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The Classical and Quantum Analysis of the Relativistic Motion of a Massive Bosonic Particle on a Conical Space-time

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Abstract

In this work we analyse from the classical and quantum point of view the relativistic motion of a massive bosonic particle on the space-time produced by a static, cylindrically symmetric cosmic string. To do that we develop a complete geometric approach to the problem of gravitational self-interaction. We obtain classical orbits and quantum states related with the particle's motion.

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1 Introduction

The classical and quantum analysis of the motion of a massive particle on a conical space-time has attracted attention in recent years and has been considered from different points of view in the nonrelativistic limit[1, 2].

In this work we study the classical and quantum relativistic motion of a massive spin-0 particle in the conical space-time. We develop a complete geometric approach to the problem of gravitational self-interaction in order to consider a relativistic motion. To do that we calculate the corrections, due to the presence of a point-like massive particle, not only to the g_{00} component of the string metric tensor but to all other components too. The interval in corresponding a background space-time is given by

$$ds^2 = -dt^2 + dr^2 + \alpha^2 r^2 d\varphi^2 + dz^2, \quad (1)$$

where $\alpha = 1 - 4\mu G < 1$ is the parameter which codify the presence of the cosmic string.

A few years ago Linet[3] has found that a test charged particle placed at rest outside the string becomes subject to a repulsive electrostatic self-interaction. This effect is due to distortion in the particle's own electromagnetic field caused by the lack of global flatness of the conical space-time. Also Smith[4] has found similar phenomenon for the gravitational self-interaction in the Newtonian limit¹. In their calculations they assumed that the test particle does not disturb the string's metric. So the electrostatic and Newtonian potentials produced by this charged and massive particle in the conical space-time φ_{Elec} and φ_{Grav} , respectively, obey the equations below:

$$\nabla^2 \varphi_{Elec}(\vec{r}) = -4\pi q \delta^{(3)}(\vec{r} - \vec{r}_o)$$

and

$$\nabla^2 \varphi_{Grav}(\vec{r}) = 4\pi m G \delta^{(3)}(\vec{r} - \vec{r}_o).$$

Here $\vec{r}_o$ is the position of the particle and q and m are its charge and mass,

$$\nabla^2 = \frac{1}{r} \partial_r r \partial_r + \frac{1}{\alpha^2 r^2} \partial_\varphi^2 + \partial_z^2$$

¹In this limit the time-time component of the metric tensor has the form $g_{00} = 1 + 2\varphi(\vec{r})$, where $\varphi(\vec{r})$ plays the role of the Newtonian potential.

is the Laplace operator in this geometry in cylindrical coordinates.

The self-interaction can be obtained observing that both solutions, for φ_{Elec} and φ_{Grav} in the neighbourhood of the point particle can always be written as the sum of two terms[3]: $\varphi(r) = \varphi_I + \varphi_R$, where the first term, φ_I , is irregular at the position of the particle, whereas the second term φ_R is regular. Ignoring the infinite contributions due to φ_I , because this space is locally flat, it remains φ_R which can be considered as an "external" electrostatic or gravitational potential, exerting a force on the charge q or mass m .

Gibbons et al[2] have studied, by the use of Hamilton-Jacobi formalism and the Schrödinger equation, the classical and quantum motion of a massive or charged particle in the space-time produced by a cosmic string. They considered the electrostatic or gravitational self-interaction as an external potential acting on the particle.

In this work we return to their analysis studying the influence of the self-gravitational field on the test massive particle moving in a conical space-time produced by a cosmic string. In distinction to Ref.[2] a more complete treatment is made here which takes into account not only the correction in the time-time component of the metric tensor, which is the Newtonian limit due to a massive particle placed in this geometry, but also the corrections in the other components which become relevant in a relativistic treatment.

