Diffeomorphism algebra of two dimensional free massless scalar field with signature change

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Abstract

We study a model of free massless scalar fields on a two dimensional cylinder with metric that admits a change of signature between Lorentzian and Euclidean type (ET), across the two timelike hypersurfaces (with respect to Lorentzian region). Considering a long strip-shaped region of the cylinder, denoted by an angle $\theta$, as the signature changed region it is shown that the energy spectrum depends on the angle $\theta$ and in a sense differs from ordinary one for low energies. Moreover diffeomorphism algebra of corresponding infinite conserved charges is different from “virasoso” algebra and approaches to it at higher energies. The central term is also modified but does not approach to the ordinary one at higher energies.

1 Introduction

The initial idea of signature change is due to Hartle, Hawking and Sakharov [1] which makes it possible to have a spacetime with Euclidean and Lorentzian regions in quantum gravity. It has been shown that the signature change may happen even in classical general relativity [2]. There are two different approaches to this problem: continuous and discontinuous signature changes. In the continuous approach, in going through Euclidean region to Lorentzian one the signature of metric changes continuously. Hence the metric becomes degenerate at the border of these regions. But in the discontinuous approach the metric is nondegenerate everywhere and discontinuous at the border of Euclidean and Lorentzian regions. In quest for effect of signature

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change in physical problems, mostly its effect on the propagation of massless boson has been rather deeply studied [3]. Dray and etal have shown that the phenomena of particle production can happen for propagation of scalar particle in spacetime with heteric signature. They have also obtained a rule for propagation of massless scalar field on a two dimensional space-time with signature change [3]. The effect of signature change on the propagation of plane waves in going from Lorentzian to Kleinian region has been studied by Alty [4]. They have shown that this kind of signature change leads to unlimited energy extraction from Kleinian region.

There are of course rather a bit of works about the subject of signature change, which we do not follow them here. In this paper we follow the idea similar to Kleinian type signature change [4] in which it is a spacelike coordinate whose associated metric component changes sign discontinuously. We study the effect of such signature change over $M = R \times S^1$ manifold on the free massless scalar field (boson field), where $R$ represents timelike coordinate $\tau$ and the change of signature is induced from the change of sign in the metric component associated to the spacelike coordinate $\sigma$ on the circle $S^1$. The model is similar to that of Dray and et al [3] with the a difference that the role of space and time are changed. An idea about the topology change of the manifold $M = R \times S^1$ over the spacelike hypersurface (circle) [5] has been one of our motivations in studying the effects that such a signature change produces. The standard investigation of ordinary free massless scalar field on the manifold $M = R \times S^1$ with a pure Lorentzian signature gives rise to discrete energy spectrum of integer values and an infinite number of conserved charges called “virasoro generators” [6]. These generators form a diffeomorphism algebra called “virasoro algebra” which gets a central term after quantization [6]. Introducing a region of Euclidean type (ET) signature, denoted by an angle $\theta$ as a long strip-shaped region of the cylinder $M = R \times S^1$ enclosed by a Lorentzian region, affects these standard results such that the energy spectrum and diffeomorphism algebra are modified; specially the corresponding central term is a complicate function of energy and the angle $\theta$. The modifications of spectrum and diffeomorphism algebra tends to disappear at higher values of energy $\omega$ but the central term does not approach to the standard central term at higher energies. Thus it appears to be worth investigating this model. The paper is organized as follows: In section 2, using the notation of reference [3], the signature change over $S^1$ has been thoroughly studied, where by requiring the continuity of currents accross the border of (ET) and Lorentzian regions we reach a junction condition. In section 3, the equation of motion for massless scalar boson has been solved in both regions. Then imposing the appropriate junction conditions, a quantization condition is imposed on the values of spectrum, $\omega$, and real distributional solutions have been constructed. The section is ended by investigating their completeness and orthogonality. In section 4, using the Poisson bracket of real scalar field with its conjugate momentum in both regions, the Poisson bracket of normal modes $\alpha_\omega$ has been obtained. In section 5, using the conservation of energy-momentum tensor distributions on both sides of the border of signature change in light cone coordinates, infinite number of conserved quantities $L_\omega$ have been obtained. In section 6, the diffeomorphism algebra of $L_\omega$ is constructed. In section 7, we expand $L_\omega$ in normal modes. Finally in section 8, we quantize this model by Dirac canonical quantization method and show that the diffeomorphism algebra gets a central term which has been calculated for higher energies. The paper ends with a conclusion and an appendix explaining the derivation of structure constants of the algebra.
2 Definition of the model

In this section we use the symbols of Ref. [3]. We take the lagrangian of free massless scalar field $\phi$ with signature change in two dimensions as

$$\mathcal{L} \star 1 = d\phi \wedge \star d\phi$$  \hspace{1cm} (1)

where $d$ is the exterior derivative and $\star$ is the Hodge star given by

$$\star 1_{U^\pm} = \epsilon^\pm \sqrt{|g|} d\tau \wedge d\sigma \mid_{U^\pm}$$  \hspace{1cm} (2)

where $\epsilon^\pm$ takes the values $\pm 1$ according to the orientation of the coordinates $\tau$ and $\sigma$ in both regions $U^\pm$ of different signatures. Moreover we assume that the free massless scalar field propagates on a 2-dimensional manifold $M = \mathbb{R} \times S^1$ (the circle $S^1$ represents “space” and the real line represents “time”) with the following metric

$$dS^2 = -d\tau^2 + g(\sigma)d\sigma^2$$  \hspace{1cm} (3)

where $\tau$ is timelike coordinate and $\sigma$ is a kind of periodic spacelike coordinate with period $2\pi$, and $g(\sigma)$ is a periodic function of $\sigma$, which takes +1 for Lorentzian region and -1 for (ET) one. Let us introduce an angle $\theta$ as a segment of the coordinate $\sigma$ for which $g(\sigma)$ is given by

$$g(\sigma) = \begin{cases} 
-1 & 0 < \sigma < \theta + \text{Mod} 2\pi \\
+1 & \theta < \sigma < 2\pi + \text{Mod} 2\pi 
\end{cases}$$  \hspace{1cm} (4)

The situation is shown in Fig.1 in which the shaded region has (ET) metric and a Lorentzian metric is governed elsewhere. Obviously for every killing vector $X$, we have a conserved current as

$$J_X = i_X d\phi \wedge \star d\phi + d\phi \wedge i_X \star d\phi$$  \hspace{1cm} (5)

where $i_X$ denotes the contraction with $X$. Now, using the definition of Hodge star in (2), the currents $J^\pm$ can be written as

$$J^\pm = \epsilon^\pm [(\partial_\sigma \phi)^2 - (\partial_\tau \phi)^2] \sqrt{|g|} d\sigma - 2\epsilon^\pm \partial_\tau \phi \partial_\sigma \phi \sqrt{|g|} sgn(g) d\tau$$  \hspace{1cm} (6)

where $sgn(g)$ denotes the definition (4). In order the closed one-form $J_X$ to be defined consistently on the manifold $M$, the pullbacks of $J^\pm$ to each hypersurface of signature change $\Sigma$, $\Sigma'$ must agree, using of stokes’ theorem [3]. The agreement of pullbacks $-2\epsilon^\pm \partial_\tau \phi \partial_\Sigma \phi sgn(g) d\tau$ at each hypersurface $\Sigma$, $\Sigma'$ together with the assumption of continuity of $\phi$ across these hypersurfaces leads to the following “junction condition” at each $\Sigma$ and $\Sigma'$

$$\partial_{\sigma_E} \phi \mid_{\Sigma, \Sigma'} = -\epsilon \partial_{\sigma_L} \phi \mid_{\Sigma, \Sigma'} \hspace{1cm} \epsilon = \frac{\epsilon^+}{\epsilon^-}$$  \hspace{1cm} (7)

