Hamilton-Jacobi approach to Berezinian singular systems

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Abstract

In this work we present a formal generalization of the Hamilton-Jacobi formalism, recently developed for singular systems, to include the case of Lagrangians containing variables which are elements of Berezin algebra. We derive the Hamilton-Jacobi equation for such systems, analyzing the singular case in order to obtain the equations of motion as total differential equations and study the integrability conditions for such equations. An example is solved using both Hamilton-Jacobi and Dirac’s Hamiltonian formalisms and the results are compared.
1 Introduction

In this work we intend to study singular systems with Lagrangians containing elements of Berezin algebra from the point of view of the Hamilton-Jacobi formalism recently developed [1, 2]. The study of such systems through Dirac’s generalized Hamiltonian formalism has already been extensively developed in literature [3, 4, 5] and will be used for comparative purposes.

Despite the success of Dirac’s approach in studying singular systems, which is demonstrated by the wide number of physical systems to which this formalism has been applied, it is always instructive to study singular systems through other formalisms, since different procedures will provide different views for the same problems, even for nonsingular systems. The Hamilton-Jacobi formalism that we study in this work has been already generalized to higher order singular systems [6, 7] and applied only to a few number of physical examples as the electromagnetic field [8], relativistic particle in an external Electromagnetic field [9] and Podolsky’s Electrodynamics [6]. But a better understanding of this approach utility in the studying singular systems is still lacking, and such understanding can only be achieved through its application to other interesting physical systems.

Besides that, Berezin algebra is a useful way to deal simultaneously with bosonic and fermionic variables in a unique and compact notation, what justifies the interest in studying systems composed by its elements using new formalisms.

The aim of this work is not only to generalize the Hamilton-Jacobi approach for singular systems to the case of Lagrangians containing Berezinian variables but to present an example of its application to a well known physical system, comparing the results to those obtained through Dirac’s method.
We will start in section 2 with some basic definitions and next, in section 3, we will introduce the Hamilton-Jacobi formalism to Berezinian systems using Carathéodory’s equivalent Lagrangians method. In section 4 the singular case is considered and the equations of motion are obtained as a system of total differential equations whose integrability conditions are analyzed in section 5. The equivalence among these integrability conditions and Dirac’s consistency conditions will be discussed separately in the appendix. In section 6 we present, as an example, the electromagnetic field coupled to a spinor, which is studied using both the formalism presented in this work and Dirac’s Hamiltonian one. Finally, the conclusions are presented in section 7.

2 Basic definitions

We will start from a Lagrangian \( L(q, \dot{q}) \) that must be an even function of a set of \( N \) variables \( q^i \) that are elements of Berezin algebra. For a basic introduction in such algebra we suggest the reader to refer to ref. [3], appendix D, from which we took the definitions used in this paper. A more complete treatment can be found in ref. [10].

The Lagrangian equations of motion can be obtained through variational principles from the action \( S = \int L dt \)

\[
\frac{\delta S}{\delta q^i} = \frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} = 0, \tag{1}
\]

were we must call attention to the use of right derivatives.

The passage to Hamiltonian formalism is made, as usual, by defining the momenta variables through right derivatives as

\[
p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \tag{2}
\]
and introducing the Hamiltonian function as (summing over repeated indexes)

$$H = p_i \dot{q}^i - L,$$

(3)

were the ordering of momenta to the left of velocities shall be observed since they were defined as right derivatives. This ordering will be, of course, irrelevant when we deal with even elements of the Berezin algebra. The Hamiltonian equations of motion will be given by

$$\dot{q}^i = \frac{\partial H}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial H}{\partial q^i} = -(-1)^{P_i} \frac{\partial H}{\partial q^i}. \quad (4)$$

If we use the Poisson bracket in the Berezin algebra, given by

$$\{F, G\}_B = \frac{\partial_F F}{\partial q^i} \frac{\partial_G G}{\partial p_i} - (-1)^{P(F)P(G)} \frac{\partial_F G}{\partial q^i} \frac{\partial_G F}{\partial p_i}, \quad (5)$$

we get the known expressions

$$\dot{q}^i = \{q^i, H\}_B; \quad \dot{p}_i = \{p_i, H\}_B. \quad (6)$$

For simplicity and clarity we will refer to these brackets as Berezin brackets. These brackets have similar properties to the usual Poisson brackets

$$\{F, G\}_B = -(-1)^{P(F)P(G)} \{G, F\}_B, \quad (7)$$

$$\{F, GK\}_B = \{F, G\}_B K + (-1)^{P(F)P(G)} G \{F, K\}_B, \quad (8)$$

$$(-1)^{P(F)P(K)} \{F, \{G, K\}_B\}_B + (-1)^{P(G)P(F)} \{G, \{K, F\}_B\}_B$$

$$+ (-1)^{P(K)P(G)} \{K, \{F, G\}_B\}_B = 0, \quad (9)$$

were the last expression is the analogue of Jacobi’s identity.

Similarly to the usual case, the transition to phase space is only possible if the momenta variables, given by Eq. (2), are independent variables among themselves
so that we can express all velocities $\dot{q}_i$ as functions of canonical variables $(q^i, p_i)$. Such necessity implies that the **Hessian supermatrix**

$$
H_{ij} \equiv \frac{\partial r}{\partial \dot{q}^i} = \frac{\partial^2 L}{\partial q^i \partial q^j} \quad (10)
$$

must be nonsingular. Otherwise, if the Hessian has a rank $P = N - R$, there will be $R$ relations among the momenta variables and coordinates $q^i$ that are primary constraints (that we suppose to have definite parity), while $R$ velocities $\dot{q}^i$ will remain arbitrary variables in the theory. The development of Dirac’s Generalized Hamiltonian Formalism is straightforward: the primary constraints have to be added to the Hamiltonian, we have to work out the consistency conditions, separate the constraints in first and second-class ones and define the Dirac brackets using the *supermatrix* whose elements are the Poisson brackets among the second-class constraints [3].

### 3 Hamilton-Jacobi formalism

From Carathéodory’s equivalent Lagrangians method [11] we can obtain the Hamilton-Jacobi equation for the even Lagrangian $L\left(q, \dot{q}\right)$. The procedure is similar to the one applied to usual variables: given a Lagrangian $L$, we can obtain a completely equivalent one given by

$$
L' = L\left(q^i, \dot{q}^i\right) - \frac{dS\left(q^i, t\right)}{dt},
$$

drew $S\left(q^i, t\right)$ is an even function in order to keep the equivalent Lagrangian even.