So, we construct a set of linear differential equations from the Einstein equation in the weak field approximation, involving all the corrections to the background metric (1) due to a static point massive particle. We obtain a class of solutions and adopt the regularization procedure given in previous papers[3, 4] to obtain the self-interacting gravitational field and consequently the metric tensor associated with the effective manifold in which the massive particle is moving.

This paper is organized as follows: In section 2 we develop the weak field approximation technique around the cosmic string background metric and apply this formalism to calculate the correction due to a static and massive point particle placed at some position outside the string. We also present a particular solution for the set of inhomogeneous linear differential equations involving these corrections and obtain, after we have applied the regularization procedure above mentioned, the particle's self-gravitational field and consequently the effective space-time metric which this particle sees. In section 3 we use the Hamilton-Jacobi formalism to study the relativistic classical motion of the particle in this effective geometry. In section 4 we

study the quantum motion of this particle by the use of the Klein-Gordon equation. We obtain bound and scattering quantum states. Finally our discussions and conclusions are left for the section 5.

2 First Order Correction About the Cosmic String Metric

In this section we obtain the first order correction to the cosmic string metric, considered as our background one, due to the presence of a massive particle placed in its neighbourhood. In order to do that we shall assume that the modified metric tensor is now given by the form

$$g_{\mu\nu} = g_{\mu\nu}^{(s)} + h_{\mu\nu}, \quad (2)$$

where $g_{\mu\nu}^{(s)}$ is the cosmic string metric tensor defined in (1) and $h_{\mu\nu}$ represents a very small time-independent perturbation which goes to zero very far from the body.

Now we want to solve the Einstein equation

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{g_{\mu\nu}}{2}T) \quad (3)$$

at linearized level in order to obtain the first-order metric perturbation, admitting that the energy-momentum tensor takes into account the presence of a static massive point particle located at the position $\vec{r}_o$.

The extra energy-momentum tensor which corresponds to the static source is

$$T_{\mu\nu}^{(1)} = m\delta^{(3)}(\vec{r} - \vec{r}_o)diag(1, 0, 0, 0). \quad (4)$$

Because we are considering a static source, the most general static space-time metric under the above mentioned situation can be written in the form:

$$ds^2 = -e^{2N_0}dt^2 + e^{2N_1}dr^2 + 2Hdrd\varphi + 2Wdrdz + \alpha^2r^2e^{2N_2}d\varphi^2 + 2Sd\varphi dz + e^{2N_3}dz^2, \quad (5)$$

where the seven unknown functions depend on the space coordinates (r, φ, z) .

The linearized Einstein equation (3) provides for the above space-time metric the following set of inhomogeneous linear differential equations: ²

$$\nabla^2 N_o(\vec{r}) = 4\pi m G \delta^{(3)}(\vec{r} - \vec{r}_o) \quad (6)$$

$$-\frac{d^2}{dr^2}(N_o + N_2 + N_3) + \frac{1}{r} \frac{dN_1}{dr} - \frac{1}{\alpha^2 r^2} \frac{d^2 N_1}{d\varphi^2} - \frac{d^2 N_1}{dz^2} - \frac{2}{r} \frac{dN_2}{dr} + \frac{1}{\alpha^2 r^2} \frac{d^2 H}{dr d\varphi} + \frac{d^2 W}{dr dz} = 4\pi m G \delta^{(3)}(\vec{r} - \vec{r}_o) \quad (7)$$

$$-\frac{d^2 N_o}{dr d\varphi} + \frac{1}{r} \frac{d}{d\varphi}(N_o + N_3) - \frac{d^2 N_3}{dr d\varphi} - \frac{1}{r} \frac{dS}{dz} + \frac{1}{2} \left(\frac{d^2 S}{dr dz} - \frac{d^2 H}{dz^2} + \frac{d^2 W}{d\varphi dz} \right) = 0, \quad (8)$$