Of course the junction condition (7) is similar to the one in [3] with the difference that the roles of space and time are interchanged.
3 Solution of wave equations

The wave equations obtained from the variation of Lagrangian (1) in (ET) and Lorentzian regions are

\[ (\partial_{\tau}^2 + \partial_{\sigma}^2) \phi^E(\sigma, \tau) = 0 \]
\[ (\partial_{\tau}^2 - \partial_{\sigma}^2) \phi^L(\sigma, \tau) = 0 \]

where \( \phi^E(\sigma, \tau) \) and \( \phi^L(\sigma, \tau) \) are solutions in (ET) and Lorentzian regions respectively. Obviously these equations can be solved exactly and their solutions are

\[ \phi^E(\sigma, \tau) = A_{\omega} \exp(-i\omega(\tau + i\sigma)) + B_{\omega} \exp(-i\omega(\tau - i\sigma)) \]
\[ \phi^L(\sigma, \tau) = C_{\omega} \exp(-i\omega(\tau + \sigma)) + D_{\omega} \exp(-i\omega(\tau - \sigma)) \]

where the coefficients \( A_{\omega}, B_{\omega}, C_{\omega} \) and \( D_{\omega} \), as well as the energy \( \omega \) can be determined by using the junction condition (7) over the solutions (9). Assuming \( \epsilon^+ = +1 \) and \( \epsilon^- = +1 \) for (ET) and Lorentzian regions respectively; the continuity of \( \phi_{\omega} \) and the junction condition (7) are written as

\[ \phi^E_{\omega}|_0 = \phi^L_{\omega}|_{2\pi} \quad \phi^E_{\omega}|_\theta = \phi^L_{\omega}|_{\theta} \]
\[ \partial_\sigma \phi^E_{\omega}|_0 = -\partial_\sigma \phi^L_{\omega}|_{2\pi} \quad \partial_\sigma \phi^E_{\omega}|_{\theta} = -\partial_\sigma \phi^L_{\omega}|_{\theta} \]

Here we have used the fact that these conditions are satisfied at all times along the coordinate \( \tau \). Now, these conditions have nontrivial solutions for coefficients \( A_{\omega}, B_{\omega}, C_{\omega} \) and \( D_{\omega} \), only if \( \omega \) satisfies the following “quantization condition”

\[ \cosh\omega\theta \cos(\omega(\theta - 2\pi)) = 1 \]

For a given root of the equation (11), the coefficients \( A_{\omega}, B_{\omega}, C_{\omega} \) and \( D_{\omega} \) are

\[ A_{\omega} = \frac{1}{2i} \left[ \exp(i\omega(\theta - 2\pi)) - \cosh\omega\theta \right] / \left( \sinh\omega\theta \right) + 1 \exp(2\pi i \omega) D_{\omega} = a_{\omega} D_{\omega} \]
\[ B_{\omega} = \frac{1}{2} \left[ - \exp(i\omega(\theta - 2\pi)) - \cosh\omega\theta \right] / \left( \sinh\omega\theta \right) + 1 \exp(2\pi i \omega) D_{\omega} = b_{\omega} D_{\omega} \]
\[ C_{\omega} = \frac{-i\exp(i\omega\theta) - \exp(2\pi i \omega) \cosh\omega\theta}{\exp(-2\pi i \omega) \sinh\omega\theta} \]
\[ D_{\omega} = \text{arbitrary} \]

where, the arbitrariness of \( D_{\omega} \) makes it possible to fix other coefficients \( A_{\omega}, B_{\omega} \) and \( C_{\omega} \). The spectrum of this model, \( \omega \), is real and \( \theta \)-dependent. It differs from ordinary spectrum (with pure Lorentzian signature) at low energies and, as is seen in Fig.2, at high energies it coincides with the roots of \( \cos\omega(2\pi - \theta) \); therefore it’s “sum over energies” approaches to “sum over integers” at higher energies.

Now, we come to the orthogonality of our solutions for different values of \( \omega \). Let us consider two solutions \( \phi^E_{\omega}(\sigma, \tau) \) and \( \phi^{E'}_{\omega}(\sigma, \tau) \), corresponding to two different roots of equation (11). Obviously they satisfy the following equations in the (ET) region

\[ \{\partial_\sigma^2 \phi^E_{\omega}(\sigma) - \omega^2 \phi^E_{\omega}(\sigma)\} \exp(-i\omega\tau) = 0 \]
\{ \partial^2_{\sigma} \phi^E_{\omega}(\sigma) - \omega^2 \phi^E_{\omega}(\sigma) \} exp(-i\omega \tau) = 0 \quad (13 - b)

where \( \phi^E_{\omega}(\sigma) \) and \( \phi^L_{\omega}(\sigma) \) are \( \sigma \)-dependent separable solutions. Multiplying the first equation by \(-\phi^E_{\omega}(\sigma)\) and the second one by \(\phi^E_{\omega}(\sigma)\), integrating them in \((ET)\) region from zero to \(\theta\) and finally adding them up, we get

\[
(\omega^2 - \omega'^2) \int_{0}^{\theta} \phi^E_{\omega}(\sigma) \phi^E_{\omega}(\sigma) d\sigma - \int_{0}^{\theta} \partial_\sigma (\phi^E_{\omega}(\sigma) \partial_\sigma \phi^E_{\omega}(\sigma) - \phi^E_{\omega}(\sigma) \partial_\sigma \phi^E_{\omega}(\sigma)) d\sigma = 0 \quad (14)
\]

note that we have ignored the \( \tau \) integration, since it does not affect the result (14). Similarly, the solutions \( \phi^L_{\omega}(\sigma) \) and \( \phi'^L_{\omega}(\sigma) \) satisfy the following equations in the Lorentzian region

\[
\begin{align*}
\partial^2_{\sigma} \phi^L_{\omega}(\sigma) + \omega^2 \phi^L_{\omega}(\sigma) &= 0 \quad (15 - a) \\
\partial^2_{\sigma} \phi'^L_{\omega}(\sigma) + \omega^2 \phi'^L_{\omega}(\sigma) &= 0 \quad (15 - b)
\end{align*}
\]