These Lagrangians are equivalent because their respective action integrals have simultaneous extremum. Then we choose the function $S\left(q^i, t\right)$ in such a way that we get an extremum of $L'$ and, consequently, an extremum of the Lagrangian $L$. 
For this purpose, it is enough to find a set of functions \( \beta^i(q^i, t) \) and \( S(q^i, t) \) such that

\[
L'(q^i, \dot{q}^i = \beta^i(q^i, t), t) = 0
\]  \hspace{1cm} (11)

and for all neighborhood of \( \dot{q}^i = \beta^i(q^i, t) \)

\[
L'(q^i, \dot{q}^i) > 0.
\]  \hspace{1cm} (12)

With these conditions satisfied, the Lagrangian \( L' \) (and consequently \( L \)) will have a minimum in \( \dot{q}^i = \beta^i(q^i, t) \) so that the solutions of the differential equations given by

\[
\dot{q}^i = \beta^i(q^i, t),
\]

will correspond to an extremum of the action integral.

From the definition of \( L' \) we have

\[
L' = L(q^i, \dot{q}^i) - \frac{\partial S(q^i, t)}{\partial t} - \frac{\partial_r S(q^i, t)}{\partial q^i} \frac{dq^i}{dt},
\]

where again we must call attention to the use of the right derivative.

Using condition (11) we have

\[
\left[ \left. \left[ L(q^i, \dot{q}^i) - \frac{\partial S(q^i, t)}{\partial t} - \frac{\partial_r S(q^i, t)}{\partial q^i} \dot{q}^i \right] \right|_{\dot{q}^i = \beta^i} \right] = 0,
\]

\[
\left. \frac{\partial S}{\partial t} \right|_{\dot{q}^i = \beta^i} = \left[ \left. \left[ L(q^i, \dot{q}^i) - \frac{\partial_r S(q^i, t)}{\partial q^i} \dot{q}^i \right] \right|_{\dot{q}^i = \beta^i} \right].
\]  \hspace{1cm} (13)

In addition, since \( L' \) has a minimum in \( \dot{q}^i = \beta^i \), we must have

\[
\left. \left[ \left. \frac{\partial_r L}{\partial \dot{q}^i} \right|_{\dot{q}^i = \beta^i} = 0 \Rightarrow \left[ \left. \frac{\partial_r L}{\partial \dot{q}^i} - \frac{\partial_r S(q^i, t)}{\partial q^i} \frac{dt}{dt} \right] \right|_{\dot{q}^i = \beta^i} \right] = 0,
\]

\[
\left. \left[ \left. \frac{\partial_r L}{\partial \dot{q}^i} - \frac{\partial_r S(q^i, t)}{\partial q^i} \right|_{\dot{q}^i = \beta^i} \right|_{\dot{q}^i = \beta^i} \right] = 0,
\]

or

\[
\left. \frac{\partial_r S(q^i, t)}{\partial q^i} \right|_{\dot{q}^i = \beta^i} = \left. \frac{\partial_r L}{\partial \dot{q}^i} \right|_{\dot{q}^i = \beta^i}.
\]  \hspace{1cm} (14)
Now, using the definitions for the conjugated momenta given by Eq. (2), we get
\[ p_i = \frac{\partial r S(q^j, t)}{\partial q^i}. \] (15)

We can see from this result and from Eq. (13) that, in order to obtaining an extremum of the action, we must get a function \( S(q^j, t) \) such that
\[ \frac{\partial S}{\partial t} = -H_0 \] (16)
where \( H_0 \) is the canonical Hamiltonian
\[ H_0 = p_i \dot{q}^i - L(q^j, q^j) \] (17)
and the momenta \( p_i \) are given by Eq. (15). Besides, Eq. (16) is the Hamilton-Jacobi partial differential equation (HJPDE).

4 The singular case

We now consider the case of a system with a singular Lagrangian. When the Hessian supermatrix is singular with a rank \( P = N - R \) we can define the variables \( q^i \) in such order that the \( P \times P \) supermatrix in the right bottom corner of the Hessian supermatrix be nonsingular, i.e.
\[ \det |H_{ab}| \neq 0; \quad H_{ab} = \frac{\partial^2 L}{\partial q^a \partial q^b}; \quad a, b = R + 1, ..., N. \] (18)

This allows us to solve the velocities \( \dot{q}^a \) as functions of coordinates \( q^i \)'s and momenta \( p_a \), i.e., \( \dot{q}^a = f^a (q^i, p_b) \).

There will remain \( R \) momenta variables \( p_\alpha \) dependent upon the other canonical variables, and we can always \([3, 4, 12]\) write expressions like
\[ p_\alpha = -H_\alpha (q^i; p_\alpha); \quad \alpha = 1, ..., R; \] (19)
that correspond to the Dirac’s primary constraints $\Phi_\alpha \equiv p_\alpha + H_\alpha (q^i; p_a) \approx 0$.

The Hamiltonian $H_0$, given by Eq. (17), becomes

$$H_0 = p_a f^a + p_\alpha|_{p_\beta=-H_\beta} \cdot q^\alpha - L \left( q^i, \dot{q}_\alpha, \dot{q}_a = f^a \right), \quad (20)$$

where $\alpha, \beta = 1, ..., R; a = R + 1, ..., N$. On the other hand we have

$$\frac{\partial_r H_0}{\partial q^\alpha} = p_a \frac{\partial_r f^a}{\partial q^\alpha} + p_\alpha - \frac{\partial_r L}{\partial q^\alpha} - \frac{\partial_r L}{\partial q^\alpha} \frac{\partial_r f^a}{\partial q^\alpha} = 0,$$

so the Hamiltonian $H_0$ does not depend explicitly upon the velocities $\dot{q}_\alpha$.

Now we will adopt the following notation: the time parameter $t$ will be called $t^0 \equiv q^0$; the coordinates $q^\alpha$ will be called $t^\alpha$; the momenta $p_\alpha$ will be called $P_\alpha$ and the momentum $p_0 \equiv P_0$ will be defined as

$$P_0 \equiv \frac{\partial S}{\partial t}. \quad (21)$$

Then, to get an extremum of the action integral, we must find a function $S(t^\alpha; q^a, t)$ that satisfies the following set of HJPDE

$$H'_0 \equiv P_0 + H_0 \left( t, t^\beta; q^a; p_\alpha = \frac{\partial_r S}{\partial q^a} \right) = 0, \quad (22)$$

$$H'_\alpha \equiv P_\alpha + H_\alpha \left( t, t^\beta; q^a; p_\alpha = \frac{\partial_r S}{\partial q^a} \right) = 0. \quad (23)$$

where $\alpha, \beta = 1, ..., R$. If we let the indexes $\alpha$ and $\beta$ run from 0 to $R$ we can write both equations as

$$H'_\alpha \equiv P_\alpha + H_\alpha \left( t^\beta; q^a; p_\alpha = \frac{\partial_r S}{\partial q^a} \right) = 0. \quad (24)$$

From the above definition and Eq. (20) we have

$$\frac{\partial_r H'_0}{\partial p_b} = - \frac{\partial_r L}{\partial q^a} \frac{\partial_r f^a}{\partial p_b} - (-1)^{P_\beta P_\alpha} \frac{\partial_r H_\alpha}{\partial p_b} q^\alpha + p_\alpha \frac{\partial_r f^a}{\partial p_b} + (-1)^{P_\beta} q^b,$$

$$\frac{\partial_r H'_0}{\partial p_b} = (-1)^{P_\beta} q^b - (-1)^{P_\beta P_\alpha} \frac{\partial_r H_\alpha}{\partial p_b} q^\alpha,$$
where we came back to $\alpha = 1, ..., R$.

