SELF-FORCES IN THE SPACETIME OF MULTIPLE COSMIC STRINGS

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Abstract

We calculate the electromagnetic self-force on a stationary linear
distribution of four-current in the spacetime of multiple cosmic strings.
It is shown that if the current is infinitely thin and stretched along
a line which is parallel to the strings the problem admits an explicit
solution.

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1 Introduction

The investigations in quantum and classical field theory in curved spacetime have shown that in many cases the local observables of the theory can not be expressed in terms of local geometry, but depend on the global structure of the space. This fact was interpreted as the manifestation of nonlocal (topological) influence of gravity on matter fields. The further researches have revealed that some of these nonlocal processes are not of pure theoretical interest, but may play an important role in modern particle physics, astrophysics, and cosmology [1, 2]. From the general point of view, nonlocal aspects are of fundamental importance in describing the physics of a given system.

Spaces with conical singularities are the new object of such an investigation. As a rule these spaces are associated with cosmic string spacetimes. The detailed survey of the problems associated with cosmic strings and an extensive list of references can be found in [3, 4]. As a matter of fact spaces with similar features appear in some other important physical applications. It is well known, for example, that some of the linear defects in crystalls, namely disclinations, are described geometrically as linear conical defects in a locally flat space [5]. Conical singularities appear in the off-shell calculations near the horizon in the Euclidean section of the Schwarzchild spacetime [6], and in connection with thermodynamics in the presence of cosmological horizons [7]. Some of conical spaces possess a trivial local geometry and this enables one to expose pure nonlocal effects in the gravitational interaction. In the particular case of the static straight infinitely thin cosmic string the spacetime looks like the direct product $M_2 \times \text{Cone}$ of the two-dimensional Minkowski space and a cone. The corresponding Riemann tensor vanishes everywhere except on the symmetry axis, where it has a $\delta$ - like singularity [8]. So, infinitely thin cosmic strings do not affect the local geometry of the spacetime but change its global properties, and therefore their effect on the matter fields is purely topological.

In recent years, the interest in cosmic strings has been associated with investigations in the quantum field theory mainly, and a lot of interesting quantum effects have been studied. The vacuum expectation values of the energy-momentum tensor for fields of different spins have been calculated and the phenomenon of vacuum polarization was considered in detail [9]. These calculations have also been extended to the case of finite temperatures.
Recently it was shown that the presence of other strings modifies the spectrum of vacuum fluctuations of a scalar field in such a way that a Casimir-like interaction arises between the strings, and two parallel cosmic strings are attracted to each other with a force that decreases with third power of the distance between the strings \cite{11}. Later the results of the paper \cite{11} were used to study the vacuum polarization of an electromagnetic field \cite{12}.

As it was mentioned above, the effect of the straight string on quantized fields is purely topological. One often says that it is due to the deformation of a Green function by the nontrivial global structure of the space. It is clear that similar effects must take place in the classical theory too if we consider both point particles and classical fields. Indeed in quantum and classical field theories we work with Green functions which are the solutions of the same equation. So, if some distortion of one of the functions, say Feynman one, is found, we must expect that the others will be distorted too. In fact some nontrivial classical effects which are due to the global distinction of the conical space from the Minkowski one have been found. The repulsion of a charged test particle from a cosmic string is an example of this type of effects \cite{13}. A similar effect was discovered for a gravitating pointlike particle \cite{14}. In the case of the linear distribution of a current the situation is somewhat more interesting because both electric and magnetic self-forces are induced \cite{15}.

In all the works mentioned previously, the effects were studied in the spacetime of a single linear defect. In typical cosmological scenarios, however, strings are generated in the form of random network of straight moving segments. Computer simulations show \cite{16} that there are about ten straight cosmic strings under the Hubble horizon at our era. Thus, the multistring spacetime is much more appropriate model for the real spacetime than the particular case of only one cosmic string. Another motivation comes from the solid state physics because the number of disclinations per unit volume in a real crystal exceeds the unity by far, and for this reason, it is necessary to consider spaces with two and more conical singularities. This problem is much more complicated than the problem with only one linear defect. Indeed the spacetime of a straight infinitely thin string possesses four Killing vectors, and three of them, namely $\partial/\partial t$, $\partial/\partial z$ and $\partial/\partial \phi$ give us the possibility to separate the variables in the field equation. The loss of the azimuthal symmetry, as it is in the multistring case, makes impossible to proceed along this line. One possible approach to this problem is associated with the use
of perturbation theory [11], another one is based on the possibility to solve some of the problems explicitly in low-dimensional gravity [17].

