We investigate the phenomenon of particle creation of the massless scalar field in the model of spacetime in which, depending on the model’s parameter $a_0$, the chronology horizon could be formed. The model represents a two-dimensional curved spacetime with the topology $R^1 \times S^1$ which is asymptotically flat in the past and in the future. The spacetime is globally hyperbolic and has no causal pathologies if $a_0 < 1$, and closed timelike curves appear in the spacetime if $a_0 \geq 1$.

We obtain the spectrum of created particles in the case $a_0 < 1$. In the limit $a_0 \to 1$ this spectrum gives the number of particles created into mode $n$ near the chronology horizon. The main result we have obtained is that the number of scalar particles created into each mode as well as the full number of particles remain finite at the moment of forming of the chronology horizon.

I. INTRODUCTION

In 1988 Morris, Thorne, and Yurtsever [1] had demonstrated how one could manufacture closed timelike curves in a spacetime containing two relatively moving traversable wormholes. That work kindled considerable interest in the question whether it is possible, in principle, to construct a “time machine” — i.e. whether, by performing operations in a bounded region of an initially “ordinary” spacetime, it is possible to bring about a “future” in which there will be closed timelike curves. So far, in spite of many-year attempts to answer this question, there does not exist full and clear understanding of the problem. On the first stage of investigations, it seemed that the quantum field theory could give a mechanism which would be able to protect chronology. Hawking has suggested the chronology protection conjecture which states that the laws of physics will always prevent the formation of closed timelike curves [2,3]. His arguments were based on the supposition that the renormalized vacuum expectation values of a stress-energy tensor of a quantum field must diverge at the chronology horizon which separates a region with closed timelike curves from a region without them. Various attempts at proving Hawking’s conjecture have been made [4], culminating in singularity theorems of Kay, Radzikowski, and Wald [6].

However, in recent time, a number of examples of configurations with the bounded renormalized stress-energy tensor near the chronology horizon have been given [7–11]. For instance, in our works [8,9] it was shown that, in the case of automorphic fields, there exists a specific choice of quantum state for which the renormalized stress-energy tensor vanishes at the chronology horizon (in fact it is zero in the whole spacetime); Krasnikov [7] exhibited several 2D models, and Visser [10] presented a 4D model of spacetime – the “Roman ring” of traversable wormholes – for which the vacuum polarization can be made arbitrarily small all the way to the chronology horizon; in addition, Li [11] showed that the renormalized stress-energy tensor may be smoothed out by introducing absorption material, such that the spacetime with a time machine may be stable against vacuum fluctuations.

These examples indicate that it is impossible to prove the chronology protection conjecture taking into account only effects of the vacuum polarization. Recently Visser [12,10] has given arguments that the solving of the problem of chronology protection is impossible within the context of semi-classical theory of gravity and requires a fully developed theory of quantum gravity.

Nevertheless in this work we remain in the framework of semi-classical quantum gravity. We shall discuss such the aspect of the quantum field theory in spacetimes with the time machine which was so far not investigated. Up to this time, various aspects of vacuum polarization near the chronology horizon have mainly been investigating. Here we consider dynamics of the chronology horizon forming. As is known, time-dependent processes in the presence of quantum fields are accompanied by a pair creation. So one may expect that particles of quantized fields will be created in the process of chronology forming. The aim of this work is to determine the spectrum of particles and try to answer the question: Can the particle creation stop the chronology horizon forming?

We use the units $c = G = \hbar = 1$ throughout the paper.

II. A MODEL OF SPACETIME

Here we shall discuss a model of spacetime in which the chronology horizon is being formed. Let us consider
a strip \( \{ \eta \in (\infty, +\infty), \xi \in [0, L] \} \) on the \( \eta \)-\( \xi \) plane and assume that the points on the bounds \( \gamma^-: \xi = 0 \) and \( \gamma^+: \xi = L \) are to be identified: \((\eta, 0) \equiv (\eta, L)\). After this procedure we obtain a manifold \( \mathcal{M} \) with a topology of cylinder: \( R^1 \times S^1 \). Introduce on this manifold the metric

\[
ds^2 = d\eta^2 + 2a(\eta)d\eta d\xi - (1 - a^2(\eta))d\xi^2, \tag{1}\]

where \( a(\eta) \) is a monotonically increasing function of \( \eta \) which has the following asymptotic behavior:

\[
\begin{align*}
  a(\eta) \to 0 & \quad \text{if } \eta \to -\infty, \\
  a(\eta) \to a_0 & \quad \text{if } \eta \to +\infty,
\end{align*} \tag{2}
\]

where \( a_0 \) is some constant. Further we shall call the manifold \( \mathcal{M} \) as the spacetime \( \mathcal{M} \) with the metric (1). Note that the metric’s coefficients in Eq. (1) do not depend on \( \xi \), so they themself and their derivatives are taking the same values at the points \((\eta, 0)\) and \((\eta, L)\). Hence, the internal metrics and external curvatures at both lines \( \gamma^- \) and \( \gamma^+ \) are identical. This guarantees the regularity of the spacetime \( \mathcal{M} \).