Multiplying this equation by $dt = dt^0$ and $(-1)^{P(b)}$, we have

$$dq^b = (-1)^{P(b)} \frac{\partial_r H'_{\alpha}}{\partial p_b} dt^0 + (-1)^{P(b)+P(\alpha)} \frac{\partial_r H'_{\alpha}}{\partial p_b} dq^\alpha.$$  

Using $t^\alpha \equiv q^\alpha$, letting the index $\alpha$ run again from 0 to $R$ and considering $P_0 = P_{(0)} = 0$ we have

$$dq^b = (-1)^{P(b)} \frac{\partial_r H'_{\alpha}}{\partial p_b} dt^0.$$  

(25)

Noticing that we have the expressions

$$dq^i = (-1)^{P(i)+P(\alpha)} \frac{\partial_r H'_{\alpha}}{\partial p_i} dt^0; \quad i = 0, 1, ..., N.$$  

(26)

If we consider that we have a solution $S(q^j, t)$ of the HJPDE given by Eq. (24) then, differentiating that equation with respect to $q^i$, we obtain

$$\frac{\partial_r H'_{\alpha}}{\partial q^i} + \frac{\partial_r H'_{\alpha}}{\partial P_\beta} \frac{\partial^2 S}{\partial t^\beta \partial q^i} + \frac{\partial_r H'_{\alpha}}{\partial p_a} \frac{\partial^2 S}{\partial q^a \partial q^i} = 0$$  

(27)

for $\alpha, \beta = 0, 1, ..., R$.

From the momenta definitions we can obtain

$$dp_i = \frac{\partial^2 S}{\partial q^i \partial t^\beta} dt^\beta + \frac{\partial^2 S}{\partial q^i \partial q^a} dq^a; \quad i = 0, 1, ..., N.$$  

(28)

Now, contracting equation (27) with $dt^\alpha$ (from the right), multiplying by $(-1)^{P(i)+P(\alpha)}$ and adding the result to equation (28) we get

$$dp_i + (-1)^{P(i)+P(\alpha)} \frac{\partial_r H'_{\alpha}}{\partial q^i} dt^\alpha =$$

$$= \left[ \frac{\partial^2 S}{\partial q^a \partial q^a} \left( dq^a - (-1)^{P(i)+P(\alpha)} (-1)^{(P(\alpha)+P(\alpha))(P(i)+P(\alpha))} (-1)^{P(i)+P(\alpha)} \frac{\partial_r H'_{\alpha}}{\partial p_a} dt^\alpha \right) \right]$$

$$+ \frac{\partial^2 S}{\partial q^i \partial t^\beta} \left( dt^\beta - (-1)^{P(i)+P(\alpha)} (-1)^{(P(\alpha)+P(\alpha))(P(i)+P(\alpha))} (-1)^{P(i)+P(\alpha)} \frac{\partial_r H'_{\alpha}}{\partial P_\beta} dt^\alpha \right),$$
\[ dp_i + (-1)^{P_{(i)}P_{(a)}} \frac{\partial H_{\alpha}'}{\partial q^i} dt^\alpha = \left[ \frac{\partial^2 S}{\partial q^i \partial q^a} \left( dq^a - (-1)^{P_{(a)}P_{(\alpha)} + P_{(\alpha)}P_{(\beta)}} \frac{\partial H_{\alpha}'}{\partial p_{\alpha}} dt^\alpha \right) 
\right. \\
\left. + \frac{\partial^2 S}{\partial q^i \partial t^\beta} \left( dt^\beta - (-1)^{P_{(\alpha)}P_{(\beta)} + P_{(\beta)}P_{(\alpha)}} \frac{\partial H_{\alpha}'}{\partial p_{\beta}} dt^\alpha \right) \right], \quad (29) \]

where we used the fact that

\[ \frac{\partial^2 S}{\partial q^j \partial q^i} = (-1)^{P_{(i)}P_{(j)}} \frac{\partial^2 S}{\partial q^i \partial q^j}, \]

and that we have the following parities

\[ P \left( \frac{\partial^2 S}{\partial q^i \partial q^j} \right) = P_{(i)} + P_{(j)}, \]

\[ P \left( \frac{\partial H_{\alpha}'}{\partial p_{j}} \right) = P_{(\alpha)} + P_{(j)}. \]

If the total differential equation given by Eq. (26) applies, the above equation becomes

\[ dp_i = -(-1)^{P_{(i)}P_{(\alpha)}} \frac{\partial H_{\alpha}'}{\partial q^i} dt^\alpha; \quad i = 0, 1, \ldots, N. \quad (30) \]

Making \( Z \equiv S (t^\alpha; q^\alpha) \) and using the momenta definitions together with Eq. (26) we have

\[ dZ = \frac{\partial S}{\partial t^\beta} dt^\beta + \frac{\partial S}{\partial q^a} dq^a, \]

\[ dZ = -H_\beta dt^\beta + p_{\alpha} \left( (-1)^{P_{(\alpha)}P_{(\alpha)} + P_{(\alpha)}P_{(\beta)}} \frac{\partial H_{\alpha}'}{\partial p_{\alpha}} dt^\alpha \right). \]

With a little change of indexes we get

\[ dZ = \left( -H_\beta + (-1)^{P_{(\alpha)}P_{(\alpha)}P_{(\beta)}P_{(\alpha)}} \frac{\partial H_{\alpha}'}{\partial p_{\alpha}} \right) dt^\beta. \quad (31) \]

This equation together with Eq. (26) and Eq. (30) are the total differential equations for the characteristics curves of the HJPDE given by Eq. (24) and, if they form a completely integrable set, their simultaneous solutions determine \( S (t^\alpha; q^\alpha) \) uniquely from the initial conditions.
Besides that, Eq. (26) and Eq. (30) are the equations of motion of the system written as total differential equations. It is important to observe that, in the nonsingular case, we have only $H'_0 \neq 0$ and no others $H'_\alpha$; so that these equations of motion will reduce naturally to the usual expressions given by Eq. (4).

5 Integrability conditions

The analysis of integrability conditions of the total differential equations (26), (30) and (31) can be carried out using standard techniques. This have already been made [2, 13, 14] for systems with usual variables, and here we will present the analysis of the integrability conditions for Berezinian singular systems.

To a given set of total differential equations

$$dq^i = \Lambda^i_{\alpha} \left( t^\beta, q^i \right) dt^\alpha$$

$$\left( i, j = 0, 1, ..., N \text{ and } \alpha, \beta = 0, 1, ..., R < N \right)$$

we may associate a set of partial differential equations [15]

$$X_\alpha f = \frac{\partial f}{\partial q^i} \Lambda^i_{\alpha} = 0,$$

where $X_\alpha$ are linear operators.

