A simple method for multi-leg loop calculations 2: a general algorithm

R. Pittau
Theoretical Physics Division, CERN
CH-1211 Geneva 23, Switzerland

Abstract

The method introduced in a previous paper to simplify the tensorial reduction in multi-leg loop calculations is extended to generic one-loop integrals, with arbitrary internal masses and external momenta.
1 Introduction

In a previous paper [1], a technique was presented to simplify the tensorial reduction of $m$-point one-loop diagrams of the type

\[
\mathcal{M}(p_1, \cdots, p_r; k_1, \cdots, k_{m-1}) = \sum_a \int d^n q \frac{\text{Tr}^{(a)}[\hat{g} \cdots \hat{g} \cdots]}{D_1 \cdots D_m},
\]

where $p_1, \cdots, p_r$ are the external momenta of the diagram, $k_1, \cdots, k_{m-1}$ the momenta in the loop denominators, defined as

\[
D_i = (q + s_{i-1})^2 - m_i^2, \quad s_i = \sum_{j=0}^i k_j \quad (k_0 = 0),
\]

and $\text{Tr}^{(a)}$ traces over $\gamma$ matrices, which may contain an arbitrary number of $\hat{g}$’s.

It was shown that, by assuming at least two massless momenta in the set $k_1, \cdots, k_{m-1}$, the traces in eq. (1) can be rewritten in terms of the denominators appearing in the diagram, therefore simplifying the calculation.

Starting from $m$-point rank-$l$ tensor integrals, the algorithm gave at most rank-1 $m$-point functions, plus $n$-point rank-$p$ tensor integrals with $n < m$ and $p < l$.

In this paper, I show how to extend this technique when the momenta $k_1, \cdots, k_{m-1}$ are generic. On the one hand, this allows to apply the method to more general problems. On the other hand, the reduction procedure can therefore be iterated in such a way that, usually, only rank-1 integrals and scalar functions remain at the end.

In the next section, I introduce the algorithm and in section 3, I apply it to a specific example.

2 The general algorithm

The basic idea is simple. Given two vectors $\ell_1$ and $\ell_2$, one can ‘extract’ the $q$ dependence from the traces with the help of the identity

\[
\hat{g} = \frac{1}{2(\ell_1 \cdot \ell_2)} [2(q \cdot \ell_2) \ell_1 + 2(q \cdot \ell_1) \ell_2 - \ell_1 \ell_2 - \ell_2 \ell_1].
\]
By further assuming \( \ell_1^2 = \ell_2^2 = 0 \), and making use of the completeness relations for massless spinors, the following result is obtained

\[
\text{Tr}[q \Gamma] = \frac{1}{2(\ell_1 \cdot \ell_2)} \left[ 2(q \cdot \ell_2) \text{Tr}[\ell_1 \Gamma] - \{q\}^+_{1,2} \{\Gamma\}^+_{1} - \{q\}^{-+}_{1,2} \{\Gamma\}^{-+}_{1} + (\ell_1 \leftrightarrow \ell_2) \right],
\]

(4)

where \( \Gamma \) represents a generic string of \( \gamma \) matrices and

\[
\{\ell_1 \ell_2 \cdots \ell_n\}^{+}_{i,j} \equiv \{12 \cdots n\}^{+}_{i,j} \equiv \bar{v}_+(\ell_i) f_1 f_2 \cdots f_n u_-(\ell_j). \quad (5)
\]

By iteratively applying the above procedure, together with the equations [1]

\[
\{q\}^{-+}_{1,2} \{q\}^{+}_{1,2} = 4(q \cdot \ell_1)(q \cdot \ell_2) - 2q^2(\ell_1 \cdot \ell_2)
\]

(6)

only one \( \{q\}^{+}_{1,2} \) (or its complex conjugate \( \{q\}^{-+}_{1,2} \)) survives in each term, and powers of \( q^2, (q \cdot \ell_1), (q \cdot \ell_2) \) and \( (q \cdot b) \) factorize out.

The next step is to reconstruct the denominators from the above scalar products. By choosing, for example, \( b = k_3 \) one trivially gets

\[
\begin{align*}
q^2 & = D_1 + m_1^2, \\
2(q \cdot b) & = D_4 - D_3 + m_4^2 - m_3^2 - (k_1 + k_2 + k_3)^2 + (k_1 + k_2)^2,
\end{align*}
\]

(7)

but \( (q \cdot \ell_1) \) and \( (q \cdot \ell_2) \) still remain.

