understand the function \( f \) by running time dependent models for a range in values of \( r_{\text{outer}} \) and \( \dot{M}_T \).
3. THE MODELS

We use the computer model described in previous works (C93b, Cannizzo 1994, Cannizzo et al. 1995, hereafter CCL). This is a one dimensional time dependent numerical model which solves explicitly for the evolution of surface density and midplane temperature in the accretion disk. We carry this out by solving the mass and energy conservation equations written in cylindrical coordinates and averaged over disk thickness. The scalings which characterize the steady state relationship between $T_{\text{eff}}$ and $\Sigma$ were taken from the vertical structure calculations (Cannizzo & Wheeler 1984, C93b). For the WD mass we adopt $M_1 = 1 M_\odot$, for the inner disk radius we take $r_{\text{inner}} = 5 \times 10^8$ cm, and for the number of grid points $N = 300$. Our grid spacing is such that $\Delta r = \sqrt{r}$. We compute the visual flux as described in C93b by assuming a face-on oriented disk and taking the standard Johnson $V$–band filters. C93b presents many tests of the numerical model to assess systematic effects.

For the $\alpha$ parameter which characterizes the strength of viscous dissipation and angular momentum transport within the accretion disk we utilize the form given in CCL, $\alpha = \alpha_0 (h/r)^n$, where $n \simeq 1.5$. CCL quantified the use of this form based on the observed ubiquitous exponential decays seen in soft X-ray transients and dwarf novae, but they were not the first to use it; it was introduced by Meyer & Meyer-Hofmeister (1984). The normalization constant $\alpha_0 \simeq 50$ is based on the magnitude of the $e$–folding time constant, and the exponent $n$ determines the functional form of the decay: $n < 1.5$ gives a faster-than-exponential decay, and $n > 1.5$ gives a slower-than-exponential decay. The reason for this particular functional form being the preferred one, based on a detailed examination of the departure from steady state conditions within the hot part of the accretion disk, has been recently provided by Vishniac & Wheeler (1996). There is one shortcoming in adopting this form for dwarf novae which must be addressed. This
shortcoming can be rectified if one imposes an upper limit on $\alpha$, which seems physically reasonable. Figure 7 of CCL shows the failure of the form $\alpha \propto (h/r)^{1.5}$ to effect a change in the $e$–folding decay time in an outburst with outer disk radius, or equivalently, orbital period. For dwarf novae, however, it is well known that there exists a linear relation between the $e$–folding time constant associated with the decay of the dwarf nova outbursts and the orbital period (Bailey 1975). Therefore, the CCL $\alpha$ value cannot be applicable to dwarf novae without modification. Recent computations applying the form $\alpha = \alpha_0(h/r)^{1.5}$ to integrations of the vertical structure show that, if $\alpha$ is computed self-consistently within this formalism for dwarf novae, one derives large values of $\alpha$ in the “outburst state” — meaning the point on the upper branch of the $\log T_{\text{eff}} - \log \Sigma$ curve which lies at the vertical extrapolation of the $\Sigma_{\text{max}}$ value (Gu & Vishniac 1998). In fact, the values are considerably larger than can be tolerated within the framework derived from the theoretical constraint imposed by the Bailey relation (Smak 1984, Cannizzo 1994) — this constraint being that $\alpha_{\text{hot}}$ cannot exceed $\sim 0.1 - 0.2$. This limit will be reached, unfortunately, when $h/r \simeq 0.016 - 0.025$ if we strictly take $\alpha = \alpha_0(h/r)^{1.5}$. Since this value of $h/r$ is exceeded on the upper stable branch of steady state solutions for dwarf novae, we conclude that the physical mechanism responsible for generating the viscous dissipation and transporting angular momentum in accretion disks must saturate to some $\alpha_{\text{limit}} \sim 0.1 - 0.2$ so as to give the Bailey relation (Smak 1984). Disks tend to be thinner in systems with larger central masses, generally speaking, so systems such as X-ray novae (and AGN) do not run into this limit (Gu & Vishniac 1998). This explains why X-ray novae do not show a “Bailey relation” (see Fig. 3 of Tanaka & Shibazaki 1996). In light of these considerations, we utilize the CCL $\alpha$ form in our computations, but in the limit of large $\alpha$ we do not allow it to exceed 0.2.