Here we present an explicitly solvable problem in the (3+1)-dimensional multistring spacetime. To our knowledge this is the first explicitly solvable nonlocal problem which has been considered on the multistring space till now. Our paper completes the investigations of topological effects in multistring spaces that were initiated in the papers [11, 18, 19].

The paper is organized as follows. In Sec.2 we develop formal expressions for the self-energy and self-force on a four-current placed in a spacetime which is the direct product of the two-dimensional Minkowski spacetime and a Riemannian surface. The regularization procedure is described in Sec.3. In Sec.4 we apply the results obtained in previous sections in order to consider the effect of topological self-action on a current in the spacetime of multiple cosmic strings. In Sec.5 we add some discussion and conclusion remarks about our results and their possible application.

2 Formal expressions for self-energy and self-force

Let us start with the metric of background spacetime which is supposed to be the direct product of the two-dimensional Minkowski space and a Riemannian surface. This is given by the interval

\[ ds^2 = dt^2 - dz^2 - \gamma_{ab} dx^a dx^b, \quad a, b, c = 1, 2. \]  

(1)

The multistring space is a particular case of the solution (1) when the section \( t = \text{const}, z = \text{const} \) is a locally flat surface with a set of conical singularities [20].

On this background we shall consider a static electromagnetic field which is associated with a stationary distribution of a four-current of the form

\[ j^\mu(x_1) = (J^t, 0, 0, J^z) \frac{\delta^2(x_1 - x)}{\sqrt{\gamma(x)}}, \]  

(2)

where the components \( J^t \) and \( J^z \) are the linear charge density and the electric current which are measured by an inertial observer with a fixed \( z \) coordinate.
The spacetime under consideration possesses a global timelike Killing vector $\zeta(t) = \partial/\partial t$. This enables us to determine a total energy of the field. We consider the infinite current parallel to the strings, and the spacetime which metric is invariant under translations along the $z$ axis. So, the energy per unit length along the current is independent on $z$, and the total energy is infinite. But the linear density is well defined, and corresponding expression reads

$$E \int \frac{dz}{d} = \int d^2 x \sqrt{\gamma(x)} T_{tt}(x).$$  \hspace{1cm} (3)

In Eq.(3) $T_{tt}$ is the $tt$-component of the electromagnetic energy-momentum tensor. To proceed further let us consider the $T_{tt}$ - component taking into account the symmetries of the spacetime and the current distribution.

In the Lorentz gauge $\nabla_{\nu} A^{\nu} = 0$ the components of a four-potential satisfy the equation

$$\nabla_{\nu} \nabla^{\nu} A_{\mu} - R^{\nu}_{\mu} A_{\nu} = 4\pi j_{\mu}, \hspace{1cm} (4)$$

where $R^{\nu}_{\mu}$ is the Ricci tensor. In the case of multistring space it has a $\delta$-like singularities at the tops of the cones, and it seems that we have to solve the wave equation with a singular potential.

Fortunately the metric and the distribution of current are both invariant under translations along $t$ and $z$. This invariance and the residual freedom in the gauge transformations enables us to choose a gauge in which the components of the four-potential depend on the coordinates on the Riemannian surface only and $A_1 = 0 = A_2$.

In this gauge the $tt$-component of the energy-momentum tensor has the following form

$$T_{tt} = \frac{1}{8\pi} \gamma^{ab} \left( \partial_a A_t \partial_b A_t + \partial_a A_z \partial_b A_z \right). \hspace{1cm} (5)$$

Then, $R^{b}_{a}$ ($a, b = 1, 2$) are the only components of the Ricci tensor which are not equal to zero in our coordinate system. So, in the gauge above the second term in the lhs of the Eq.(4) is equal to zero, and we get

$$A_{\mu}(x_1) = -4\pi \int d^2 x_2 \sqrt{\gamma(x_2)} G^{(2)}(x_1, x_2) j_{\mu}(x_2). \hspace{1cm} (6)$$

In the last expression $G^{(2)}(x_1, x_2)$ stands for the Green function fulfilling the two-dimensional Poisson equation

$$\frac{1}{\sqrt{\gamma(x_1)}} \partial_{1a} \left( \sqrt{\gamma(x_1)} \gamma^{ab}(x_1) \partial_{1b} G^{(2)}(x_1, x_2) \right) = \delta^2(x_1, x_2), \hspace{1cm} (7)$$
where $\delta^2(x_1, x_2)$ is a covariant two-dimensional $\delta$ - function and $\partial_{i_1 b}$ stands for the partial derivative $\partial/\partial x^b_{i_1}$.