By a similar procedure we will have

\[
(\omega^2 - \omega'^2) \int_{\theta}^{2\pi} \phi^L_{\omega}(\sigma) \phi^L_{\omega}(\sigma) d\sigma + \int_{\theta}^{2\pi} \partial_\sigma (\phi^L_{\omega}(\sigma) \partial_\sigma \phi^L_{\omega}(\sigma) - \phi^L_{\omega}(\sigma) \partial_\sigma \phi^L_{\omega}(\sigma)) d\sigma = 0 \quad (16)
\]

for the Lorentzian region. Adding equations (14) and (16), and then imposing the junction conditions (10) we get

\[
(\omega^2 - \omega'^2) \left( \int_{0}^{\theta} \phi^E_{\omega}(\sigma) \phi^E_{\omega}(\sigma) d\sigma + \int_{\theta}^{2\pi} \phi^L_{\omega}(\sigma) \phi^L_{\omega}(\sigma) d\sigma \right) = 0 \quad (17)
\]

Now, if we introduce the distribution [3]

\[
\phi_{\omega}(\sigma) = \Theta^+ \phi^E_{\omega} + \Theta^- \phi^L_{\omega} \quad (18)
\]

where \( \Theta^\pm \) are the Heaviside distribution with support in \( U^\pm \), then we get the following orthogonality condition on the \( \tau = \text{const} \) hypersurface

\[
\frac{1}{2\pi} \int_{0}^{2\pi} \phi_{\omega}(\sigma) \phi'_{\omega}(\sigma) \ast 1 = (\delta_{\omega,\omega'} + \delta_{\omega',-\omega'}) < \phi_{\omega}, \phi_{\omega'} > \quad (19)
\]

where the factor \( 1/2\pi \) is introduced for convenience and the symbol \(<,>\) denotes the inner product of solutions. Choosing \( D_\omega = 1/b_\omega \) and substituting \( A_\omega, B_\omega, C_\omega \) given by equation (12) in (9), we get

\[
\begin{align*}
\phi^E_{\omega}(\sigma, \tau) &= [(a/b)_\omega \exp(\omega \sigma) + \exp(-\omega \sigma)] \exp(-i\omega \tau) \\
\phi'^L_{\omega}(\sigma, \tau) &= [(c/b)_\omega \exp(-i\omega \sigma) + (1/b)_\omega \exp(i\omega \sigma)] \exp(-i\omega \tau) \\
\phi^E_0 &= [(a/b)_0 + 1] \\
\phi'^L_0 &= [(c/b)_0 + (1/b)_0] \quad (20 - 21)
\end{align*}
\]
where \((a/b)_,(c/b)_,\) and \((1/b)\mO\) are given by
\[
(a/b)_\omega = \frac{\sin(\omega - 2\pi)}{\cosh\theta + \sinh\theta - \cos(\omega - 2\pi)}
\]
\[
(c/b)_\omega = \frac{(1+i)e^{2i\omega x}(\sinh\theta + \cosh\theta - \exp(i\omega - 2\pi))}{2(\cosh\theta + \sinh\theta - \cos(\omega - 2\pi))}
\]
\[
(1/b)_\omega = \frac{(1-i)e^{-2i\omega x}(\sinh\theta + \cosh\theta - \exp(-i\omega - 2\pi))}{2(\sinh\theta - \cos(\omega - 2\pi) + \cosh\theta)}
\]  
\( (a/b)_\omega \) is real while \((c/b)_\omega \) and \((1/b)_\omega \) satisfy the following relation
\[
(c/b)_\omega^* + (1/b)_\omega^* = (c/b)_\omega - (1/b)_\omega
\]
where \(\ast\) denotes the complex conjugation. If \(\omega = 0\) then we have
\[
(a/b)_{\omega=0} + 1 = 2(\theta - \pi)/\theta \quad (c/b)_{\omega=0} + (1/b)_{\omega=0} = 2(\theta - \pi)/\theta
\]
Now by using \((23)\) and the reality of \((a/b)_\omega \) we can define, a set of real “distributional” orthogonal functions on the \(tau = const\) hypersurface
\[
\Phi_\omega(\sigma) = \Theta^+ \Phi_\omega^F + \Theta^- \Phi_\omega^L
\]  
where \(\Phi_\omega^F = (\phi_\omega^F(\sigma) + \phi_\omega^F(\sigma))\), \(\Phi_\omega^F = (\phi_\omega^F(\sigma) + \phi_\omega^F(\sigma))\) are symmetric under the interchange of \(\omega \leftrightarrow -\omega\). Then the orthogonality condition of the functions \(\Phi_\omega\) is given by
\[
\frac{1}{2\pi} \int_0^{2\pi} \Phi_\omega(\sigma) \Phi_\omega'(\sigma) * 1 = (\delta_{\omega',\omega} + \delta_{\omega,-\omega'}) < \Phi_\omega, \Phi_\omega'>
\]  
They also form a set of complete functions over the hypersurface \(\tau = const\). Therefore we can expand any real function \(f(\sigma)\) in terms of them
\[
f(\sigma) = \sum \beta_\omega \Phi_\omega(\sigma)
\]  
where the coefficients of expansion \(\beta_\omega\) and \(\beta_-\omega\) can be determined by using the orthogonality of \(\Phi_\omega\) and reality of \(f(\sigma)\). That is, we have
\[
\beta_0 = (1/2\pi < \Phi_0, \Phi_0>) \int_0^{2\pi} f(\sigma) \Phi_0 * 1 \quad \beta_\omega = \beta_-\omega = (1/4\pi < \Phi_\omega, \Phi_-\omega>) \int_0^{2\pi} f(\sigma) \Phi_\omega * 1
\]  
with
\[
\Phi_0 = 4(\theta - \pi)/\theta \quad < \Phi_0, \Phi_0 > = 16(\theta - \pi)^2/\theta^2
\]  
where eq \((24)\) and the relation \((\Theta^+ + \Theta^-) = 1\) have been used. Therefore the completeness of \(\Phi_\omega\) can be written as
\[
\frac{\Phi_0^2}{2\pi < \Phi_0, \Phi_0>} + \sum_{\omega \neq 0} \frac{1}{4\pi} < \Phi_\omega, \Phi_\omega > = \delta(\sigma - \sigma')
\]  
or
\[
\delta(\sigma - \sigma') = \frac{1}{2\pi} + non\ zero\ modes
\]
4 Poisson bracket structure

The general solution of this model can be given as a superposition of its special solutions which are consistent with the junction condition (10), that is

\[ \Phi(\sigma, \tau) = \frac{1}{2\pi} \varphi_0(\sigma, \tau) + \sum_{\omega \neq 0} \frac{i}{\omega} \alpha_{\omega} \exp(-i\omega \tau) \Phi_{\omega}(\sigma) \] (32)

where \( \varphi_0(\sigma, \tau) \) is the zero mode, and \( \alpha_{\omega} \) is the normal mode; such that the reality of \( \Phi(\sigma, \tau) \) imposes the following condition on \( \alpha_{\omega} \)

\[ \alpha_{\omega}^* = \alpha_{-\omega} \] (33)

The momentum conjugate of \( \Phi(\sigma, \tau) \) is given by

\[ \Pi(\sigma, \tau) = \partial_{\tau} \Phi(\sigma, \tau) \sqrt{|g|} \] (34)

or

\[ \Pi(\sigma, \tau) = \sqrt{|g|} \left\{ \partial_{\tau} \varphi_0 + \sum_{\omega \neq 0} \alpha_{\omega} \Phi_{\omega}(\sigma) \exp(-i\omega \tau) \right\} \] (35)