The metric (1) describes a curved spacetime which is asymptotically flat in the past, \( \eta \to -\infty \) (the “in-region”), and in the future, \( \eta \to +\infty \) (the “out-region”). Really, as follows from (2), if \( \eta \to -\infty \), the metric (1) takes exactly the Minkowski form

\[
ds^2 = d\eta^2 - d\xi^2. \tag{3}\]

In the future, \( \eta \to +\infty \), the metric (1) reads

\[
ds^2 = d\eta^2 + 2a_0 d\eta d\xi - (1 - a_0^2) d\xi^2, \tag{4}\]

and takes the Minkowski form

\[
ds^2 = dt^2 - dx^2 \tag{5}\]

in new ‘Minkowski’ coordinates

\[
t = \eta + a_0 \xi, \quad x = \xi. \tag{6}\]

Note that the spacetime \( \mathcal{M} \) could be considered as the factor space: \( \mathcal{M} = \mathcal{M} / \mathcal{R} \). Here \( \mathcal{M} \) is an universal covering spacetime for \( \mathcal{M} \), in our case it is the whole plane \((\eta, \xi)\), and \( \mathcal{R} \) is the equivalence relation:

\[
(\eta, \xi + L) \equiv (\eta, \xi) \tag{7}\]

Consider now the causal structure of the spacetime \( \mathcal{M} \). With this aim we have to define null (lighthlike) curves which form a light cone at a point. The equations for null curves could be found from the condition \( ds^2 = 0 \). There exist exactly two null curves which go via each point of the spacetime and their equations read

\[
\begin{align*}
  \xi + \int^{\eta} \frac{d\eta}{1 + a(\eta)} = & \quad \text{const}, \\
  \xi - \int^{\eta} \frac{d\eta}{1 - a(\eta)} = & \quad \text{const.} \tag{8}\end{align*}
\]

As follows from the asymptotical properties (2) of \( a(\eta) \) in the in-region the equations (8) take the form:

\[
\eta + \xi = \text{const}, \quad \eta - \xi = \text{const}, \tag{9}\]

and in the out-region they are

\[
\eta + (1 + a_0)\xi = \text{const}, \quad \eta - (1 - a_0)\xi = \text{const}. \tag{10}\]

Analysing of the equations (8) together with their asymptotical forms (9) and (10) reveals the following qualitative picture: The light future cone in the past (see Eqs.(9)) has the angle \( \alpha_1 = 90^\circ \) and has no inclination (i.e., the angle between the cone’s axis and the \( \eta \)-axis is zero); then, the cone is enlarging and inclining so that in the future (see Eqs.(10)) its angle becomes equal to \( \alpha_2 = \arctan \left( 1 + a_0 \right) - \arctan \left( 1 - a_0 \right) \) and the inclination’s angle becomes equal to \( \alpha_2 = \frac{\pi}{4} \left[ \arctan \left( 1 + a_0 \right) - \arctan \left( 1 - a_0 \right) \right] \). There are two qualitatively different cases: (i) \( a_0 < 1 \) and (ii) \( a_0 \geq 1 \). In the first case, \( a_0 < 1 \), this rotation of the cone on the \( \eta \)-\( \xi \) plane does not lead to the appearance of causal pathologies, i.e. all future-directed world lines remain unclosed. In the second case, \( a_0 \geq 1 \), the situation is cardinally changed (this case is illustrated by the spacetime diagram in the figure 1).

![Fig.1](image-url)

Namely, now there is such a moment of time (the value of \( \eta = \eta_* \)) when one of the cone’s side takes a ‘horizontal’ position, i.e. one of the null curve’s equations becomes \( \eta = \text{const}. \) But these lines are closed (see the Eq.(7)), and hence, at the moment \( \eta_* \) the closed null curves appear in our model. We shall speak that a time machine is being formed at this moment of time. If \( \eta_* = \infty \) (\( a_0 = 1 \)) the time machine is formed in the infinitely far future. Otherwise, it is formed at the time \( \eta_* < \infty \). In this case at later times \( \eta > \eta_* \) the closed line \( \eta = \text{const} \) lies inside
of the light cone, i.e. the region $\eta > \eta_*$ contains closed timelike curves.