Given any twice differentiable solution of the set (33), it should also satisfy the equation

$$[X_\alpha, X_\beta] f = 0,$$

where

$$[X_\alpha, X_\beta] f = \left( X_\alpha X_\beta - (-1)^{P(\alpha)P(\beta)} X_\beta X_\alpha \right) f$$

is the bracket among the operators $X_\alpha$. This implies that we should have

$$[X_\alpha, X_\beta] f = C_{\alpha\beta\gamma} X_\gamma f.$$
So, the commutation relations (35) will give the maximal number of linearly independent equations. Any commutator that results in an expression that can’t be written as Eq. (36) must be written as a new operator \( X \) and be joined to the original set (33), having all commutators with the other \( X \)'s calculated. The process is repeated until all operators \( X \) satisfy Eq. (36).

If all operators \( X_\alpha \) satisfy the commutation relations given by Eq. (36) the system of partial differential equations (33) is said to be *complete* and the corresponding system of total differential equations (32) is integrable if, and only if, the system (33) is complete.

Now, we consider the system of differential equations obtained in the previous section. First we shall observe that if the total differential equations (26) and (30) are integrable the solutions of Eq. (31) can be obtained by a quadrature, so we only need to analyze the integrability conditions for the last ones, since the former will be integrable as a consequence.

The operators \( X_\alpha \) corresponding to the system of total differential equations formed by Eq. (26) and Eq. (30) are given by,

\[
X_\alpha f(t^\beta, q^a, p^a) = \frac{\partial_r f}{\partial q^i} (-1)^{P(\alpha)+P(\alpha)} \frac{\partial_r H'_\alpha}{\partial p_i} - \frac{\partial_f}{\partial p_i} (-1)^{P(\alpha)+P(\alpha)} \frac{\partial_r H'_\alpha}{\partial q^i};
\]

\[
X_\alpha f(t^\beta, q^a, p^a) = \frac{\partial_r f}{\partial q^i} \frac{\partial_r H'_a}{\partial p_i} (-1)^{P(\alpha)+P(\alpha)} (-1)^{(P(\alpha)+P(\alpha))(P(\alpha)+P(\alpha))} \frac{\partial_r f}{\partial q^i} \frac{\partial_r H'_\alpha}{\partial p_i};
\]

\[
X_\alpha f(t^\beta, q^a, p^a) = \frac{\partial_r f}{\partial q^i} \frac{\partial_r H'_a}{\partial p_i} (-1)^{P(\alpha)+P(\beta)} \frac{\partial_r f}{\partial q^i} \frac{\partial_r H'_\alpha}{\partial p_i};
\]

\[
X_\alpha f(t^\beta, q^a, p^a) = \{f, H'_a\}_B;
\]

where \( i = 0, 1, ..., N; \alpha, \beta = 0, 1, ..., R; a = R + 1, ..., N \) and we have used Eq. (32) and Eq. (33) together with the result

\[
\frac{\partial_r A}{\partial q^i} = (-1)^{P(\alpha)} (-1)^{P(\alpha)+P(\alpha)} \frac{\partial_r A}{\partial q^i}.
\]
It is important to notice that the Berezin bracket in Eq. (37) is defined in a $2N + 2$ dimensional phase space, since we are including $q^0 = t$ as a “coordinate”.

Now, the integrability condition will be

$$[X_\alpha, X_\beta] f = \left(X_\alpha X_\beta - (-1)^{p(\alpha)p(\beta)} X_\beta X_\alpha\right) f$$

$$= X_\alpha \left\{ f, H'_\beta \right\}_B - (-1)^{p(\alpha)p(\beta)} X_\beta \left\{ f, H'_\alpha \right\}_B = 0,$$

$$[X_\alpha, X_\beta] f = \left\{ \left\{ f, H'_\beta \right\}_B, H'_\alpha \right\}_B - (-1)^{p(\alpha)p(\beta)} \left\{ \left\{ f, H'_\alpha \right\}_B, H'_\beta \right\}_B = 0,$$

that will reduce to

$$[X_\alpha, X_\beta] f = -(-1)^{(p(f) + p(\beta))p(\alpha)} \left\{ H'_\alpha, \left\{ f, H'_\beta \right\}_B \right\}_B$$

$$- (-1)^{p(\alpha)p(\beta)} (-1)^{(p(f) + p(\alpha))p(\beta) + p(f)p(\alpha)} \left\{ H'_\beta, \left\{ H'_\alpha, f \right\}_B \right\}_B = 0,$$

$$[X_\alpha, X_\beta] f = -(-1)^{p(f)p(\alpha)} \left[ (-1)^{p(\beta)p(\alpha)} \left\{ H'_\alpha, \left\{ f, H'_\beta \right\}_B \right\}_B$$

$$+ (-1)^{p(f)p(\beta)} \left\{ H'_\beta, \left\{ H'_\alpha, f \right\}_B \right\}_B \right] = 0,$$

$$[X_\alpha, X_\beta] f = -(-1)^{p(f)p(\alpha)} \left[ -(-1)^{p(\beta)p(\alpha)} \left\{ f, \left\{ H'_\beta, H'_\alpha \right\}_B \right\}_B \right],$$

$$[X_\alpha, X_\beta] f = \left\{ f, \left\{ H'_\beta, H'_\alpha \right\}_B \right\}_B = 0,$$

(40)

when using the Jacobi relations for Berezin brackets given by Eq. (9) and the fact that

$$P(\{A, B\}_B) = P(A) + P(B).$$

(41)

So, the integrability condition will be

$$\left\{ H'_\beta, H'_\alpha \right\}_B = 0; \forall \alpha, \beta.$$
It is important to notice that the above condition can be shown to be equivalent to the consistency conditions in Dirac’s Hamiltonian formalism but, to keep the continuity of the presentation, we will postpone the demonstration of this equivalence to the appendix.

Now, the total differential for any function $F(t^\beta, q^a, p^a)$ can be written as

$$dF = \frac{\partial F}{\partial q^a} dq^a + \frac{\partial F}{\partial p^a} dp^a + \frac{\partial F}{\partial t^\alpha} dt^\alpha,$$

where the Berezin bracket above is the one defined in the $2N + 2$ phase space used in Eq. (37). Using this result we have

$$dH'_{\beta} = \{H'_\beta, H'_\alpha\}_B dt^\alpha$$

and, consequently, the integrability condition (42) reduces to

$$dH'_\alpha = 0, \ \forall \alpha.$$
6 Example

As an example we analyze the case of the electromagnetic field coupled to a spinor, whose Hamiltonian formalism was analyzed in references [3, 4]. We will consider the Lagrangian density written as

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \tilde{\psi} \gamma^\mu (\partial_\mu + ieA_\mu) \psi - m \tilde{\psi} \psi, \]

where \( A_\mu \) are even variables while \( \psi \) and \( \tilde{\psi} \) are odd ones. The electromagnetic tensor is defined as \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and we are adopting the Minkowski metric \( \eta_{\mu\nu} = diag(+1, -1, -1, -1) \).