So, for the distribution of current (2) all the nonzero components of the four-potential can be expressed in terms of the two-dimensional Green function (7). We shall see later that this enables one to solve the problem of topological self-action explicitly in much more general cases that it has been performed before.

Substituting (6) into (5) we get

$$\int d^2 x_1 d^2 x_2 \sqrt{\gamma(x_1)} \sqrt{\gamma(x_2)} \times$$

$$\left( j_t(x_1) G^{(2)}(x_1, x_2) j_t(x_2) + j_z(x_1) G^{(2)}(x_1, x_2) j_z(x_2) \right),$$

The expression above is valid for any distribution of four-current, which is stretched along $z$-axis and depends on the coordinates on the Riemannian surface $x^a$ only. For the infinitely thin distribution of current (2) we obtain from it

$$E_{stat} \int dz = -2\pi \left( J_t^2 + J_z^2 \right) G^{(2)}(x, x).$$

Equation (9) is the energy of a static electromagnetic field per unit length along the strings. But an observer measures the force, and corresponding expression is needed. In the case of the pure electrostatic self-forces the situation is more or less clear [15], the force which is measured in a locally inertial frame can be obtained by taking the negative gradient of the self-energy. In the case of magnetic force, as it is not a potential one, another procedure must be adopted. So let us start from the force itself.

In the case of infinitely thin current and for our particular geometry, the force per unit length of the current measured by a locally inertial observer is well defined. In an arbitrary coordinate system it is given by the relation

$$f_\mu = \int d^2 x \sqrt{\gamma(x)} F_{\mu\nu}(x) j^\nu(x).$$

In terms of the two-dimensional Green function this expression has the following form

$$f_\mu = -4\pi \int d^2 x_1 d^2 x_2 \sqrt{\gamma(x_1)} \sqrt{\gamma(x_2)} \left( j_\nu(x_1) j^\nu(x_2) \partial_{1\mu} G^{(2)}(x_1, x_2) \right).$$
\[ j_\mu(x_2) j_\nu(x_1) \partial_{1\nu} G^{(2)}(x_1, x_2) \].

Integrating by parts the second term in the integrand and using the current conservation law, we obtain

\[ f_\mu = -4\pi \int d^2 x_1 d^2 x_2 \sqrt{\gamma(x_1)} \sqrt{\gamma(x_2)} j_\nu(x_1) j^\nu(x_2) \partial_{1\mu} G^{(2)}(x_1, x_2). \] (12)

Taking into account the explicit form of the current distribution (2) we get the following formal result

\[ f_\mu = -4\pi J^2 \partial_\mu G^{(2)}(x, x), \] (13)

where \( J^2 = (J_t^2 - J_z^2) \) is the squared invariant amplitude of the current.

We are in a position to calculate the self-energy and self-force now, but it is necessary to solve two problems. Indeed both expressions for the self-energy (9) and for the self-force (13) diverge because of the divergence of the Green function and its derivatives in the coincidence limit. So, it is necessary to find the solution of the Poisson equation (7). After that we have to adopt some regularization procedure and calculate the regularized values of the Green function and its derivatives in the coincidence limit.

### 3 Green function of the Poisson equation and its derivatives

As it was shown by one of the authors [17] the problem of finding \( G^{(2)}(x_1, x_2) \) is greatly simplified by the fact that any two-dimensional Riemannian surface is locally conformal to the Euclidean plane. This means that the coordinate system can be introduced, in which in the vicinity of any point the metric takes the form

\[ \gamma_{ab}(x^c) = e^{-\Omega(x)} \delta_{ab}, \quad a, b, c = 1, 2. \] (14)

The primary goal of such a choice is to simplify the equation for the Green function as far as possible. Indeed in conformal coordinates the two-dimensional Poisson equation (7) reduces to the one which has exactly the same form as the Poisson equation on the Euclidean plane

\[ \Delta_E G^{(2)}(\vec{x}_1, \vec{x}_2) = \delta^2(\vec{x}_1 - \vec{x}_2), \] (15)
where $\Delta_E$ is the Laplace operator on the Euclidean plane.