$$-\frac{d^2}{dr dz}(N_o + N_2) + \frac{1}{r} \frac{d}{dz}(N_1 - N_2) + \frac{1}{2\alpha^2 r^2} \left(\frac{d^2 H}{d\varphi dz} + \frac{d^2 S}{dr d\varphi} - \frac{d^2 W}{d\varphi^2} \right) = 0, \quad (9)$$

$$-\frac{1}{r} \frac{d}{dr}(N_o - N_1 + 2N_2 + N_3) - \frac{d^2 N_2}{dr^2} - \frac{d^2 N_2}{dz^2} - \frac{1}{\alpha^2 r^2} \frac{d^2}{d\varphi^2}(N_o + N_1 + N_3) + \frac{1}{r} \frac{dW}{dz} + \frac{1}{\alpha^2 r^2} \left(\frac{d^2 S}{d\varphi dz} + \frac{d^2 H}{dr d\varphi} \right) = 4\pi m G \delta^{(3)}(\vec{r} - \vec{r}_o), \quad (10)$$

$$-\frac{d^2}{d\varphi dz}(N_o + N_1) + \frac{1}{2} \left(\frac{d^2 H}{dr dz} - \frac{d^2 S}{dr^2} + \frac{d^2 W}{dr d\varphi} \right) + \frac{1}{2r} \left(\frac{dH}{dz} + \frac{dS}{dr} - \frac{dW}{d\varphi} \right) = 0, \quad (11)$$

²The linearized equations were obtained using the algebraic computer program SHEEP[5].

and

$$\begin{aligned}
-\frac{d^2}{dz^2}(N_o + N_1 + N_2) - \frac{d^2 N_3}{dr^2} - \frac{1}{r} \frac{dN_3}{dr} - \frac{1}{\alpha^2 r^2} \frac{d^2 N_3}{d\varphi^2} + \frac{d^2 W}{dr dz} + \\
\frac{1}{r} \frac{dW}{dz} + \frac{1}{\alpha^2 r^2} \frac{d^2 S}{d\varphi dz} = 4\pi m G \delta^{(3)}(\vec{r} - \vec{r}_o).
\end{aligned} \tag{12}$$

Although this set of differential equations seems a simple one, we were unable to obtain a general solution for it. On the other hand a particular solution can be obtained, which corresponds a particular choice of the symmetry axis in the perturbed metric:

$$N_o(\vec{r}) = -N_1(\vec{r}) = -N_2(\vec{r}) = -N_3(\vec{r}) = N(\vec{r}) \tag{13}$$

and

$$S(\vec{r}) = H(\vec{r}) = W(\vec{r}) = 0. \tag{14}$$

In fact choosing the first condition (13) we get consequently (14) and vice-versa.

So, the new metric tensor, taking into account only the first order correction on the cosmic string background metric (1), is:

$$ds^2 = -(1 + 2N(\vec{r}))dt^2 + (1 - 2N(\vec{r}))(dr^2 + \alpha^2 r^2 d\varphi^2 + dz^2), \tag{15}$$

where the unknown function $N(\vec{r})$ obeys the Poisson equation

$$\nabla^2 N(\vec{r}) = 4\pi m G \delta^{(3)}(\vec{r} - \vec{r}_o). \tag{16}$$

The solution of this equation is already known[3, 4]:

$$N(\vec{r}) = -\frac{mG}{\alpha\pi\sqrt{2rr_o}} \int_{u_o}^{\infty} \frac{du}{(\cosh u - \cosh u_o)^{1/2}} \frac{\sinh(u/\alpha)}{\cosh(u/\alpha) - \cos(\frac{\theta-\theta_o}{\alpha})}, \tag{17}$$

with

$$\cosh u_o = \frac{r^2 + r_o^2 + (z - z_o)^2}{2rr_o}. \tag{18}$$

As it was pointed out by Linet[3], it is always possible to write the function $N(\vec{r})$ in the neighbourhood of the particle's position in the form