6.1 Hamiltonian formalism

Let’s first review Dirac’s Hamiltonian formalism. The momenta variables conjugated, respectively, to \( A_\mu, \psi \) and \( \tilde{\psi} \), are

\[ p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = -F_{0\mu}, \]

\[ p_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i \tilde{\psi} \gamma^0, \quad p_\tilde{\psi} = \frac{\partial \mathcal{L}}{\partial \dot{\tilde{\psi}}} = 0, \]

where we must call attention to the necessity of being careful with the spinor indexes. Considering, as usual, \( \psi \) as a column vector and \( \tilde{\psi} \) as a row vector implies that \( p_\psi \) will be a row vector while \( p_\tilde{\psi} \) will be a column vector.

From the momenta expressions we have the primary constraints

\[ \phi_1 = p_0 \approx 0, \]

\[ \phi_2 = p_\psi - i \tilde{\psi} \gamma^0, \quad \phi_3 = p_\tilde{\psi} \approx 0. \]

The canonical Hamiltonian is given by

\[ H_C = \int \mathcal{H}_C d^3 x = \int \left( p_\mu \dot{A}_\mu + (p_\psi)_\alpha \left( \dot{\psi} \right)_\alpha + (p_\tilde{\psi})_\alpha \left( \dot{\tilde{\psi}} \right)_\alpha - \mathcal{L} \right) d^3 x, \]
\[
H_C = \int H_C d^3x = \int \left[ \frac{1}{4} F_{ij} F^{ij} - \frac{1}{2} p_i p^i - A_0 \left( \partial_i p^i - e \bar{\psi} \gamma^0 \psi \right) - i \bar{\psi} \gamma^j \left( \partial_j + ie A_j \right) \psi + m \bar{\psi} \psi \right] d^3x. \tag{53}
\]

The primary Hamiltonian is
\[
H_P = \int \left( H_C + \lambda_1 \phi_1 + \phi_2 \lambda_2 + \lambda_3 \phi_3 \right) d^3x, \tag{54}
\]
where \(\lambda_1\) is an even variable and \(\lambda_2, \lambda_3\) are odd variables, \(\lambda_2\) being a column vector and \(\lambda_3\) a row vector. The fundamental nonvanishing Berezin brackets (here the brackets are the ones defined by Eq. (5) in the 2N phase space) are
\[
\{ A_\mu (x), p^\nu (y) \}_B = \delta_\mu^\nu \delta^3 (x - y), \tag{55}
\]
\[
\{ \psi (x), p_\psi (y) \}_B = \delta^3 (x - y), \tag{56}
\]
\[
\{ \bar{\psi} (x), p_{\bar{\psi}} (y) \}_B = \delta^3 (x - y). \tag{57}
\]

The consistency conditions are
\[
\{ \phi_1, H_P \}_B = \partial_i p^i - e \bar{\psi} \gamma^0 \psi \approx 0, \tag{58}
\]
\[
\{ \phi_2, H_P \}_B = -i (\partial_j - ie A_j) \bar{\psi} \gamma^j - e \bar{\psi} \gamma^0 A_0 - m \bar{\psi} + i \lambda_3 \gamma^0 \approx 0, \tag{59}
\]
\[
\{ \phi_3, H_P \}_B = -i (\partial_j + ie A_j) \gamma^j \psi + e \gamma^0 A_0 \psi + m \psi - i \gamma^0 \lambda_2 \approx 0. \tag{60}
\]

The last two ones will determine \(\lambda_2\) and \(\lambda_3\) while the first one will give rise to the secondary constraint
\[
\chi = \partial_i p^i - e \bar{\psi} \gamma^0 \psi \approx 0, \tag{61}
\]
for which the consistency condition will be identically satisfied with the use of the expressions for \(\lambda_2\) and \(\lambda_3\) given by Eq. (59) and Eq. (60). Taking the Berezin brackets among the constraints we have as nonvanishing results
\[
\{ (\phi_2)_\alpha, (\phi_3)_\beta \}_B = -i \left\{ \left( \bar{\psi} \gamma^0 \right)_\alpha, (p_{\bar{\psi}})_\beta \right\}_B = -i (\gamma^0)_{\beta\alpha}. \tag{62}
\]
\{ (\phi_2)_\alpha , \chi \}_B = -e \left\{ (p\psi)_\alpha , \bar{\psi} \gamma^0 \psi \right\}_B = e \left( \bar{\psi} \gamma^0 \right)_\alpha , \quad (63)

\{ (\phi_3)_\alpha , \chi \}_B = -e \left\{ (p\bar{\psi})_\alpha , \bar{\psi} \gamma^0 \psi \right\}_B = -e \left( \gamma^0 \psi \right)_\alpha , \quad (64)

where we explicitly wrote the spinor indexes. Obviously the \( \phi_1 \) constraint is first class, but we have another first class constraint. This can be seen from the supermatrix \( \Delta \) formed by the Berezin brackets among the second class constraints \( \chi, \phi_2 \) and \( \phi_3 \). Numbering the constraints as \( \Phi_1 = \chi, \Phi_2 = \phi_2 \) and \( \Phi_3 = \phi_3 \) we have this supermatrix in normal form (see ref.[3], appendix D) given, with spinor indexes indicated, by

\[
\Delta = \begin{pmatrix}
\{ \Phi_1, \Phi_1 \}_B & \{ \Phi_1, (\Phi_2)_\alpha \}_B & \{ \Phi_1, (\Phi_3)_\beta \}_B \\
\{ (\Phi_2)_\delta , \Phi_1 \}_B & \{ (\Phi_2)_\delta , (\Phi_2)_\alpha \}_B & \{ (\Phi_2)_\delta , (\Phi_3)_\beta \}_B \\
\{ (\Phi_3)_\mu , \Phi_1 \}_B & \{ (\Phi_3)_\mu , (\Phi_2)_\alpha \}_B & \{ (\Phi_3)_\mu , (\Phi_3)_\beta \}_B 
\end{pmatrix}, \quad (65)
\]

\[
\Delta = \begin{pmatrix}
0 & -e \left( \bar{\psi} \gamma^0 \right)_\alpha & e \left( \psi \gamma^0 \right)_\beta \\
e \left( \bar{\psi} \gamma^0 \right)_\delta & 0 & -i \left( \psi \gamma^0 \right)_{\beta \delta} \\
-e \left( \gamma^0 \psi \right)_\mu & -i \left( \psi \gamma^0 \right)_{\mu \alpha} & 0
\end{pmatrix}. \quad (66)
\]