The solution of (15) is given by

$$G^{(2)}(\vec{x}_1, \vec{x}_2) = \frac{1}{4\pi} \log |\vec{x}_1 - \vec{x}_2|^2 + f(\vec{x}_1 - \vec{x}_2),$$  \hspace{1cm} (16)

an arbitrary analytic function of $(\vec{x}_1 - \vec{x}_2)$,

where $|\vec{x}_1 - \vec{x}_2|$ denotes the Euclidean norm of the conformal vector $(\vec{x}_1 - \vec{x}_2)$.

The choice of the analytic function in (16) depends on the boundary conditions. It seems to be impossible to formulate these conditions in the general case of Riemannian surfaces. So, let us consider the interesting case of an infinite two-dimensional surface which is covered by the conformal coordinates globally. It is reasonable to demand that the field of a point source tends to zero at the infinity. This boundary condition reduces the choice of the analytic function to an arbitrary constant which can be taken equal to zero without loss of generality. Thus the Green function amounts to

$$G^{(2)}(\vec{x}_1, \vec{x}_2) = \frac{1}{4\pi} \log |\vec{x}_1 - \vec{x}_2|^2.$$  \hspace{1cm} (17)

Under the above assumptions this is an explicit solution. It is necessary to emphasize that in contrast with all the previous papers no symmetries of the two-dimensional surface were used to obtain it.

We are interested in the behaviour of the Green function and its derivatives in the coincidence limit. In this limit both these quantities diverge. To obtain a regularized value of the Green function it is necessary to subtract from (17) the Green function on the Euclidean plane \[1, 2\]

$$G^{(2)}_{\text{reg}}(\vec{x}_1, \vec{x}_2) = G^{(2)}(\vec{x}_1, \vec{x}_2) - G^{(2)}_E(\vec{x}_1, \vec{x}_2),$$  \hspace{1cm} (18)

where $G^{(2)}_E(\vec{x}_1, \vec{x}_2)$ is the Euclidean Green function, which can be written as

$$G^{(2)}_E(\vec{x}_1, \vec{x}_2) = \frac{1}{4\pi} \log 2\sigma(\vec{x}_1, \vec{x}_2).$$  \hspace{1cm} (19)

In the expression above $\sigma(\vec{x}_1, \vec{x}_2)$ is the half of the squared geodesic distance between two points with the conformal vectors $\vec{x}_1$ and $\vec{x}_2$. 

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Thus it is necessary to obtain an approximate expression for $G^{(2)}$ which is valid for a small enough $\sigma$. The well known way to obtain the desired result is the use of the Riemannian coordinates. Let us choose the origin of Riemannian frame at the centre of the geodesic, connecting the points $\vec{x}_1$ and $\vec{x}_2$, and let $\vec{x}$ be the conformal radius of this central point and $t^a$ be a tangent vector to the geodesic at the point $\vec{x}$ directed towards the point $\vec{x}_2$. Proceeding as in [17], we can write

$$x_2^a = x^a + \sqrt{\frac{\sigma}{2}} t^a - \frac{\sigma}{8} \left( \nabla^a \Omega - 2 t^a t^b \nabla_b \Omega \right) + \frac{(2\sigma)^{3/2}}{48} \left( t^a t^c \nabla_{bc} \Omega + \right. \left. \right) + O(\sigma^{5/2}).$$

(20)

Corresponding equation for $x_1^a$ can be obtained from (20) by changing the sign before the vector $t^a$.

From (20) we obtain that the Euclidean norm of the vector $(\vec{x}_1 - \vec{x}_2)$ has the following form

$$|\vec{x}_1 - \vec{x}_2|^2 = e^{\Omega(\vec{x})} 2\sigma(\vec{x}_1, \vec{x}_2) \left( 1 + \frac{\sigma}{12} t^a t^b (\nabla_a \nabla_b \Omega + \nabla_a \Omega \nabla_b \Omega - \frac{1}{2} \gamma_{ab} \nabla_c \Omega \nabla^c \Omega) \right) + O(\sigma^3).$$

(21)