$$N(\vec{r}) = N_I(\vec{r}) + N_R(\vec{r}), \quad (19)$$

where the first term, N_I , is irregular at the particle's position and the second one, N_R , is regular. In order to obtain a well defined self-gravitational field at the particle's position, we shall adopt the standard regularization procedure: we discard the infinite contribution arising from the irregular term because our background space-time metric is locally Minkowsky. So the induced gravitational self-energy is

$$U_{Grav}(\vec{r}_o) = \frac{m}{2} \lim_{\vec{r} \rightarrow \vec{r}_o} [N(\vec{r}) - N_I(\vec{r})] = -\frac{m^2 G K(\alpha)}{2r_o}, \quad (20)$$

where

$$K(\alpha) = \frac{1}{\pi\alpha} \int_0^\infty \frac{\coth(x/\alpha) - \alpha \coth(x)}{\sinh x} dx \quad (21)$$

is a positive numerical factor[4]. So, from (20) and (21), we get that self-gravitational field is then

$$\varphi(r_o) = -\frac{M}{r_o} \quad (22)$$

with

$$M = \frac{m G K(\alpha)}{2}. \quad (23)$$

In our formalism $\varphi(r)$ should be understood as an external gravitational field producing, by its turn, a correction of the background (cosmic string) space-time metric where the particle is placed. So, we shall consider that the space-time metric where our particle is placed, is effectively described by the following line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 + \frac{2M}{r}\right) (dr^2 + \alpha^2 r^2 d\varphi^2 + dz^2), \quad (24)$$

which preserves the cylindrical symmetry. (We expect that even in the case that we have found a general solution for the set of linear differential equations presented above, the effective space-time metric seen by the particle should be described by (24).)

The above expression for the metric tensor is only acceptable as long as the first-order perturbation correction is smaller than ours, i.e., $2M/r \ll 1$.

Now, after we have derived (24), we proceed in the next two sections the analysis of the relativistic motion of a test particle near a cosmic string, taking into account the complete self-gravitational effect. We start first with the classical analysis.

3 Classical Motion

In this section we study the relativistic classical motion of a massive particle in the neighborhood of cosmic string taking into account its self-gravitational field represented by the metric tensor given in (24). In order to do that we shall use the Hamilton-Jacobi formalism, expressed by two sets of equations. The first one to be used is :

$$m \frac{dx^\mu}{d\lambda} = g^{\mu\nu} \frac{\partial S}{\partial x^\nu} \quad (25)$$

with λ being a parameter along the trajectory of the particle. The second set is given by

$$g^{\alpha\beta} \frac{\partial S}{\partial x^\alpha} \frac{\partial S}{\partial x^\beta} = -m^2. \quad (26)$$

For this system we shall employ the cylindrically symmetric ansatz below, ignoring the motion in the z-direction

$$S = Et + J\varphi + R(r), \quad (27)$$

where $R(r)$ is an unknown function, E and J are constants of motion.

From the component $\mu = 1$ of Eq. (25), combined with the Eqs. (26), (27) and using (24), we get

$$m^2 \left(1 + \frac{2M}{r}\right) \left(\frac{dr}{d\lambda}\right)^2 = \frac{E^2}{1 - \frac{2M}{r}} - \frac{J^2}{\alpha^2 r^2} \frac{1}{1 + \frac{2M}{r}} - m^2. \quad (28)$$

We also have from (25)

$$m\alpha^2 r^2 \frac{d\varphi}{d\lambda} = \frac{J}{1 + \frac{2M}{r}}. \quad (29)$$

Combining (28) and (29) and defining a new variable $u = \frac{1}{r}$ we get

$$(1 - 2Mu) \left(\frac{du}{d\varphi} \right)^2 = \frac{E^2 \alpha^4}{J^2} (1 + 2Mu) - \alpha^2 u^2 (1 - 2Mu) - \frac{m^2 \alpha^4}{J^2} (1 + 2Mu)(1 - 2Mu). \quad (30)$$