Now the poisson bracket is given by

\[ \{ \Phi(\sigma), \Pi(\sigma') \} = \delta(\sigma - \sigma') \]

\[ \{ \Phi(\sigma), \Phi(\sigma') \} \{ \Pi(\sigma), \Pi(\sigma') \} = 0 \] (36)

By substituting the normal mode expansion of \( \Phi, \Pi \) and the expansion of \( \delta(\sigma - \sigma') \) (30) in (36) we obtain

\[ \{ \varphi_0, \Pi_0 \} = 1 \]

\[ \{ \alpha_{\omega}, \alpha_{\omega'} \} = -i\omega \delta_{\omega + \omega', 0}/4\pi < \Phi_{\omega}, \Phi_{\omega} > \] (37)

where \( \Pi_0 = \sqrt{|g|} \partial_{\tau} \varphi_0 \). Let us define \( \tilde{\alpha}_{\omega} \) as

\[ \tilde{\alpha}_{\omega} = \alpha_{\omega} \sqrt{4\pi < \Phi_{\omega}, \Phi_{\omega} >} \] (38)

Hence the relations (37) and (32) can be rewritten as

\[ \{ \tilde{\alpha}_{\omega}, \tilde{\alpha}_{\omega'} \} = -i\omega \delta_{\omega + \omega', 0} \] (39)

\[ \Phi(\sigma, \tau) = \frac{1}{2\pi} \varphi_0(\sigma, \tau) + \sum_{\omega \neq 0} \frac{i}{\omega} \tilde{\alpha}_{\omega} \exp(-i\omega \tau) \tilde{\Phi}_{\omega}(\sigma) \] (40)

respectively, where

\[ \tilde{\Phi}_{\omega}(\sigma) = \Phi_{\omega}(\sigma)/\sqrt{4\pi < \Phi_{\omega}, \Phi_{\omega} >} \] (41)
5 Infinite conserved charges

Energy-momentum tensors are conserved in both (ET) and Lorentzian regions, that is to say

\[ \partial_{\mu} T_{\mu\nu}^E = 0 \quad 0 < \sigma < \theta \]
\[ \partial_{\mu} T_{\mu\nu}^L = 0 \quad \theta < \sigma < 2\pi \] (42)

where by using of (6) and (25) we get the following expressions for the components of energy-momentum tensors associated to real scalar field \( \Phi(\sigma, \tau) \)

\[ T_{00}^E = [ (\partial_{\tau} \Phi^E)^2 - (\partial_{\sigma} \Phi^E)^2 ] \]
\[ T_{00}^L = [ (\partial_{\tau} \Phi^L)^2 + (\partial_{\sigma} \Phi^L)^2 ] \]
\[ T_{01}^E = 2 \partial_{\tau} \Phi^E \partial_{\sigma} \Phi^E \]
\[ T_{01}^L = 2 \partial_{\tau} \Phi^L \partial_{\sigma} \Phi^L \] (43)

Introduce new coordinates \( \sigma_{+}^E \) and \( \sigma_{-}^E \) in (ET) region

\[ \sigma_{+}^E = \tau + i \sigma \]
\[ \sigma_{-}^E = \tau - i \sigma \] (44)

and \( \sigma_{+}^L \) and \( \sigma_{-}^L \) in Lorentzian region

\[ \sigma_{+}^L = \tau + \sigma \]
\[ \sigma_{-}^L = \tau - \sigma \] (45)

In terms of these coordinates in (ET) region the conservation law can be written as

\[ \partial_{+}^E T_{--}^E = 0 \]
\[ \partial_{-}^E T_{++}^E = 0 \] (46)

and similarly in Lorentzian region we have

\[ \partial_{+}^L T_{--}^L = 0 \]
\[ \partial_{-}^L T_{++}^L = 0 \] (47)

where

\[ T_{++}^E = (T_{00}^E - iT_{01}^E)/2 \]
\[ T_{--}^E = (T_{00}^E + iT_{01}^E)/2 \]
\[ T_{++}^E = T_{--}^E = 0 \] (48)

and

\[ T_{++}^L = (T_{00}^L + T_{01}^L)/2 \]
\[ T_{--}^L = (T_{00}^L - T_{01}^L)/2 \]
\[ T_{++}^L = T_{--}^L = 0 \] (49)
Now we can rewrite the equations (46) and (47) as
\[ \partial_+^E(f^E(-) T^E_{-}) = 0 \]
\[ \partial_-^E(f^E(+) T^E_{+}) = 0 \] respectively, where \( f(\pm)^E \) and \( f(\pm)^L \) are arbitrary functions of \( \sigma^E_{\pm} \) and \( \sigma^L_{\pm} \), respectively. So we can get infinite number of conserved quantities. Adding the equations (50) and (51) in both regions and writing the partial derivatives \( \partial_\pm \) in terms of \( \partial_\tau \) and \( \partial_\sigma \), we get the following current conservations
\[ \partial_\tau[(f^E(-)+f^E(+))T^E_{00}+i(f^E(-)-f^E(+))T^E_{01}]+\partial_\sigma[i(f^E(+)-f^E(-))T^E_{00}+(f^E(-)+f^E(+))T^E_{01}] = 0 \]
in (ET) region, and
\[ \partial_\tau[(f^L(-)+f^L(+))T^L_{00}-(f^L(-)-f^L(+))T^L_{01}]+\partial_\sigma[(f^L(-)-f^L(+))T^L_{00}-(f^L(-)+f^L(+))T^L_{01}] = 0 \]
in Lorentzian region, respectively. As in section 2 in order to have a conserved current across the hypersurface of signature change, \( \Sigma \), one finds
\[ (f^E(-)+f^E(+)) \bigg|_{\Sigma} = -(f^L(-)+f^L(+)) \bigg|_{\Sigma} \]
where, because of similar condition at both hypersurfaces \( \Sigma, \Sigma' \), only the junction condition at \( \Sigma \) is considered.

Now, using the junction condition (10) imposed on the \( T_{01} \) components in (43), leads to the following junction condition over \( f,s \)
\[ (f^E(-)+f^E(+)) \bigg|_{\Sigma} = (f^L(-)+f^L(+)) \bigg|_{\Sigma} \]
It is quite easy to show that
\[ \tilde{f}_E^E(\sigma^\pm) = [(a/b)\omega + 1]\exp(-i\omega\sigma^E_{\pm})/\sqrt{4\pi < \Phi_{\omega}, \Phi_{\omega}>} \]
\[ \tilde{f}_L^E(\sigma^\pm) = [(c/b)\omega + (1/b)\omega]\exp(-i\omega\sigma^L_{\pm})/\sqrt{4\pi < \Phi_{\omega}, \Phi_{\omega}>} \]
and
\[ \tilde{f}_E^0 = [(a/b)\omega + 1]/\sqrt{< \Phi_0, \Phi_0>} \]
\[ \tilde{f}_L^0 = [(c/b)\omega + (1/b)\omega]/\sqrt{< \Phi_0, \Phi_0>} \]
satisfy the condition (55). Now one can define the following quantities
\[ \tilde{L}_\omega = \int_0^{2\pi} \{ \Theta^+[\tilde{f}_E^E(+)+\tilde{f}_E^E(-)-\tilde{f}_E^L(-)+\Theta^{-}[\tilde{f}_L^L(+)+\tilde{f}_L^L(-)-\tilde{f}_L^L(-)] \} d\sigma \]
as the required conserved quantities, that is

\[ \frac{d\tilde{L}_\omega}{d\tau} = 0 \]  