Substituting this expression into (17) and supposing $\sigma$ to be small enough we get

$$G^{(2)}(\vec{x}_1, \vec{x}_2) = G^{(2)}_E(\vec{x}_1, \vec{x}_2) + \frac{\Omega(\vec{x})}{4\pi} + \frac{\sigma}{48\pi} t^a t^b \left( \nabla_a \nabla_b \Omega + \nabla_a \Omega \nabla_b \Omega - \frac{1}{2} \gamma_{ab} \nabla_c \Omega \nabla^c \Omega \right) + O(\sigma^2).$$

(22)

From the last expression we can obtain for the coincidence limit of the regularized value of the Green function the following result

$$G^{(2)}_{reg}(\vec{x}, \vec{x}) = \lim_{\vec{x}_1, \vec{x}_2 \to \vec{x}} \left( G^{(2)}(\vec{x}_1, \vec{x}_2) - G^{(2)}_E(\vec{x}_1, \vec{x}_2) \right) = \frac{\Omega(\vec{x})}{4\pi}. \quad (23)$$

The result is expressed in terms of conformal factor only, which depends on the global structure of the surface. So we can say that the coincidence limit
of the regularized Green function plays the role of the local test of the global structure of the surface.

There are some ways to obtain the regularized value of the derivative of \( G^{(2)}(\vec{x}_1, \vec{x}_2) \). One can start from the explicit expression (17) and after the differentiation with respect to \( x_1 \) proceed along the same lines as above, or we can use the decomposition (22). In this last case we must subtract \( G^{(2)}_E \) from (22), and after that, perform the differentiation. If we intend to differentiate with respect to the coordinates of the points \( \vec{x}_1 \) or \( \vec{x}_2 \), it is necessary to remember that all the tensor quantities in (22) are taken at the middle point and that the coordinates of this middle point are functions of \( \vec{x}_1 \) and \( \vec{x}_2 \). To take into account this dependence let us return to the Eq.(20). This equation gives us that

\[
x^a = \frac{1}{2}(x^a_1 + x^a_2) + \frac{\sigma}{8} \left( \nabla^a \Omega - 2 t^a t^b \nabla_b \Omega \right) + O(\sigma^2) .
\] (24)

¿From this equality one can obtain

\[
\frac{\partial x^a}{\partial x^1_1} = \frac{1}{2} - \frac{\sqrt{2}\sigma}{8} t^b \left( \nabla^a \Omega - 2 t^a t^c \nabla_c \Omega \right) + O(\sigma) . \quad (25)
\]

Differentiating (22) with respect to \( x_1 \) and taking into account (25) we get

\[
\frac{\partial}{\partial x^1_1} \left( G^{(2)}(\vec{x}_1, \vec{x}_2) - G^{(2)}_E(\vec{x}_1, \vec{x}_2) \right) = \frac{\nabla^a \Omega}{8\pi} + \frac{\sqrt{2}\sigma}{48\pi} t^a \left( \nabla_b \Omega \nabla^b \Omega - 2(t^b \nabla_b \Omega)^2 + t^b t^c \nabla^2_{bc} \Omega \right) + O(\sigma) . \quad (26)
\]

¿From the last expression we obtain a very interesting correspondence

\[
\lim_{\vec{x}_1, \vec{x}_2 \to \vec{x}} \frac{\partial}{\partial x^1_1} \left( G^{(2)}(\vec{x}_1, \vec{x}_2) - G^{(2)}_E(\vec{x}_1, \vec{x}_2) \right) = \frac{1}{2} \frac{\partial}{\partial x} \lim_{\vec{x}_1, \vec{x}_2 \to \vec{x}} \left( G^{(2)}(\vec{x}_1, \vec{x}_2) - G^{(2)}_E(\vec{x}_1, \vec{x}_2) \right) , \quad (27)
\]

which shows that the coincidence limit and the differentiation with respect to coordinate are noncomutative operations.
So, the regularized value of the derivative is equal to
\[
\left( \frac{\partial}{\partial x^a} G^{(2)} \right)_{\text{reg}} (\vec{x}, \vec{x}) = \frac{\nabla_a \Omega(\vec{x})}{8\pi} .
\] (28)

It is clear that the same procedure permits us to calculate the coincidence limit of all the other derivatives of the Green function. It may be necessary if we want to consider the self-forces on a current distributions with some inner structure.

4 Explicit solution for topological self-force on multistring spacetime

The results obtained in the previous sections enable us to conclude the considerations on the problem of topological self-action on a linear distribution of four-current placed in the multistring spacetime and even to consider a more general case of the spacetime with a metric of the form (1).