The fact that we see a system as a dwarf nova means that the mass transfer rate into the outer accretion disk from the secondary star cannot be too great or else the disk would
be in permanent outburst (Smak 1983). This reasoning must apply to the systems used by Warner (1987) in his compilation. Shafter (1992) considered the relative frequency of dwarf novae as a fraction of all cataclysmic variables in different period bins longward of the $2 - 3$ h period gap in an attempt to understand the variation of the rate of mass transfer $\dot{M}_T$ from the secondary star (feeding into the outer accretion disk) with orbital period which we observe in the dwarf novae. Shafter concluded that, by restricting our attention solely to dwarf novae, we are probably not sampling the long term $\dot{M}_T(P_{\text{orbital}})$ value which characterizes the secular evolution of cataclysmic variables as a whole (Kolb 1993). Therefore we need not feel reticent in adopting an $\dot{M}_T(P_{\text{orbital}})$ law which serves only to ensure $\dot{M}_T < \dot{M}_{\text{crit}}$, where $\dot{M}_{\text{crit}}$ is the value of the secondary mass transfer $\dot{M}_T$ which must be exceeded for the entire disk to be stable in the high state of the "S" curve. Stated another way, our concern in this work is to adopt a law for $\dot{M}_T(P_{\text{orbital}})$ which will produce dwarf nova outbursts. This law is not related to that which characterizes the entire class of cataclysmic variables. From a practical standpoint, we find that our computed values of $M_V(\text{peak})$ are relatively insensitive to the specific $\dot{M}_T$ at a given orbital period.

We run computations for five values of $r_{\text{outer}}/10^{10}$ cm $-2, 3, 4, 5, \text{and } 6 -$ while at the same time scaling the value of the rate of mass transfer $\dot{M}_T$ feeding into the outer disk so that $\dot{M}_T < \dot{M}_{\text{crit}} = \dot{M}(\Sigma_{\text{min}}) \approx 10^{16}$ g s$^{-1}$ $(r_{\text{outer}}/10^{10}$ cm$)^{2.6}m_1^{-0.87}$ (Cannizzo & Wheeler 1984). For our canonical SS Cyg model, we take $r_{\text{outer}} = 4 \times 10^{10}$ cm and $\dot{M}_T = 6.3 \times 10^{16}$ g s$^{-1}$ (C93b, Cannizzo 1996). At this value, the system is about a factor of 4 below $\dot{M}_{\text{crit}}$. Therefore, to determine $\dot{M}_T$ for other $r_{\text{outer}}$ values we scale the SS Cyg $\dot{M}_T$ by $r_{\text{outer}}^{2.6}$. We also ran a second series of computations with the normalization on $\dot{M}_T$ a factor of two smaller.

Figure 1 shows the computations from the first series of models. In Figure 1a we give the light curves, and in Figure 1b we give the accretion disk masses, shown in units of $M_{\text{max}}$. 
for the relevant $r_{\text{outer}}$ value. The assumption implicit in eqn. (2) that $f$ is constant with orbital period appears to be quite good: there is no noticeable drifting of the disk mass relative to $M_{\text{max}}$ as $r_{\text{outer}}$ changes. Figure 2 shows the computations from the second series. The results are similar to the first series. The equilibrium disk masses shift to slightly lower values, as do the $M_V(\text{peak})$ values.

Figure 3 shows the results taken from the previous figures of $M_V(\text{peak})$ versus orbital period. The results from the first series are indicated by the hatched area with hatched lines inclined at $\pm 45^\circ$ with respect to the $x$–axis, whereas those from the second series have hatched lines inclined at $90^\circ$ and $180^\circ$. The conversion between $r_{\text{outer}}$ and $P_{\text{orbital}}$ was carried out using the fitting formula given in Eggleton (1983), assuming a secondary mass $M_2 = 0.1 M_\odot P_{\text{orbital}}(\text{h})$. The hatched area accompanying each series shows the range allowed by taking the disk to fill between 0.7 and 0.8 of the Roche lobe of the primary.

4. DISCUSSION AND CONCLUSION

We have run models using the accretion disk limit cycle model for dwarf novae in an attempt to understand Warner’s relation for dwarf nova outbursts $M_V(\text{peak}) = 5.64 - 0.259 P_{\text{orbital}}(\text{h})$. As noted in C93b, the observed upper limit is a natural consequence of the “maximum mass” of the accretion disk that is allowed by the critical surface density $\Sigma_{\text{max}}$. This value increases steeply with orbital period, therefore the amount of fuel available in a dwarf nova outburst also scales with orbital period. An important finding is that, for a constant value of $\dot{M}_T/\dot{M}_{\text{crit}}$, the value $f$ which is the ratio of the accretion disk mass at outburst onset to the “maximum mass” is relatively constant with orbital period. This gives some confidence in the analytical estimate given by C93b. The fact that the theoretical variation of $M_V(\text{peak})$ with orbital period is flatter than might have been expected from C93b’s scaling is due in part to the variation of the orbital period.
with outer disk radius in the limit where the mass of the secondary star starts to become comparable to the primary mass. This effect contributes to the arc-like shape of the shaded regions shown in Fig. 3.