(59)

One can also show that:

\[ H = 2\tilde{L}_0 \]  

(60)

where the Hamiltonian is defined as:

\[ H = \int_0^{2\pi} T_{00} \times 1 = \int_0^\theta T_{00}^E d\sigma + \int_0^{2\pi} T_{00}^L d\sigma \]  

(61)

6 Diffeomorphism algebra

It is important to note that the conserved quantities \( L_\omega \) are closed under the Poisson bracket algebra. Considering the following poisson brackets

\[ \{T^E_+, T^E_+(\sigma')\} = i/2 (T^E_+(\sigma) + T^E_+(\sigma')) \partial_\sigma \delta(\sigma - \sigma') \]

\[ \{T^E_-(\sigma), T^E_-(\sigma')\} = -i/2 (T^E_-(\sigma) + T^E_-(\sigma')) \partial_\sigma \delta(\sigma - \sigma') \]

\[ \{T^L_+, (T^L_+ (\sigma) + T^L_+ (\sigma')) \partial_\sigma \delta(\sigma - \sigma') \]

\[ \{T^L_-, (T^L_- (\sigma) + T^L_- (\sigma')) \partial_\sigma \delta(\sigma - \sigma') \]

\[ \{T^E_+(\sigma), T^L_+(\sigma')\} = \{T^E_+(\sigma), T^L_+(\sigma')\} = 0 \]

where (48),(49),(43) and (36) has been used, and redefining

\[ \tilde{L}_\omega = L(\tilde{f}^E_+, \tilde{f}^E_-, \tilde{f}^L_+, \tilde{f}^L_-) \]

(63)

the Poisson bracket of \( \tilde{L}_\omega \),s takes the following form after some manipulation

\[ \{ \tilde{L}_\omega, \tilde{L}_{\omega'} \} = -i(\omega - \omega')L(\tilde{f}^E_+, \tilde{f}^E_-, \tilde{f}^L_+, \tilde{f}^L_-) \]

(64)

It is easy to see that \( \tilde{f}^E_+ \) and \( \tilde{f}^F_+ \) satisfy the equations (13-a,b) and also \( \tilde{f}^L_+ \) and \( \tilde{f}^L_+ \) satisfy the equations (15-a,b), therefore one can expand \( \tilde{f}^E_+ \) \( \tilde{f}^E_+ \) and \( \tilde{f}^L_+ \) \( \tilde{f}^L_+ \) in terms of \( \tilde{f}^E_+ \) \( \tilde{f}^E_+ \) and \( \tilde{f}^L_+ \) \( \tilde{f}^L_+ \) respectively (2,3-A). Hence we can write \( \{ \tilde{L}_\omega, \tilde{L}_{\omega'} \} \) in terms of \( \tilde{L}_{\omega''} \); or \( \tilde{L}_{\omega''} \) are closed under Poisson bracket

\[ \{ \tilde{L}_\omega, \tilde{L}_{\omega'} \} = -i(\omega - \omega') \sum_{\omega''} O^{\omega''}_{\omega', \omega''} \tilde{L}_{\omega''} \]

(65)

where, the structure constants \( O^{\omega''}_{\omega', \omega''} \) are real and given by (see appendix A)

\[ O^{\omega''}_{\omega', \omega''} = \frac{\omega + \omega' + \omega''}{4\pi \omega''[<\phi_{\omega''}, \phi_{\omega''}> + <\phi_{\omega''}, \phi_{\omega''}>]} \int_0^\theta \tilde{f}^E_+(\sigma) \tilde{f}^E_+(\sigma) \tilde{f}^E_-(\sigma) \tilde{f}^E_-(\sigma) d\sigma \]

(66)
momentum tensors as normal modes $\tilde{L}_{\alpha}$. Mode expansion of the following asymptotic forms, using of (15,19-A), for higher values of $\omega$ are normalized similar to $\tilde{\omega}$ and:

$$O_{\omega}^{0} = \frac{1}{4\pi} \int_{0}^{\theta} (\tilde{f}_{w}^{E}(\sigma) \tilde{f}_{w}^{E}(\sigma) + \tilde{f}_{w}^{E}(\sigma) \tilde{f}_{w}^{E}(\sigma)) d\sigma + \int_{\theta}^{2\pi} (\tilde{f}_{w}^{E}(\sigma) \tilde{f}_{w}^{E}(\sigma) + \tilde{f}_{w}^{E}(\sigma) \tilde{f}_{w}^{E}(\sigma)) d\sigma$$

(67)

for $\omega'' = 0$, where $\tilde{\phi}_{w}$ are normalized similar to $\tilde{\omega}$ by dividing by $\sqrt{4\pi < \Phi_{w}, \Phi_{w}>}$ and $\tilde{f}(\sigma)_{w}$s are $\sigma$ dependent terms in (56). These structure constants have asymptotic forms (15,19-A) for higher values of $\omega$ or $\omega'$. So, by redefinition of $\tilde{L}_{\omega}$ and $\tilde{L}_{0}$ as

$$L_{\omega} = \frac{1}{\sqrt{\omega}} \tilde{L}_{\omega}, \quad L_{0} = \tilde{L}_{0}$$

(68)

the poisson bracket (65) is rewritten as

$$\{L_{\omega}, L_{\omega'}\} = -i(\omega - \omega') \sum_{w''} C_{w''}^{\omega} L_{w''}$$

(69)

where the redefined real structure constants $C_{w''}^{\omega} = O_{\omega}^{0} \sqrt{\frac{|\omega'|}{|\omega|}}$ and $C_{w''}^{0} = O_{\omega}^{0} \sqrt{\frac{1}{|\omega''|}}$ have the following asymptotic forms, using of (15,19-A), for higher values of $\omega$ or $\omega'$

$$C_{w''}^{\omega} \approx \delta_{\omega}^{0} C_{w''}^{\omega'} \approx \delta_{\omega''}^{0}$$

(70)

which are the structure constants of “virasoro” algebra. In table.1 and table.2 a set of numerical structure constants are given for two generic values of $\theta$, which confirm (70). Note that for smaller $\theta$ the coincidence of structure constants with asymptotic forms are more clear than that of larger $\theta$. Interpretation of these asymptotic behaviours will be given in section 7.