This supermatrix has one eigenvector with null eigenvalue that is

\[
\begin{pmatrix}
1 \\
ie \left( \bar{\psi} \right)_\alpha \\
-ie \left( \psi \right)_\beta
\end{pmatrix}, \quad (67)
\]

so there is another first class constraint given by

\[
\phi = \chi + ie \left( \Phi_2 \right)_\alpha \left( \bar{\psi} \right)_\alpha - ie \left( \Phi_3 \right)_\beta \left( \psi \right)_\beta, \quad (68)
\]

\[
\phi = \partial_i p^i + ie \left( p\psi \psi + \bar{\psi} \bar{p} \bar{\psi} \right), \quad (69)
\]

that we will substitute for \( \chi \). So, we have the first class constraints \( \phi_1 \) and \( \phi \), and the second class ones \( \phi_2 \) and \( \phi_3 \). The supermatrix \( \Delta \) now reduces to the Berezin
brackets among the second class constraints $\phi_2$ and $\phi_3$ and is given by

$$
\Delta = \begin{cases}
\{\Phi_2, \Phi_2\}_B & \{\Phi_3, \Phi_3\}_B
\end{cases} = \begin{pmatrix}
0 & -i (\gamma^0)_{\beta \delta} \\
-i (\gamma^0)_{\mu \alpha} & 0
\end{pmatrix},
$$

having as inverse

$$
\Delta^{-1} = \begin{pmatrix}
0 & \frac{i (\gamma^0)_{\alpha \lambda}}{i (\gamma^0)_{\delta \gamma}} \\
\frac{i (\gamma^0)_{\alpha \lambda}}{i (\gamma^0)_{\delta \gamma}} & 0
\end{pmatrix}.
$$

With these result, the Dirac brackets among any variables $F$ and $G$ are

$$
\{F(x), G(y)\}_D = \{F(x), G(y)\}_B
$$

$$
-i \int d^3z \left( \{F(x), (\Phi_2)_a(z)\}_B (\gamma^0)_{a \beta} \{\Phi_3, \Phi_3\}_B (\gamma^0)_{\beta \alpha} \{G(y)\}_B \right)
$$

$$
+ \{F(x), (\Phi_3)_a(z)\}_B (\gamma^0)_{\beta \alpha} \{\Phi_2, \Phi_2\}_B (\gamma^0)_{\alpha \lambda} \{G(y)\}_B.
$$

The nonvanishing fundamental brackets now will be

$$
\{A_\mu(x), p^\nu(y)\}_D = \delta_\mu^\nu \delta^3(x - y),
$$

$$
\{(\psi)_\lambda(x), (\bar{\psi})_\alpha(y)\}_D = -i (\gamma^0)_{\lambda \alpha} \delta^3(x - y),
$$

$$
\{(\psi)_\lambda(x), (p_\psi)_\alpha(y)\}_D = \delta_{\lambda \alpha} \delta^3(x - y).
$$

Now we can make the second class constraints as strong equalities and write the equations of motion in terms of the Dirac brackets and the extended Hamiltonian given by

$$
H_E = \int H_E d^3x = \int (H_C + \lambda_1 \phi_1 + \alpha \varphi) d^3x.
$$

We must remember that, when making $\phi_2 = \phi_3 \equiv 0$ the constraint $\varphi$ becomes identical to the original secondary constraint $\chi$. Then, the equations of motion will be

$$
\dot{A}^i \approx \{A^i, H_E\}_D = -p^i + \partial^i A_0 - \partial^i \alpha,
$$
\[ A^0 \approx \{ A^0, H_E \}_D = \lambda_1, \] (78)

\[ \dot{p}^i \approx \{ p^i, H_E \}_D = \partial_j F^{ji} - e \bar{\psi} \gamma^i \psi, \] (79)

\[ \dot{\psi} \approx \{ \psi, H_E \}_D = -ieA_0 \psi - \gamma^0 \gamma^j (\partial_j + ieA_j) \psi - im\gamma^0 \psi, \] (80)

\[ \dot{\bar{\psi}} \approx \{ \bar{\psi}, H_E \}_D = ieA_0 \bar{\psi} - (\partial_j - ieA_j) \bar{\psi} \gamma^j \gamma^0 + im \bar{\psi} \gamma^0. \] (81)

Multiplying Eq. (80) from the left by \( i\gamma^0 \) we get

\[ i\gamma^0 \dot{\psi} = eA_0 \gamma^0 \psi - i\gamma^j (\partial_j + ieA_j) \psi + m\psi, \] (82)

\[ i(\partial_\mu + ieA_\mu) \gamma^\mu \psi - m\psi = 0, \] (83)

while multiplying Eq. (81) from the right by \( i\gamma^0 \) we get

\[ i\dot{\bar{\psi}} \gamma^0 = -e \bar{\psi} \gamma^0 A_0 - i(\partial_j + ieA_j) \bar{\psi} \gamma^j - m \bar{\psi}, \] (84)

\[ i\bar{\psi} \gamma^\mu \left( \bar{\partial}_\mu - ieA_\mu \right) + m \bar{\psi} = 0. \] (85)

These are the equations of motions with full gauge freedom. It can be seen, from Eq. (78), that \( A^0 \) is an arbitrary (gauge dependent) variable since its time derivative is arbitrary. Besides that, Eq. (77) shows the gauge dependence of \( A^i \) and, taking the curl of its vector form, leads to the known Maxwell equation

\[ \frac{\partial \vec{A}}{\partial t} = -\vec{E} - \vec{\nabla} (A_0 - \alpha) \Rightarrow \frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}. \] (86)

Writing \( j^\mu = e \bar{\psi} \gamma^\mu \psi \) we get, from Eq. (79), the inhomogeneous Maxwell equation

\[ \frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \times \vec{B} - \vec{j}, \] (87)

while the other inhomogeneous equation

\[ \vec{\nabla} \cdot \vec{E} = j^0 \] (88)

follows from the secondary constraint (61). Expressions (83) and (85) are the known Dirac’s equations for the spinor fields \( \psi \) and \( \bar{\psi} \).
6.2 Hamilton-Jacobi formalism

Now we apply the formalism presented in the previous sections. From the momenta definition we have the “Hamiltonians”

\[
H_0' = \int (P_0 + H_C) d^3x, \quad (89)
\]

\[
H_1' = \int p_0 d^3x, \quad (90)
\]

\[
H_2' = \int (p_\psi - i \bar{\psi} \gamma^0) d^3x, \quad (91)
\]

\[
H_3' = \int p_\psi d^3x, \quad (92)
\]

which are associated, respectively, to \( t = t_0 \) (remember that \( P_0 \) is the momentum conjugated to \( t \)), \( A_0, \psi \) and \( \bar{\psi} \). The first two \( H' \) are even variables, while the last two are odd. Then, using Eq. (26), we have

\[
dA_i = \frac{\delta H_0'}{\delta p_i} dt = (-p_i + \partial^i A_0) dt. \quad (93)
\]

From equation (30) we have

\[
dp^j = -\frac{\delta H_0'}{\delta A_i} dt = (\partial_j F^{ji} - e \bar{\psi} \gamma^i \psi) dt, \quad (94)
\]

\[
dp^0 = -\frac{\delta H_0'}{\delta A_0} dt = (\partial_j p^j - e \bar{\psi} \gamma^0 \psi) dt, \quad (95)
\]

\[
dp_\psi = -\frac{\delta H_0'}{\delta \psi} dt = (-i \partial_j \bar{\psi} \gamma^j - e \bar{\psi} \gamma^0 A_\mu - m \bar{\psi}) dt, \quad (96)
\]

\[
dp_\bar{\psi} = -\frac{\delta H_0'}{\delta \bar{\psi}} dt + \frac{\delta H_2'}{\delta \bar{\psi}} d\psi = (-i \gamma^j \partial_j \psi + e A_\mu \gamma^\mu \psi + m \psi) dt - i \gamma^0 d\bar{\psi}. \quad (97)
\]