The mean level of $M_V(\text{peak})$ in our models exceeds Warner’s empirical line by $\sim 0.3 - 0.5$ mag over most of the range shown. It is probable that our method for computing the $V$ band flux is too crude to expect consistency with observations at this level of detail — for instance we do not include limb darkening in the models, an effect mentioned by Warner (1987). Also, we have utilized Planckian flux distributions for the disk. Wade (1988) has shown that, to varying degrees, both Planckian distributions and stellar or Kurucz type distributions fail to represent adequately the observations. Although Wade found that the Planckian distributions can account for both UV flux and UV color in a sample of nova-like variables, the failure of the model in other respects leads to the conclusion that “one cannot rely on model fitting to give the correct luminosities or mass-transfer rates (within an order of magnitude).” Unless there are systematic effects which depend strongly on orbital period, however, our computed $M_V(\text{peak})-P_{\text{orbital}}$ slope should have some physical relevance. When one goes beyond this to considering the normalization level of $M_V(\text{peak})$, it would seem that a better flux model must be utilized.

It is also interesting to note the relative insensitivity of $M_V(\text{peak})$ to $\dot{M}_T$ at a given orbital period. This would imply that the observed scatter at a given orbital period must be due largely to variations in distance and inclination. We varied the normalization constant on $\dot{M}_T$ by a factor of two between our two series in this work, and found the resulting $M_V(\text{peak})$ values to differ by $\lesssim 0.3$ mag.

The soft X-ray transients — interacting binary stars containing a neutron star or black hole as the accreting object — also seem to obey a relation between $M_V(\text{peak})$ and $P_{\text{orbital}}$ for outbursts. Van Paradijs & McClintock (1994) noted a relation between $M_V$ and
$(L_X/L_{\text{Edd}})^{1/2} P_{\text{orbital}}^{2/3}$, where $L_X/L_{\text{Edd}}$ is the X-ray luminosity in units of the Eddington luminosity. Irradiation is a complicating factor in these systems, and by comparing the $M_V$ values between Figure 1 of Warner (1987) and Figure 2 of van Paradijs & McClintock (1994), one can see that the X-ray binaries are $\sim 1 - 7$ mag brighter than the dwarf novae at maximum light. Clearly most of the optical flux is reprocessed X-ray radiation coming from large radii in the disk. For the black hole X-ray binaries, however, a large fraction of the difference comes from having larger disks due to the larger orbital separations, at a given orbital period (Cannizzo 1998). For the neutron star systems (in which the primary mass is $\sim 1 M_\odot$ as in dwarf novae), the difference is entirely due to the irradiation.

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FIGURE CAPTIONS

Figure 1. (a) The light curves from the first series for which (top to bottom) the values \((r_{outer}/10^{10} \text{ cm}, \dot{M}_T/10^{16} \text{ g s}^{-1}) = (2, 1.04), (3, 2.98), (4, 6.30), (5, 11.3), \text{ and } (6, 18.1)\). The dashed lines indicate the approximate absolute magnitude of the secondary star \(M_V = 22.0 - 17.46 \log P_{\text{orbital}}(h)\) (Patterson 1984). The two values shown in each panel are consistent with a disk which fills 0.7 and 0.8, respectively, of the Roche lobe of the primary. In the first two panels these lines are off-scale toward fainter magnitudes; in the third panel only the upper line is visible.

(b) The accretion disk masses corresponding to the light curves in (a), scaled to the “maximum mass” values (top to bottom) \(M_{\text{max}} = 2.713 \times 10^{23} \text{ g}, 8.013 \times 10^{23} \text{ g}, 1.727 \times 10^{24} \text{ g}, 3.135 \times 10^{24} \text{ g}, \text{ and } 5.103 \times 10^{24} \text{ g}\.\)

Figure 2. (a) The light curves from the second series for which the \(\dot{M}_T\) values are a factor of two smaller than in Fig. 1. The \(r_{outer}\) values are the same.

(b) The accretion disk masses corresponding to the light curves in (a), scaled to the same maximum mass values as in Fig. 1.

Figure 3. The peak absolute magnitudes of the outbursts shown in Figs. 1a and 2a. The upper hatched region is for the first series and the lower hatched region is for the second series. The normalization on the \(\dot{M}_T\) values used for the two series is different by a factor of two, and yet the offset between the two regions is quite small. To convert between \(r_{outer}\) and \(P_{\text{orbital}}\), we use the fitting formula for the Roche radius given by Eggleton (1983), assuming either 0.8 (left boundary of each hatched region) or 0.7 (right boundary of each hatched region) for the fraction of the Roche lobe which \(r_{outer}\) occupies. We also take \(M_2 = 0.1M_\odot P_{\text{orbital}}(h)\). The dashed line indicates \(M_V(\text{peak}) = 5.64 - 0.259P_{\text{orbital}}(h)\) (Warner 1987). Warner quotes an rms scatter about this relation of \(\pm 0.23\) mag.