7 Mode expansion of $L_{\omega}$

Similar to the ordinary 2-dimensional massless boson model, we can expand $L_{\omega}$ in terms of normal modes $\tilde{\alpha}_{w}$. Hence, using expansion of $\Phi(\sigma, \tau)$, given in (32), we can expand energy momentum tensors as

$$T_{++}^{E} = \frac{1}{2}((\partial_{\tau} \Phi - i \partial_{\tau} \Phi))^{2} = 2 \sum_{w''} \tilde{f}_{w}^{E}(+) \tilde{f}_{w'}^{E}(+) \tilde{\alpha}_{w} \tilde{\alpha}_{w'}$$

$$T_{-}^{E} = \frac{1}{2}((\partial_{\tau} \Phi + i \partial_{\tau} \Phi))^{2} = 2 \sum_{w''} \tilde{f}_{w}^{E}(-) \tilde{f}_{w'}^{E}(-) \tilde{\alpha}_{w} \tilde{\alpha}_{w'}$$

$$T_{++}^{L} = \frac{1}{2}((\partial_{\sigma} \Phi + \partial_{\omega} \Phi))^{2} = 2 \sum_{w''} \tilde{f}_{w}^{L}(+) \tilde{f}_{w'}^{L}(+) \tilde{\alpha}_{w} \tilde{\alpha}_{w'}$$

$$T_{-}^{L} = \frac{1}{2}((\partial_{\sigma} \Phi - \partial_{\omega} \Phi))^{2} = 2 \sum_{w''} \tilde{f}_{w}^{L}(-) \tilde{f}_{w'}^{L}(-) \tilde{\alpha}_{w} \tilde{\alpha}_{w'}$$

(71)

Inserting these expansions in formula (58), and using the relations (2,3 - A) for the products of $\tilde{f}_{w}$, and also the orthogonality of $\tilde{\Phi}_{0}$ and $\tilde{\Phi}_{w}$, we get the following expansion for $L_{\omega}$s

$$L_{\omega} = 8\pi \sum_{\omega''} \sqrt{\omega' \omega''} C_{w''}^{\omega''} C_{w''}^{\omega'} \tilde{\alpha}_{w'} \tilde{\alpha}_{w''}$$

(72)
$L_\omega$ takes the following form for large values of $\omega$

$$L_\omega = 8\pi \sum_{\omega',\omega''} \sqrt{|\omega'\omega''|} C_{\omega',\omega''} \tilde{\alpha}_{\omega'} \tilde{\alpha}_{\omega''}$$  \hspace{1cm} (73)

Using the reality of structure constants $C_{\omega',\omega''}$ and the relation $\tilde{\alpha}_\omega^* = \tilde{\alpha}_{-\omega}$ it is easy to show that $L_\omega$ satisfies the following relations

$$L_\omega^* = L_{-\omega}$$  \hspace{1cm} (74)

Also the Hamiltonian $H$ has the expansion

$$H = 16\pi \sum_{\omega'} \sqrt{|\omega|} C_{\omega,\omega'} \tilde{\alpha}_\omega \tilde{\alpha}_\omega'$$  \hspace{1cm} (75)

where we have used eq (70) as

$$C_{\omega,\omega'}^0 \approx \delta_{\omega'}$$  \hspace{1cm} (76)

Due to the asymptotic forms of $C_{\omega,\omega'}^0$ and $C_{\omega',\omega''}$ given in eq (70), the higher frequency terms of expansion of $H$ and $L_\omega$ (eqs (75),(73)) will look similar to the ordinary two dimensional boson model. Therefore our model is almost similar to the ordinary boson model for higher modes. Physically this is due to the fact that the signature change effect in this model is similar to the presentation of an (ET) wall in the Lorentzian background [4]; so that these modifications are reminiscent of a tunneling effect related to the (ET) region, in which the rate of tunneling increases for higher energies. Notice that due to the existence of a sharp discontinuity in this model it is not expected to get the standard results, corresponding to a pure Lorentzian signature, at the continuous limit of $\theta \to 0$; rather, it is seen from quantization condition (11) that for small values of $\theta$ the “sum over energies” corresponding to the spectrum $\omega$ tends to “sum over integers” corresponding to the ordinary spectrum of boson field, at higher levels of energies (not necessarily higher values of energies) compared to the spectrum corresponding to a large value of $\theta$. This is because, as $\theta$ tends to smaller values, the cosine function in (11) oscillates more rapidly and the spectrum is more condensed.

8 Quantization

We quantize this model by Dirac canonical quantization. Using the prescription of

$$\{ , \} [P,B] \rightarrow -\frac{1}{i} [ [ , ] ]$$  \hspace{1cm} (77)

where $[,]$ denotes the commutator, we get the following poisson brackets for operators $\Phi(\sigma), \Pi(\sigma')$ and $\tilde{\alpha}_\omega$

$$[\Phi(\sigma), \Pi(\sigma')] = i\delta(\sigma - \sigma')$$

$$[\Phi(\sigma), \Phi(\sigma')] = [\Pi(\sigma), \Pi(\sigma')] = 0$$

$$[\tilde{\alpha}_\omega, \tilde{\alpha}_{\omega'}] = \omega \delta_{\omega+\omega',0}$$  \hspace{1cm} (78)

In order to avoid infinity appearing in operators $L_\omega$ and $H$, we have to define them in normal order, similar to the ordinary boson model. Hence, first we define normal ordering for $\tilde{\alpha}_\omega$ as

$$\tilde{\alpha}_{\omega_1} \tilde{\alpha}_{\omega_2} = \begin{cases} \tilde{\alpha}_{\omega_1} \tilde{\alpha}_{\omega_2} & \text{if } \omega_1 < \omega_2 \\ \tilde{\alpha}_{\omega_2} \tilde{\alpha}_{\omega_1} & \text{if } \omega_2 < \omega_1 \end{cases}$$  \hspace{1cm} (79)
so normal ordered $L_\omega$ is

$$L_\omega = \sum_{\omega''} \sum_{\omega' < \omega''} \sqrt{\omega' \omega''} |C_{\omega'' \omega'}^{\omega'' \omega'} C_{\omega'' \omega}^{\omega'' \omega} \tilde{\alpha}_{\omega''} \tilde{\alpha}_{\omega'} + \sum_{\omega' < \omega''} \sqrt{\omega' \omega''} |C_{\omega'' \omega'}^{\omega'' \omega'} C_{\omega'' \omega}^{\omega'' \omega} \tilde{\alpha}_{\omega''} \tilde{\alpha}_{\omega'}| \quad (80)$$

Obviously the normal order operator leads to the appearance of a central term or anomaly in the diffeomorphism algebra, i.e eq (69)

$$[L_\omega, L_{\omega'}] = (\omega - \omega') \sum_{\omega''} C_{\omega'' \omega'}^{\omega'' \omega'} L_{\omega''} + C(\omega, \omega') \quad (81)$$

where the central term $C(\omega, \omega')$ is a function of $\omega$ and $\omega'$. Here we calculate $C(\omega, \omega')$ for higher values of $\omega$ and $\omega'$. Using eq (73) and asymptotic behaviour of $C_{\omega'' \omega'}^{\omega'' \omega'}$ (70) for higher values of $\omega, \omega'$ and $\omega''$, $L_\omega$ takes the following form for $\omega > N$

$$L_\omega = \sum_{\omega' (\omega'' = N+1 \rightarrow \infty)}^{N} \sqrt{\omega' \omega''} |C_{\omega'' \omega'}^{\omega'' \omega'} : \tilde{\alpha}_{\omega'} \tilde{\alpha}_{\omega''} : + \sum_{\omega' = -\infty}^{-N} \sqrt{\omega' (\omega + \omega')} : \tilde{\alpha}_{\omega'} \tilde{\alpha}_{-\omega - \omega'} : \quad (82)$$