Unfortunately only an integral expression for $\varphi = \varphi(u)$ can be obtained and an approximate classical form for the orbits is not evident. (Similar situation also happens in the analysis of the motion of a massive particle in a Schwarzschild black-hole[6]). However the explicit expressions for the solutions of (30) can be obtained approximately if we neglect terms of order $(Mu)^3$ in this differential equation, which is acceptable in the weak field limit. So for this case we get:

$$\left(\frac{du}{d\varphi} \right)^2 = \frac{E^2 \alpha^4}{J^2} (1 + 4Mu + 8M^2 u^2) - \alpha^2 u^2 - \frac{m^2 \alpha^4}{J^2} (1 + 2Mu), \quad (31)$$

whose solutions are:

$$u(\varphi) = \frac{M\alpha^2(2E^2 - m^2)}{J^2 - 8M^2 E^2 \alpha^2} + \alpha \sqrt{\frac{E^2 - m^2}{J^2 - 8M^2 E^2 \alpha^2} + \frac{\alpha^2(2E^2 - m^2)^2 M^2}{(J^2 - 8M^2 E^2 \alpha^2)^2}} \times \cos \left[\frac{\alpha}{J} \sqrt{J^2 - 8M^2 E^2 \alpha^2} (\varphi - \varphi_0) \right], \quad (32)$$

for $J^2 > 8M^2 E^2 \alpha^2$,

$$u(\varphi) = -\frac{M\alpha^2(2E^2 - m^2)}{8M^2 E^2 \alpha^2 - J^2} + \alpha \sqrt{-\frac{E^2 - m^2}{8M^2 E^2 \alpha^2 - J^2} + \frac{\alpha^2(2E^2 - m^2)^2 M^2}{(8M^2 E^2 \alpha^2 - J^2)^2}} \times \cosh \left[\frac{\alpha}{J} \sqrt{8M^2 E^2 \alpha^2 - J^2} (\varphi - \varphi_0) \right], \quad (33)$$

for $J^2 < 8M^2 E^2 \alpha^2$ and

$$u(\varphi) = \frac{E^2 - m^2}{2(2E^2 - m^2)M} - \alpha^4 \frac{(2E^2 - m^2)M}{16M^2 E^2 \alpha^2} (\varphi - \varphi_0)^2, \quad (34)$$

for $J^2 \rightarrow 8M^2 E^2 \alpha^2$.

There is also a constant solution obtained by setting $\frac{du}{d\varphi} = 0$ in (25). This circular motion occurs in relativity theory just as in classical theory.

From the equations above we can see that trajectories described by (32) will be bounded below by some positive value if $E^2 < m^2$, so r will be bounded from above. If $E^2 > m^2$, r can be infinite. The trajectories described by (33) and (34) are spiral ones and r goes to zero when φ goes to infinity. In relativistic classical mechanics finite trajectories in general are not closed but rather rosette shaped[7]. These finite trajectories, given by (32), present period given by

$$T = \frac{2\pi J}{\alpha\sqrt{J^2 - 8M^2 E^2 \alpha^2}}, \quad (35)$$

which explicitly depends on the parameter α .

After this analysis about the classical solution for a test particle in the manifold given by (24), where exact solutions were obtained, we shall leave for the next section the relativistic quantum mechanical analysis for this system.

4 Quantum Motion

This section is devoted to the relativistic quantum analysis of bound and scattering states of a massive test particle in the space-time metric given by (24). In order to do that we shall use the Klein-Gordon equation written in a covariant form in this geometry

$$(\square - m^2)\psi(x) = 0 \quad (36)$$

with

$$\square\psi(x) = \frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}g^{\mu\nu}\partial_\nu\psi(x)], \quad (37)$$

$g = \det(g_{\mu\nu})$.