The integrability conditions require \( dH' = 0 \), which implies for \( H_1' \)

\[
dH_1' = d \int p_0 d^3x = 0 \Rightarrow dp_0 = \partial_j p^j - e \bar{\psi} \gamma^0 \psi = 0, \quad (98)
\]

where we made use of Eq. (95). This expression is equivalent to the secondary constraint (61) and has to satisfy

\[
dH_4' = 0; \quad H_4' = \int (\partial_j p^j - e \bar{\psi} \gamma^0 \psi) d^3x, \quad (99)
\]
which is indeed identically satisfied.

For $H'_2$ we have

$$dH'_2 = d \int (p_\psi - i \bar{\psi} \gamma^0) \, d^3x = 0 \Rightarrow dp_\psi = i \left( d \bar{\psi} \right) \gamma^0, \quad (100)$$

which can't be written as an expression like $H' = 0$ due to the presence of two differentials ($dp_\psi$ and $d \bar{\psi}$) but, substituting in Eq. (96), we get

$$i \left( d \bar{\psi} \right) \gamma^0 = \left( -i \partial_j \bar{\psi} \gamma^j - e \bar{\psi} \gamma^\mu A_\mu - m \bar{\psi} \right) dt, \quad (101)$$
i.e.

$$-i \bar{\psi} \gamma^0 - i \partial_j \bar{\psi} \gamma^j - e \bar{\psi} \gamma^\mu A_\mu - m \bar{\psi} = 0, \quad (102)$$

$$i \bar{\psi} \gamma^\mu \left( \bar{\partial}_\mu - ie A_\mu \right) + m \bar{\psi} = 0. \quad (103)$$

For $H'_3$ we have

$$dH'_3 = d \int p_\bar{\psi} d^3x = 0 \Rightarrow dp_\bar{\psi} = 0, \quad (104)$$

that, similarly to the case above, can be used in Eq. (97) giving

$$\left( -i \gamma^j \partial_j \psi + e A_\mu \gamma^\mu \psi + m \psi \right) dt - i \gamma^0 d\psi = 0, \quad (105)$$

$$i \left( \partial_\mu + ie A_\mu \right) \gamma^\mu \psi - m \psi = 0. \quad (106)$$

Finally, we can verify, using the above results, that $dH'_0 = 0$ is identically satisfied.

Equations (103) and (106) are identical, respectively, to equations (85) and (83) obtained in Hamiltonian formalism. Besides that, equations (94) and (79) are equivalent, while Eq. (93) corresponds to Eq. (77) except for an arbitrary gauge factor.
7 Conclusions

In this work we presented a formal generalization of the Hamilton-Jacobi formalism for singular systems with Berezinian variables, obtaining the equations of motion as total differential equations (26) and (30). In these equations, the coordinates $q^\alpha = t^\alpha (\alpha = 1, ..., R)$, whose momenta are constrained, play the role of evolution parameters of the system, together with the time parameter $t = t^0$. So, the system’s evolution is described by contact transformations generated by the “Hamiltonians” $H'_\alpha$ and parametrized by $t^\alpha$ (with $\alpha = 0, 1, ..., R$), were $H'_0$ is related to the canonical Hamiltonian by Eq. (22) and the other $H'_\alpha (\alpha = 1, ..., R)$ are the constraints given by Eq. (23). This evolution is considered as being always restricted to the constraints surface in phase space, there is no complete phase space treatment that is latter reduced to the constraints surface, as in Dirac’s formalism with the use of weak equalities.

We should observe that, in the case of systems composed exclusively by even variables, all parities are equal to zero and equations (26), (30), (31) reduce to the results obtained in ref.[1]. Furthermore, if the system is nonsingular, we have $H'_\alpha \equiv 0$ except for $\alpha = 0$, so the total differential equations (26) and (30) will be reduced to the expressions given by Eq. (4).

The integrability conditions (which relation to the consistency conditions in Dirac’s formalism is discussed in the appendix) were shown to be equivalent to the necessity of the vanishing of the variation of each $H'_\alpha (\alpha = 0, ..., R)$, i.e. $dH'_\alpha = 0$.

The example presented was chosen for its completeness: it is a singular system with even and odd variables and its Hamiltonian treatment contains all kinds of constraints (primary and secondary, first and second class ones). This example is very illustrative, since it allows a comparison between all features of Dirac’s and Hamilton-Jacobi formalisms. For example, the fact that the integrability
conditions $dH'_2 = 0$ and $dH'_3 = 0$ give expressions involving some differentials $dt^\alpha$
is related to the fact that the corresponding Hamiltonian constraints $\phi_2$ and $\phi_3$
are second class constraints and determine some of the arbitrary parameters in
the primary Hamiltonian (54). Similarly, the fact that the condition $dH'_1 = 0$
generated an expression like $H'_4 = 0$ is related to the fact that the corresponding
Hamiltonian constraint $\phi_1$ is a first class one (see appendix).

Finally, we must call attention to the presence of arbitrary variables in some
of the Hamiltonian equations of motion due to the fact that we have gauge de-
pendent variables and we have not made any gauge fixing. This does not occur
in Hamilton-Jacobi formalism since it provides a gauge-independent description
of the systems evolution due to the fact that the Hamilton-Jacobi function $S$
contains all the solutions that are related by gauge transformations.

8 Appendix: Equivalence among consistency
and integrability conditions

In this appendix we will show the equivalence among the integrability conditions
of the formalism showed above and the consistency conditions in Dirac’s Hamil-
tonian formalism, in a similar way to what was made for usual variables [16]. In
the notation used in this paper the Dirac’s primary constraints are written, from
Eq. (23), as

$$H'_{\alpha} \equiv P_{\alpha} + H_{\alpha} (q^i; p_a) \approx 0,$$

(107)

where $\alpha = 1, ..., R; i = 1, ..., N$. The canonical Hamiltonian is given by $H_0$ in Eq.
(20), so the primary Hamiltonian $H_P$ is

$$H_P \equiv H_0 + H'_{\alpha} v^\alpha,$$

(108)
where the $v^\alpha$ are unknown coefficients related to the undetermined velocities $\dot{q}^\alpha$ [4]. The ordering of the $v^\alpha$ with respect to the $H'_\alpha$ is a matter of choice, since it will simply produce a change of sign, but the natural procedure, that identifies $v^\alpha$ and $q^\alpha$, suggests the ordering above as a consequence of the ordering adopted in the Hamiltonian (3). This ordering is also the most natural choice to our purpose but is, of course, irrelevant for systems containing only usual variables. The consistency conditions, which demand that the constraints preserved by time evolition, are written as

$$\dot{H}'_\mu \approx \{H'_\mu, H_P\}_B = \{H'_\mu, H_0\}_B + \{H'_\mu, H'_\alpha\}_B \dot{q}^\alpha = 0, \quad (109)$$

where $\alpha, \mu = 1, \ldots, R$ and the Berezin brackets here are that given in Eq. (5) defined in the usual $2N$ dimensional phase space and we have made the explicit identification $\dot{q}^\alpha \equiv v^\alpha$.