Here $N$ is large enough so that we can take the asymptotic form of $C_{\omega' \omega''}^{\omega'' \omega'}$ (70) for $\omega, \omega'$ and $\omega''$ larger than $N$. Notice that $N$ is the root of (11) after which the roots approaches the roots of $\cos \omega (2\pi - \theta)$ such that “sum over energies” approach to “sum over integers”. Thus for $\omega_1 < N$ and $\omega > N$ we get

$$[\tilde{\alpha}_{\omega_1}, L_\omega] = 2 \sum_{\omega' = -N+1}^{N} \sqrt{\omega_1 \omega'} |C_{\omega_1 \omega'}^{\omega_1 \omega'} \tilde{\alpha}_{\omega'} + \sqrt{\omega_1 (\omega - \omega_1)} \tilde{\alpha}_{\omega_1 - \omega} \quad (83)$$

and for $\omega_1 > N$ and $\omega > N$ we have

$$[\tilde{\alpha}_{\omega_1}, L_\omega] = \{ \Theta(\omega)[3\Theta(\omega_1) + 2\Theta(-\omega_1)] + \Theta(-\omega)[2\Theta(\omega_1) + 3\Theta(-\omega_1)] \} \omega_1 \sqrt{|\omega_1 (\omega - \omega_1)|} \tilde{\alpha}_{\omega_1 - \omega} \quad (84)$$

where $\Theta$ is the step function. Finally, after a rather lengthy calculation (for simplicity we assume $\omega, \omega' > 0$) and using the asymptotic form of $L_\omega$ for higher values of $\omega$, together with the relations (83), (84) and (79) the diffeomorphism algebra given in (81) takes the following form for higher $\omega$ and $\omega'$

$$[L_\omega, L_{\omega'}] = (\omega - \omega') L_{\omega + \omega'} + C(\omega, \omega') \quad (85)$$

with

$$C(\omega, \omega') = \delta_{\omega + \omega', 0} f(\omega, \omega', \theta) - 4 \sum_{\omega_1, \omega_2 > 0} N \omega_1 \omega_2 C^{\omega_1 \omega_2}_{\omega_1, \omega_2} C^{\omega'}_{\omega_1, \omega_2} + \sum_{\omega_1 > 0} N \omega_1 (\omega - \omega_1)^2 C^{\omega'}_{\omega_1, \omega_1 - \omega_1} \quad (86)$$

where

$$f(\omega, \omega', \theta) = 3 \sum_{l = -n}^{0} [-2(l + k)a - \omega - \omega'] [2(l + k)a + \omega]^{2} [((2l + k)a + \omega + \omega')(2(l + k)a)]^{1/2}$$

and

$$N = (2k - 1)a, \quad \omega = (2n - 1)a, \quad a = \frac{\pi}{2(2\pi - \theta)}$$

with $k$ and $n$ as integers.
9 Conclusion

We have considered a two dimensional model in which space-time is a cylinder (circle $\times$ real number) where the circle represents “space” and the real line represents “time”. A segment of the circle is fixed with an angle $\theta$ and it is assumed that, inside the corresponding infinitely long strip-shaped region of the cylinder, the metric is of Euclidean type (ET) signature. On one hand, such situations are unrealistic: what happens, when an observer tries to enter the (ET) region? On the other hand, the size of (ET) region might be very small (planck scale?), and quantum effects might kill (macroscopic) inconsistencies. Nevertheless it is mathematically interesting to investigate which kinds of effects a signature changing metric of this type would produce. Hence it is worth exploring this model. For the sake of definiteness one may put a simple field on this manifold. We have chosen a free massless scalar field and have obtained the result that the energy spectrum, diffeomorphism algebra and the central term are modified as compared to the standard case (pure Lorentzian signature). At least at formal level, these modifications are reminiscent of a tunneling effect related to the (ET) region [4]. The modifications of spectrum and diffeomorphism algebra tend to disappear for higher values of energies but not the central term. So it appears that the presentation of signature change affects drastically the ordinary boson model at quantum level.
Appendix A

In order to obtain the structure constants \( O_{\omega\omega'} \), given in (65) we note that the left hand side of (65) is time independent. Therefore, the right hand side must also be time independent. This is possible only if a condition be imposed on the product \( ff \), analogous to the condition (55) on \( f \), as follows

\[
\tilde{f}^E(+) \tilde{f}^E(+) + \tilde{f}^E(-) \tilde{f}^E(-) \mid_\Sigma = \tilde{f}^L(+) \tilde{f}^L(+) + \tilde{f}^L(-) \tilde{f}^L(-) \mid_\Sigma \quad (A - 1)
\]

Since \( O_{\omega\omega'} \) is time independent, then we can find them at the hypersurface \( \tau = const \). So we take the following expansions for \( \tilde{f}^E(\sigma) \), \( \tilde{f}^E(\sigma) \) and \( \tilde{f}^L(\sigma) \), \( \tilde{f}^L(\sigma) \) respectively

\[
\tilde{f}^E(\sigma) \tilde{f}^E(\sigma) = \sum_{\omega''} O_{\omega\omega''} \tilde{f}^E(\sigma) \quad (A - 2)
\]

\[
\tilde{f}^L(\sigma) \tilde{f}^L(\sigma) = \sum_{\omega''} O_{\omega\omega''} \tilde{f}^L(\sigma) \quad (A - 3)
\]

where the coefficients \( O_{\omega\omega'} \) are assumed to be the same in both (ET) and Lorentzian regions and the functions \( f(\sigma) \), \( s \) are \( \sigma \) dependent terms in (56). Since \( \tilde{f}^E(\sigma) \) are real and \( \tilde{f}^L(\sigma) \) are complex, we have

\[
O_{\omega\omega'}^{\ast} = O_{\omega\omega'} = O_{-\omega'\omega'} \quad (A - 4)
\]

it is easily seen from eqs (20) and (56) that

\[
\tilde{f}^E(\sigma) + \tilde{f}^E(\sigma) = (\tilde{f}^E(\sigma) + \tilde{f}^E(\sigma)) \quad (A - 5)
\]

\[
\tilde{f}^L(\sigma) + \tilde{f}^L(\sigma) = (\tilde{f}^L(\sigma) + \tilde{f}^L(\sigma)) \quad (A - 6)
\]

with some algebra and with the help of (2-6-A), we get

\[
\tilde{f}^E(\sigma) \tilde{f}^E(\sigma) + \tilde{f}^E(\sigma) \tilde{f}^E(\sigma) = \sum_{\omega''} (O_{\omega\omega''} + O_{-\omega'\omega''}) \tilde{f}^E(\sigma) \quad (A - 7)
\]

\[
\tilde{f}^L(\sigma) \tilde{f}^L(\sigma) + \tilde{f}^L(\sigma) \tilde{f}^L(\sigma) = \sum_{\omega''} (O_{\omega\omega''} + O_{\omega\omega''}) \tilde{f}^L(\sigma) \quad (A - 8)
\]