In ref. [1] the simple case was studied in which the diagram in eq. (1) is such that at least two \( k \)'s (say \( k_1 \) and \( k_2 \)) are massless. A solution to the problem is then to take \( \ell_1 = k_1 \) and \( \ell_2 = k_2 \):

\[
\begin{align*}
2(q \cdot \ell_1) & = D_2 - D_1 + m_2^2 - m_1^2, \\
2(q \cdot \ell_2) & = D_3 - D_2 + m_3^2 - m_2^2 - (k_1 + k_2)^2.
\end{align*}
\]

(8)

If, in the set \( k_1 \cdots m-1 \), only one momentum (say \( k_1 \equiv \ell_1 \)) is massless, a solution can still be found by decomposing any other massive momentum (say \( k_2 \)) in terms of massless vectors:

\[
k_2 = \ell_2 + \alpha \ell_1.
\]

(9)
The requirement that also $\ell_2$ is massless, implies
\[
\alpha = \frac{k_2^2}{2(k_1 \cdot k_2)},
\]
and therefore
\[
2(q \cdot \ell_1) = D_2 - D_1 + m_2^2 - m_1^2,
\]
\[
2(q \cdot \ell_2) = D_3 - (1 + \alpha)(D_2 + m_2^2) + \alpha(D_1 + m_1^2) + m_3^2 - (k_1 + k_2)^2.
\]
When there are no massless $k$'s, a basis of massless vectors can yet be constructed:
\[
k_1 = \ell_1 + \alpha_1 \ell_2, \quad k_2 = \ell_2 + \alpha_2 \ell_1.
\]
In fact, requiring $\ell_1^2 = \ell_2^2 = 0$ gives
\[
\alpha_1 = \frac{(k_1 \cdot k_2) \pm \sqrt{\Delta}}{k_2^2}, \quad \alpha_2 = \frac{(k_1 \cdot k_2) \pm \sqrt{\Delta}}{k_1^2},
\]
\[
\ell_1 = \beta(k_1 - \alpha_1 k_2), \quad \ell_2 = \beta(k_2 - \alpha_2 k_1),
\]
\[
\Delta = (k_1 \cdot k_2)^2 - k_1^2 k_2^2, \quad \beta = \frac{1}{1 - \alpha_1 \alpha_2},
\]
from which one computes
\[
\frac{2(q \cdot \ell_1)}{\beta} = (1 + \alpha_1)(D_2 - k_1^2 + m_2^2) - (D_1 + m_1^2)
\]
\[
- \alpha_1[D_3 + m_3^2 - (k_1 + k_2)^2],
\]
\[
\frac{2(q \cdot \ell_2)}{\beta} = D_3 + \alpha_2(D_1 + m_1^2) - (1 + \alpha_2)(D_2 - k_1^2 + m_2^2)
\]
\[
+ m_3^2 - (k_1 + k_2)^2.
\]
When the loop integrals have to be evaluated in $n$ dimensions, the substitution $q \rightarrow q \equiv q + \tilde{q}$ is needed \cite{1, 2}, where $q$ lives in 4 dimensions and $\tilde{q}$ is the $(n - 4)$-dimensional part of the integration momentum, such that $(q \cdot \tilde{q}) = 0$. The only change in the previous formulas is that
\[
q^2 = D_1 - \tilde{q}^2 + m_1^2,
\]
and the additional integrals, involving powers of $\tilde{q}^2$, can be easily handled as shown in ref. [1, 3].

Therefore, the described procedure completely solves the problem, for arbitrary $k$’s appearing in the denominators of $n$-dimensional one-loop diagrams.

If, in the original trace, the number $n_q$ of $\not{q}$’s is less than the number $m$ of loop denominators, the algorithm can be iterated until rank-1 functions remain, at most. If $n_q \geq m$, owing to the lack of momenta $k$’s to perform the denominator reconstruction, residual rank-$p$ two-point integrals remain instead, with $p \leq (2 + n_q - m)$. However, two-point tensors are much easier to handle than generic $m$-point tensors, so that the diagram is anyhow simplified.

A last remark is in order. When some $k$’s become collinear, one is faced with the usual problem of singularities generated by the tensor reduction (for an exhaustive study of this topic, see ref. [4]). In fact, denominators appear in eqs. (4) and (6), which may vanish, and the quantity $\Delta$ in eq. (13) is nothing but a Gram determinant. Even if the occurrence of such singularities cannot be completely avoided, a better control on them is in general possible [1], with respect to traditional techniques [5]. In addition, the analytic expressions can be kept rather compact, avoiding, at the same time, the appearance of large-rank tensors.