We adopt for the wave function the form below

$$\psi(x) = e^{i(J\varphi - \epsilon t)} R(r), \quad (38)$$

where, as in the classical analysis, we are again ignoring the motion in the z -direction. E and J are the energy and the azimuthal quantum number, respectively, associated with the particle's motion. The unknown function $R(r)$ obeys the radial Klein-Gordon equation

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dR(r)}{dr} \right) + \left[(E^2 - m^2) + \frac{2M}{r} (2E^2 - m^2) - \frac{J^2}{\alpha^2 r^2} \right] R(r) = 0, \quad (39)$$

where, in our development, we have kept only terms linear in (M/r) [8].

We shall analyse in this section bound and scattering states of (39). First we study bound solutions, which are present when $E^2 < m^2$. This class of solutions obey appropriate boundary condition and can be expressed in terms of the confluent hypergeometric functions by

$$R_j(r) = C_j e^{-\kappa r} r^{|J|/\alpha} {}_1F_1 \left(\frac{1}{2} + \frac{|J|}{\alpha} - \sigma, 1 + 2\frac{|J|}{\alpha}; 2\kappa r \right), \quad (40)$$

where $\kappa = \sqrt{m^2 - E^2}$ and $\sigma = M(m^2 - 2E^2)/\kappa$.

In order to obtain bound states, we have to choose appropriately the parameters to terminate the series in (40). Admitting a polynomial of degree n for the hypergeometric function, we must impose the condition

$$\sigma - \frac{1}{2} - \frac{|J|}{\alpha} = n. \quad (41)$$

With this condition we get discrete values for the self-energy given by

$$E_{nj}^{(\alpha)} = \frac{m}{\sqrt{2}} \left\{ 1 + \frac{(n + 1/2 + |J|/\alpha)}{4m^2 M^2} \left[\sqrt{1 + \frac{8m^2 M^2}{(n + 1/2 + |J|/\alpha)^2}} - 1 \right] \right\}^{1/2} \quad (42)$$

with $n = 0, 1, 2$ and $J = 0, \pm 1, \pm 2$.

Before finishing this analysis we would like to make a two comments about (42): (i) The first one refers to the degeneracy presented by our solution. When $\alpha = 1$ the energy levels depend on the integer quantum number $N = n + |J|$ and $(2N + 1)$ degeneracy degree would be present. (ii) The second one refers with the self-energy. As we can see $E_{nj}^{(\alpha)}$ depends on the parameter M^2 and consequently on the square of the gravitational constant G^2 . This indicates the low order of energy involved in this process.

Expanding (42) in powers of M^2 we get

$$E_{nj}^{(\alpha)} \simeq m \left[1 - \frac{m^2 M^2}{2(n + 1/2 + |J|/\alpha)^2} \right], \quad (43)$$

which reproduces the non-relativistic result given in Ref.[2] for the self-energy associated with a particle in the neighbourhood of a cosmic string taking into account the gravitational self-interaction as an external potential. This non-relativistic procedure is quite different from ours, where the complete correction to the background metric (24) was considered in a geometric approach in the relativistic Klein-Gordon equation.

After this discussion we now proceed the analysis for scattering states of the Klein-Gordon equation. Again, taking the radial equation (39) with $E^2 > m^2$, we get

$$R_j(r) = C_j e^{ikr} r^{|J|/\alpha} {}_1F_1 \left(\frac{1}{2} + \frac{|J|}{\alpha} - i\xi, 1 + 2\frac{|J|}{\alpha}; -2ikr \right), \quad (44)$$

where $k = \sqrt{E^2 - m^2}$ and $\xi = M(2E^2 - m^2)/k$.

Following the standard procedure⁽⁹⁾ we take the asymptotic form for the confluent hypergeometric function and give the long distance behavior for the radial function:

$$R_j(r) \approx \frac{C_j \Gamma(1 + 2|J|/\alpha) e^{-\pi\xi/2}}{|\Gamma(1/2 + |J|/\alpha - i\xi)|} \times \cos \frac{[kr - \pi/2(|J|/\alpha + 1/2) + \xi \ln(2kr) - \eta_j]}{(kr)^{1/2}}, \quad (45)$$

where $\eta_j = \arg \Gamma(1/2 + |J|/\alpha + i\xi)$.