Now, we multiply both sides of (7 - A) and (8 - A) by \( \tilde{f}^E(\sigma) \) and \( \tilde{f}^L(\sigma) \) and integrate both sides from 0 to \( \theta \) and \( \theta \) to \( 2\pi \), respectively. By adding these relations, and using the orthogonality of \( \tilde{f}(\sigma) \), we get

\[
O_{\omega\omega''} + O_{-\omega'\omega''} = \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}^E(\sigma) \tilde{f}^E(\sigma) + \tilde{f}^E(\sigma) \tilde{f}^E(\sigma) d\sigma + \int_0^{2\pi} \tilde{f}^L(\sigma) \tilde{f}^L(\sigma) + \tilde{f}^L(\sigma) \tilde{f}^L(\sigma) d\sigma \quad (A - 9)
\]

On the other hand, in addition to the junction condition (55), one can find another condition over the functions \( \tilde{f}^E(\sigma) \) and \( \tilde{f}^L(\sigma) \) (56), namely

\[
\tilde{f}^E(+) - \tilde{f}^E(-) \mid_\Sigma = i(\tilde{f}^L(+) - \tilde{f}^L(-)) \mid_\Sigma \quad (A - 10)
\]
Using (2, 3 - A) and (5, 6 - A), one can show that

\[ j^E_\omega(\sigma) \tilde{j}^E_\omega(\sigma) - j^{-E}_\omega(\sigma) \tilde{j}^{-E}_\omega(\sigma) = \sum_{\omega''} \frac{1}{\omega''} (O^\omega_{\omega''} - O^{-\omega'}_{\omega''}) \partial_\sigma \tilde{\phi}^E_\omega(\sigma) \]  \hspace{1cm} (A-11)

\[ -i(j^L_\omega(\sigma) \tilde{j}^L_\omega(\sigma) - j^{-L}_\omega(\sigma) \tilde{j}^{-L}_\omega(\sigma)) = \sum_{\omega''} \frac{1}{\omega''} (O^\omega_{\omega''} - O^{-\omega'}_{\omega''}) \partial_\sigma \tilde{\phi}^L_\omega(\sigma) \]  \hspace{1cm} (A-12)

Now, we multiply both sides of (11 - A) and (12 - A) by \( \partial_\sigma \tilde{\phi}^E_\omega(\sigma) \) and \( \partial_\sigma \tilde{\phi}^L_\omega(\sigma) \) respectively and integrate in the same way as before. Then, subtracting the two integrated expressions, the surface terms cancel each other by using (10 - A) and (1 - A). Using the orthogonality of \( \tilde{\phi}_\omega(\sigma) \), we get

\[ O^\omega_{\omega''} - O^{-\omega'}_{\omega''} = \frac{\omega + \omega'}{2\pi \omega''} \left[ \phi_{\omega''} + <\phi_{\omega''}, \phi_{-\omega''}> \right] \{ \int_0^\theta \tilde{\phi}^E_{\omega''}(\sigma)(j^E_\omega(\sigma) \tilde{j}^E_\omega(\sigma) + j^{-E}_\omega(\sigma) \tilde{j}^{-E}_\omega(\sigma))d\sigma \right. \\
\left. + \int_0^{2\pi} \tilde{\phi}^L_{\omega''}(\sigma)(j^L_\omega(\sigma) \tilde{j}^L_\omega(\sigma) + j^{-L}_\omega(\sigma) \tilde{j}^{-L}_\omega(\sigma))d\sigma \right\} \]  \hspace{1cm} (A-13)

Finally, using the relations (9 - A) and (13 - A), we obtain the following expression for the coefficients given in (66)

\[ O^\omega_{\omega''} = \frac{\omega + \omega' + \omega''}{4\pi \omega''} \left[ \phi_{\omega''} + <\phi_{\omega''}, \phi_{-\omega''}> \right] \{ \int_0^\theta \tilde{\phi}^E_{\omega''}(\sigma)(j^E_\omega(\sigma) \tilde{j}^E_\omega(\sigma) + j^{-E}_\omega(\sigma) \tilde{j}^{-E}_\omega(\sigma))d\sigma \right. \\
\left. + \int_0^{2\pi} \tilde{\phi}^L_{\omega''}(\sigma)(j^L_\omega(\sigma) \tilde{j}^L_\omega(\sigma) + j^{-L}_\omega(\sigma) \tilde{j}^{-L}_\omega(\sigma))d\sigma \right\} \]  \hspace{1cm} (A-14)

one can show from (14-A) or (2,3-A) together with (56) that the limits of \( \omega, \omega' \to \infty \) tend to the asymptotic form

\[ O^\omega_{\omega''} \approx \sqrt{\frac{\omega \omega'}{\omega''}} \delta_{\omega + \omega'} \]  \hspace{1cm} (A-15)

For \( \omega'' = 0 \), we rewrite (7-A) and (8-A) as

\[ j^E_\omega(\sigma) \tilde{j}^E_\omega(\sigma) + j^{-E}_\omega(\sigma) \tilde{j}^{-E}_\omega(\sigma) = \sum_{\omega''} (O^\omega_{\omega''})(\tilde{\phi}^E_{\omega''}(\sigma) + \tilde{\phi}^{-E}_{-\omega''}(\sigma)) \]  \hspace{1cm} (A-16)

\[ j^L_\omega(\sigma) \tilde{j}^L_\omega(\sigma) + j^{-L}_\omega(\sigma) \tilde{j}^{-L}_\omega(\sigma) = \sum_{\omega''} (O^\omega_{\omega''})(\tilde{\phi}^L_{\omega''}(\sigma) + \tilde{\phi}^{-L}_{-\omega''}(\sigma)) \]  \hspace{1cm} (A-17)

where (4-A) has been used. By integrating both sides of (16-A) and (17-A) from 0 to \( \theta \) and \( \theta \) to 2\( \pi \) respectively, and adding these two relations and using the orthogonality of \( \tilde{\Phi}_\omega \) and \( \tilde{\Phi}_0 \) we finally get

\[ O^0_{\omega\omega'} = \frac{1}{4\pi} \{ \int_0^\theta (j^E_\omega(\sigma) \tilde{j}^E_\omega(\sigma) + j^{-E}_\omega(\sigma) \tilde{j}^{-E}_\omega(\sigma))d\sigma + \int_0^{2\pi} (j^L_\omega(\sigma) \tilde{j}^L_\omega(\sigma) + j^{-L}_\omega(\sigma) \tilde{j}^{-L}_\omega(\sigma))d\sigma \} \]  \hspace{1cm} (A-18)

In the limits of \( \omega, \omega' \to \infty \), equations (2.3-A) together with (56),(57) or (18-A) with the help of the orthogonality of \( \tilde{\Phi}_{\omega + \omega'} \) and \( \tilde{\Phi}_0 \) give the asymptotic form
\[ O_{0 \omega}^0 \approx \delta_{\omega + \omega'} \sqrt{\omega \omega'} \quad (A-19) \]

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**References**


### Table 1  \( \theta = 1.0000000000E-02 \) rad

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