Indeed from (9) and (23) we can obtain the expression for the regularized electromagnetic self-energy per unit length of the current
\[
\frac{E_{\text{stat}}^{\text{reg}}}{\int dz} = \Omega(\vec{x}) \left( J_t^2 + J_z^2 \right) .
\] (29)

The obtained value depends on the coordinates on the section \( t = \text{const}, z = \text{const} \). This is a trivial consequence of the absence of translational invariance in the direction perpendicular to the \( z \)-axis and means that the current must feel the action of some self-force. As it was stressed in Sec.2 to obtain the expression for the regularized self-force we must start from (13) and after the use of (28) we obtain that the nonvanishing components of the force are as follows
\[
\frac{F^a(\vec{x})}{\int dz} = \frac{J^2}{2} \gamma^{ab} \partial_b \Omega(\vec{x}) .
\] (30)

\( ^{\text{11}} \)From the last expressions for the the observer in the locally Cartesian frame we get
\[
\frac{\vec{F}}{\int dz} = \frac{J^2}{2} e^\frac{\alpha}{\pi} \nabla \Omega .
\] (31)
Let us apply the above results and consider a particular and a very interesting case of the multiconical space, when the subspace \( t = \text{const}, z = \text{const} \) is a locally flat two-dimensional surface with a number of conical singularities located at the points with conformal radii \( \vec{x}_i \). For these purposes the conformal factor in (14) must be taken in the form \([20, 21]\)

\[
\Omega(\vec{x}) = \sum_{i=1}^{N} 2(1 - b_i) \log |\vec{x} - \vec{x}_i| .
\] (32)

In the expression above \( b_i \) stands for the parameter which determines the angle deficit of the \( i \)-th conical singularity. From the cosmological point of view this solution describes the ultrastatic spacetime of \( N \) parallel cosmic strings [20]. In the same context, the current (2) may be associated with the current of superconducting cosmic strings which were predicted by Witten [22].

From (29) and (32) we see that the contributions to the self-energy from different singularities add to each other

\[
\frac{E_{\text{stat}}}{\int dz} - \frac{E_{\text{sup}}}{\int dz} = -(J^2_t + J^2_z) \sum_{i=1}^{N} (1 - b_i) \log |\vec{x} - \vec{x}_i| ,
\] (33)

One can conclude that there is a principle of superposition and that the addition of an extra string does not lead to an extra difficulty.

It is easy to understand that the real situation is much more complicated. One has to remember that \( \vec{x}_i \) in (32) are not the real geodesic distances from the corresponding singularities but the conformal radii. So, we obtained a superposition principle but in terms of the Euclidean plane, with which our two-dimensional multiconical subspace is conformally connected. When the self-energy is expressed in terms of geodesic distances from the strings it is a much more complicated function, and there is no superposition principle in the explicit sense of this word.

To illustrate this statement let us consider two-strings spacetime and a current between them (both strings and the current lie in one and the same plane \( x_2 = 0 \)). Suppose for simplicity that the strings have equal tensions. Direct calculation gives that in this case the difference between the explicit expression for the topological self-energy and the one obtained under the assumption that the superposition principle holds reads

\[
\frac{E_{\text{stat}} - E_{\text{sup}}}{\int dz} = -(1 - b)(J^2_t + J^2_z) F(x) ,
\] (34)
where

$$F(x) = \log \frac{x(1 - x)}{(\rho_1 \rho_2)^\frac{1}{2}}, \quad 0 < x < 1.$$  \hspace{1cm} (35)$$

The coordinate $x$ indicates the position of a current between the strings, and $\rho_{1,2}$ are the dimensionless geodesic distances from corresponding string to the current

$$\rho_1 = \int_0^x \frac{dx}{x^{1-b}(1-x)^{1-b}}, \quad \rho_2 = \int_x^1 \frac{dx}{x^{1-b}(1-x)^{1-b}}.$$  \hspace{1cm} (36)$$

Analytic estimates show that the value of $F(x)$ decreases with $b \to 1$ as $(1 - b)$. So for the cosmic string network $((1 - b) \sim 10^{-5})$ we indeed can use the superposition principle, but it is not the case for the disclinated medium where $(1 - b) \sim 1$.