3 An example

To illustrate the method, I compute the reduction for the following integral with $n_q = 2$:

$$I = \int d^n q \frac{1}{D_1 \cdots D_m} \text{Tr}[q \Gamma q \Lambda],$$

(16)

where, to fix the ideas, $\Gamma$ and $\Lambda$ are strings containing an odd number of four-dimensional $\gamma$ matrices. For convenience of notation, I omit to write the slashes in the traces.

Since the integration is performed in $n$ dimensions, the denominators are given by eq. (2) with the substitution $q \to q = \tilde{q} + \tilde{q}$.

When $m \geq 3$, the algorithm reduces $I$ to a sum of scalar and rank-1
integrals. In fact, by splitting $q$ in the numerator, one gets
\[ \text{Tr}[q \Gamma q \Lambda] = \text{Tr}[q \Gamma q \Lambda] - \tilde{q}^2 \text{Tr}[\Gamma \Lambda], \quad (17) \]
and, by applying the formulas in the previous section,
\[
\text{Tr}[q \Gamma q \Lambda] = \frac{1}{2(\ell_1 \cdot \ell_2)} \left[ 2(q \cdot \ell_1) E(\ell_2) + 2(q \cdot \ell_2) E(\ell_1) - q^2 A - 2(q \cdot k_3) G \right],
\]
\[
A = 2 \text{Re} \left[ \{ \Lambda \} \overline{\{ \Gamma \}} + \{ \Lambda \} \overline{\{ \Gamma \}} - C \{ k \} \right],
\]
\[
G = 2 \text{Re} \left[ C \{ q \} \right],
\]
\[
C = \frac{1}{k_3^n} \left[ \{ \Lambda \} \overline{\{ \Gamma \}} + \{ \Lambda \} \overline{\{ \Gamma \}} - C \{ k \} \right],
\]
\[
E(\ell) = \text{Tr}[\ell \Gamma q \Lambda] - \frac{1}{2(\ell_1 \cdot \ell_2)} \left\{ \text{Tr}[\ell_2 q \ell_1 \Gamma \ell \Lambda] + \text{Tr}[\ell_1 q \ell_2 \Gamma \ell \Lambda] - 2(k_3 \cdot \ell) G - (q \cdot \ell) A \right\}. \quad (18)
\]
The above equations give the final answer:
\[
I = \frac{1}{2(\ell_1 \cdot \ell_2)} \int d^nq \frac{1}{D_1 \cdots D_m} \left\{ (D_1 + m_1^2) \left[ E(\beta \alpha_2 \ell_1 - \beta \ell_2) - A \right] + (D_2 + m_2^2 - k_1^2) E(\beta \ell_2 + \beta \alpha_1 \ell_2 - \beta \ell_1 - \beta \alpha_2 \ell_1) \right. \\
+ \left. (D_3 + m_3^2 - (k_1 + k_2)^2) \left[ E(\beta \ell_1 - \beta \alpha_1 \ell_2) + G \right] - (D_4 + m_4^2 - (k_1 + k_2 + k_3)^2) G + \tilde{q}^2 (A - 2(\ell_1 \cdot \ell_2) \text{Tr}[\Gamma \Lambda]) \right\},
\]
\[
\beta = \frac{1}{{1 - \alpha_1 \alpha_2}}. \quad (19)
\]
When $k_{1,2}^2 \neq 0$, $\ell_{1,2}$ and $\alpha_{1,2}$ are as in eq. (13).
If $k_1^2 = 0$ and $k_2^2 \neq 0$, eq. (19) still holds with $\alpha_1 = 0$, $\ell_1 = k_1$ and $\ell_2 = k_2 - \alpha_2 k_1$, where $\alpha_2 = \alpha$ is given in eq. (10).
If $k_{1,2}^2 = 0$, then $\alpha_{1,2} = 0$ and $\ell_{1,2} = k_{1,2}$.
When $m = 3$, some terms vanish. This implies $C = G = 0$ and
\[
E(\ell) = \text{Tr}[\ell \Gamma q \Lambda] + A \frac{(q \cdot \ell)}{2(\ell_1 \cdot \ell_2)}. \quad (20)
\]
4 Summary

In this paper, I extended the technique introduced in ref. [1] to reduce the tensorial complexity of the diagrams appearing in multi-leg loop calculations. The method is now applicable to generic one-loop integrals, with arbitrary internal masses and external momenta.

The algorithm can usually be iterated in such a way that only scalar and rank-1 functions appear at the end of the reduction. At worst, higher-rank two-point tensors survive, independently from the initial number of denominators.

References