From the above expression it is possible to obtain the phase shift δ_j , which is the most relevant parameter in the calculation of the scattering amplitude[9]

$$\delta_j = -\pi \frac{|J|}{2} (\alpha^{-1} - 1) - \eta_j. \quad (46)$$

It worth to mention that our above result for the phase shift presents two contributions: (i) The first one comes from the conical geometry of the manifold itself and coincides with previous result given in the paper

by Deser at all in Ref.[1]; (ii) the second one is a consequence of the self-gravitational interaction which is present in our formalism. It in turn, is an indirect consequence of the nontrivial topology associated with this space-time. So the effect of the self-interaction can be detected by measuring the phase shift δ_j .

Finally we can say that the extra phase, η_j , in (46) can be evaluated by the expansion[10]

$$\eta_j = \xi \Psi(1/2 + |J|/\alpha) + \sum_{n=0}^{\infty} \left[\frac{\xi}{1/2 + |J|/\alpha + n} - \arctan \frac{\xi}{1/2 + |J|/\alpha + n} \right] \quad (47)$$

where $\Psi(z) = \frac{d}{dz} \ln \Gamma(z)$ is the psi function. Taking $\arctan x \simeq x - x^3/3 + \dots$ in the equation above we get

$$\eta_j \simeq \xi \Psi(1/2 + |J|/\alpha) + O(\xi^3). \quad (48)$$

For large azimuthal quantum number we can use the asymptotic expansion for the psi function, $\Psi(z) = \ln z - 1/2z - \sum_{n=1}^{\infty} B_{2n}/2nz^{2n}$, B_n being the Bernoulli numbers. So in this limit we obtain

$$\eta_j \simeq \xi \left[\ln(1/2 + |J|/\alpha) - \frac{1}{1 + 2|J|/\alpha} + O\left(\frac{1}{(1/2 + |J|/\alpha)^2}\right) \right]. \quad (49)$$

5 Concluding Remarks

Although the cosmic string exerts no direct gravitational forces on the matter a gravitational self-interaction on the massive body is induced by this object. Recently this effect was calculated for a point test particle in the Newtonian limit[4]. The classical and quantum analysis of the motion of a massive particle in a conical space-time produced by a cosmic string becomes very rich. This is due to the nontrivial topology of this manifold and induced gravitational self-interaction, the latter being closely connected with nontrivial topology.

Here, in this paper, we decided to investigate the effect of the gravitational self-interaction in the relativistic classical and quantum motion. In order to do that we had to obtain the complete correction to the cosmic string background metric due to the presence of a static massive particle in this manifold. Assuming the most general static metric tensor, we derived from the Einstein equation at linearized level a set of inhomogeneous differential equations. We present a solution for this set and consequently the complete self-gravitational field, which enables us to adopt a geometric approach to the study the relativistic motion of the massive particle in the neighborhood of a cosmic string.

Our analysis of the relativistic classical and quantum motion were developed using the effective metric tensor given in (13). By using the Hamilton-Jacobi formalism we derived the classical orbits and obtained finite and infinite trajectories. Finally by using the Klein-Gordon equation we also derived the self-energy for bound states and the phase shift for scattering ones.

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- [8] The first order approximation in powers of M/r adopted in this section, is compatible with the adopted in the Hamilton-Jacobi formalism, where we kept terms quadratic in M/r . As we know the Klein-Gordon equation is linear on the field, on the other hand the Hamilton-Jacobi formalism is quadratic on the function S
- [9] Schiff L I 1968 *Quantum Mechanics* 3rd ed (Mc.Graw-Hill International)
- [10] Abramowitz M and Stegun I A 1972 *Hand book of Mathematical Functions* 9th ed (New York: Dover)