Multiplying the above equation by $dt$ we get

$$dH'_\mu \approx \{H'_\mu, H_0\}_B dt + \{H'_\mu, H'_\alpha\}_B dt^\alpha = 0, \quad (110)$$

where, as before, $q^\alpha = t^\alpha$ but we are still making $\alpha = 1, \ldots, R$. At this point we can already see that, when Dirac’s consistency conditions are satisfied we have $dH'_\mu = 0$ satisfied. We must see now that we have $dH'_0 = 0$ when Dirac’s consistency conditions are satisfied. The Hamiltonian equation of motion for $H'_0$ is

$$\dot{H}'_0 \approx \{H'_0, H_P\}_B = \{H'_0, H'_0\}_B + \{H'_0, H'_\alpha\}_B \dot{q}^\alpha, \quad (111)$$

which, multiplied by $dt$ becomes

$$dH'_0 \approx \{H'_0, H_0\}_B dt + \{H'_0, H'_\alpha\}_B dt^\alpha. \quad (112)$$

Remembering that the “momentum” $P_0$ in $H'_0$ is independent of the canonical variables $q^i$ and $p_i$, we have

$$dH'_0 \approx \{H_0, H_0\}_B dt + \{H_0, H'_\alpha\}_B dt^\alpha = \{H_0, H'_\alpha\}_B dt^\alpha. \quad (113)$$
But, if Dirac’s consistency conditions are satisfied, we must have only primary first class constraints, otherwise we would have conditions imposed on the unknown velocities $\dot{q}^\alpha$. So, the preservation of constraints in time will reduce to

$$\left\{ H'_\mu, H_0 \right\}_B \approx 0,$$

and the right side of Eq. (113) will be zero. This is simply a consequence of the fact that, once all Dirac’s conditions are satisfied, the Hamiltonian is preserved. So the condition $dH'_0 = 0$ is satisfied when Dirac’s consistency conditions are satisfied.

This shows that the integrability conditions in Hamilton-Jacobi formulation will be satisfied when Dirac’s consistency are satisfied. Similarly, we can consider that we have the integrability conditions satisfied so that $dH'_0 = dH'_\mu = 0$ and then Eq. (110), which is equivalent to Eq. (109), implies that Dirac’s conditions are satisfied. So, both conditions are equivalent.

Now, we will consider that these conditions are not initially satisfied. When we have only first class constraints in Hamiltonian formalism we will simply get a new constraint from some of the conditions (109). From Eq. (110) we see that this will imply an expression like $dH'_\mu = H' dt$ ($H' \approx 0$ is the secondary Hamiltonian constraint) which means that there will be a new $H'$ in Hamilton-Jacobi formalism that have to satisfy $dH' = 0$.

If we have some second class Hamiltonian constraints the consistency conditions (109) will imply a condition over some of the velocities $\dot{q}^\alpha$. From Eq. (110) we see that, in Hamilton-Jacobi approach, there will be conditions imposed on some differentials $dt^\alpha$.

Such correspondence among the formalisms can be clearly seen in the example presented in this paper.
Besides that, Eq. (110) and Eq. (112) can be written as
\[ dH'_\mu \approx \{ H'_\mu, H'_\alpha \}_B dt^\alpha, \] (115)
were now \( \alpha, \mu = 0, 1, ..., R \) and the Berezin bracket is again defined in the \( 2N + 2 \)
dimensional phase space containing \( t^0 \) and \( P_0 \). This equation is obviously identical
to Eq. (45), that leads to the integrability condition \( dH'_\mu = 0 \), and its right hand
side was showed to correspond to Dirac’s consistency conditions. Consequently,
this expression shows directly the relation among consistency and integrability
conditions.

It’s important to notice that here we are not considering any explicity de-
pendence on time, neither of the constraints nor of the canonical Hamiltonian,
because it is an usual procedure in Hamiltonian approach. But the equations
of Hamilton-Jacobi formalism were obtained without considering this condition
and, consequently, remain valid if we consider systems with Lagrangians that are
explicitly time dependent.

But Hamiltonian approach is also applicable to such systems (see reference [3],
page 229) and in this case we can follow a procedure similar to that one showed
here and demonstrate the correspondence among Dirac’s consistency conditions
and integrability conditions.

Finally, some words about the simpletic structure. Using Eq. (38), we can
writte Eq. (26) and Eq. (30) in terms of left derivatives as
\[ dq^i = \frac{\partial H'_\alpha}{\partial p_i} dt^\alpha, \] (116)
\[ dp_i = -(-1)^{P(i)} \frac{\partial H'_\alpha}{\partial q^i} dt^\alpha, \] (117)
where, as before, \( i = 0, 1, ..., N \) and \( \alpha = 0, 1, ..., R \). These expressions can be
compactly written as
\[ d\eta^I = E^{IJ} \frac{\partial H'_\alpha}{\partial \eta^J} dt^\alpha, \] (118)
were we used the notation

\[ \eta^{1i} = q^i, \quad \eta^{2i} = p_i \Rightarrow \eta^I = \left(q^i, p_i\right), \quad I = (\zeta = 1, 2; \ i = 0, 1, ..., N) , \]  \hspace{1cm} (119)

\[ E_{IJ} = \delta^i_j \left( \delta^1_\sigma \delta^2_\sigma - (-1)^{\beta(i)} \delta^1_\sigma \delta^2_\sigma \right), \quad I = (\zeta; \ i), \quad J = (\sigma; \ j); \]  \hspace{1cm} (120)

that was introduced in page 76 of reference [3]. The Berezin brackets defined in Eq. (5) can be written as

\[ \{ F, G \}_B = \frac{\partial_r F}{\partial \eta^I} E_{IJ} \frac{\partial_r F}{\partial \eta^J} . \]  \hspace{1cm} (121)

This simplectic notation allows us to obtain the expression for the total differential for any function \( F \left(t^\beta, q^a, p^a\right) \) in a more direct way. Using it in Eq. (43) we get

\[ dF = \frac{\partial_r F}{\partial \eta^I} d \eta^I , \]  \hspace{1cm} (122)

where the use of Eq. (118) gives

\[ dF = \frac{\partial_r F}{\partial \eta^I} E_{IJ} \frac{\partial_r H'_\alpha}{\partial \eta^J} dt^\alpha = \{ F, H'_\alpha \}_B dt^\alpha , \]  \hspace{1cm} (123)

in agreement with Eq. (44).

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References