Let us consider the self-force now. From (31) it is very easy to show that in the locally inertial frame the force of topological self-action has the form

$$\vec{F}(x) = \int \frac{dx}{\rho}, \quad \rho = \frac{1 - b}{1 - x} \quad \vec{t}.$$  \hspace{1cm} (37)$$

We see again that because of the factor $e^{\Omega/2}$ the contributions from different strings are mixed and the force which is measured by a locally inertial observer is not a sum of the contributions from different strings.

To interpret the obtained result, let us consider the simplest case of only one conical string ($N = 1$) stretched along the $z$ axis. Introducing a new radial coordinate $\rho = b^{-1}|\vec{x}|^b$, we get

$$\frac{\vec{F}(x)}{\int dz} = J^2 \frac{(1 - b)}{b \rho^2} \vec{t}.$$  \hspace{1cm} (38)$$

This new coordinate $\rho$ gives the true geodesic distance from the string. Thus for the locally inertial observer the force on the four-current is the same as the force between two parallel four-currents, our one and $I^\mu(x) = (2b)^{-1}(1 - b)J^\mu$, in Minkowski space. The sign of the force depends on the sign of $J^2$. For $b < 1$ the force is attractive for the space-like current, and repulsive for the time-like one.

We see that if $N = 1$ our result coincides with the result [15], but it should be pointed out that the method of separation of variables used in
[15], can not be applied in the multistring case, and of course, it can not be used in more general cases.

To conclude this section let us add some discussion concerning the connection of the obtained results and the symmetries of the system under consideration.

The expression for the self-force in the multistring space (37) is proportional to the squared invariant amplitude of the current $J^2$, and so, the force is independent of an observer’s speed along the strings. We shall show that this fact can be explained on the pure qualitative level, and moreover for the case of only one conical string the expression for the self-force can be obtained up to a numerical coefficient.

Let us remember that the vector $z\partial/\partial t + t\partial/\partial z$ is the fourth of the four Killing vectors of the locally flat conical space mentioned in the Introduction. Indeed, in the case of a straight string such a Lorentz transform changes the parametrization of its world sheet, but the Nambu action is invariant under general reparametrizations of the world sheet [3, 4]. The existence of this symmetry means that the longitudinal motion of a Nambu string is unobservable.

To obtain a time independent infinitely thin current we can use any action for the superconducting string, say the Nielsen-Olesen one [23]. But the action for the superconducting string is invariant under the reparametrization of the string’s world sheet too. Thus, if we consider two parallel strings, and one of the strings is a superconducting string, a Lorentz boost along them leads to the reparametrization of the both world sheets, and so, must be unobservable. This means that the force may depend on the ”invariant charge” $J^5$ and the only possibility to obtain the expression for the force with the correct dimension is to write $F \sim J^2/\rho$ but this is the force between two parallel four-currents which are proportional to each other.

Then, we consider the interaction of a conical gravitational field with the electromagnetic field only. So, we can extend the qualitative result above to any current independently of the nature of its carrier.

We see that the result (38) can be predicted on a pure qualitative level. But it is not the case in the spacetime of multiple cosmic strings because in this case we have several parameters (conformal distances between the

\footnote{Similar result was obtained under consideration of radiation processes in the cosmic string network [19]}
strings) with the same dimension. This makes it impossible to obtain the result (37) by a qualitative discussion along the same line as above.

5 Conclusion

Spaces with conical singularities have a lot of physical applications in the phenomena of very different spacetime scales. In some cases the multistring solution corresponds to the real physical situation much more than the spacetime of only one cosmic string.

Our primary goal was to show that the naive point of view that some results concerning the behaviour of classical or quantized matter on the multistring space can be obtained as a superposition of individual contributions from the separate strings is wrong. Multiconical boundary conditions change the result essentially, and one can speak about the superposition principle approximately in a very limited number of cases.

In the paper we have considered an explicitly solvable problem which enables us to establish certain nonlocal effects in the spacetime of multiple cosmic strings. To our mind this is the first explicitly solvable problem in the 4D locally flat multiconical space that have been considered in scientific literature.

We hope that the results presented here can be of interest in investigating the behaviour of superconducting strings in the cosmic string network and the topological effects in disclinated media. From the observational point of view the effect of topological self-action may be of much more interest for the solid state physics because it is proportional to the parameter \((1 - b)\), which is very small for the GUT strings, but may be of the order of unity in the case of disclinations. Our results show that in this last case the collective nonlocal contribution to the effect which is due to the presence of more than one conical singularity may be of great importance and must be taken into account.

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