Thus we may conclude we have constructed the model of spacetime in which, depending on the parameter $a_0$, the chronology horizon is being formed at some moment of time.

Now let us investigate a behavior of quantized fields in this model.

III. A PARTICLE CREATION

A. Scalar field: solutions of the wave equation

Consider a conformal massless scalar field $\phi$ for which the Lagrangian is

$$ L = \frac{1}{2} \nabla \mu \phi \nabla^\mu \phi. \quad (11) $$

The scalar field $\phi$ obeys the wave equation

$$ \square \phi = 0. \quad (12) $$

In addition, it follows from the identification rule (7) and from the quadratic form of the lagrangian (11) that the scalar field has to obey the periodic condition (the ordinary field)

$$ \phi(\eta, \xi + L) = \phi(\eta, \xi) \quad (13) $$

or the antiperiodic one (the twisted field)

$$ \phi(\eta, \xi + L) = -\phi(\eta, \xi). \quad (14) $$

In the metric (1) the wave equation (12) reads

$$ \left[ (1 - a^2) \partial_\eta^2 + 2a \partial_\eta \partial_\xi - \partial_\xi^2 - 2a^2 \partial_\eta + a' \partial_\xi \right] \phi(\eta, \xi) = 0, \quad (15) $$

where $\partial_\eta = \frac{\partial}{\partial \eta}$, $\partial_\xi = \frac{\partial}{\partial \xi}$ and a prime denotes the derivative on $\eta$, $a' = da/d\eta$.

First of all, let us solve this equation in the asymptotical regions. In the in-region, where $a(\eta) \to 0$ and $a'(\eta) \to 0$, the Eq.(15) reduces to

$$ [\partial_\eta^2 - \partial_\xi^2] \phi(\eta, \xi) = 0, \quad (16) $$

The complete set of solutions of this equation is

$$ \phi^{(\pm, \text{in})}_n = D^{(\text{in})}_n e^{ik_n \xi + i\omega \eta}, \quad (17) $$

where

$$ k_n = \frac{2\pi n}{L}, \quad n = \pm 1, \pm 2, \ldots \quad \text{(ordinary field)} $$

or

$$ k_n = \frac{2\pi(n + \frac{1}{2})}{L}, \quad n = 0, \pm 1, \pm 2, \ldots \quad \text{(twisted field)}, $$

$$ \omega = |k_n| \text{ and } \phi^{(+, \text{in})}_n \text{ and } \phi^{(-, \text{in})}_n \text{ are positive and negative frequency solutions (the “in-modes”) in the in-region, respectively. The in-modes form the basis in Hilbert space $H$ with the scalar product}$$

$$ (\phi_n, \phi_n') = -i \int_0^L d\xi \left( \frac{\partial \phi_n^*}{\partial \eta} - \frac{\partial \phi_n}{\partial \eta} \right) \eta = \text{const}, \quad (18) $$

where $d\Sigma^\mu = d\Sigma n^\mu$, with $d\Sigma$ being the volume element in a given spacelike hypersurface, and $n^\mu$ being the timelike unit vector normal to this hypersurface. Choosing the hypersurface $\eta = \text{const}$ we may write down the scalar product (18) in the in-region as follows:

$$ (\phi_n, \phi_n') = -i \int_0^L d\xi \left( \frac{\partial \phi_n^*}{\partial \eta} - \frac{\partial \phi_n}{\partial \eta} \right) \eta = \text{const}. \quad (19) $$

The in-modes are orthonormal provided

$$ D^{(\text{in})}_n = \frac{1}{\sqrt{\pi \alpha}} \delta_{\alpha \beta} \quad (20) $$

where $\alpha = 0$ for an ordinary scalar field and $\alpha = \frac{1}{2}$ for a twisted one.

Analogously we may find a solution of the wave equation (15) in the out-region. There $a(\eta) \to a_0$ and $a'(\eta) \to 0$, and the Eq.(15) reduces to

$$ \left[ (1 - a^2_0) \partial_\eta^2 + 2a_0 \partial_\eta \partial_\xi - \partial_\xi^2 \right] \phi(\eta, \xi) = 0, \quad (21) $$

The complete set of solutions of this equation is

$$ \phi^{(+, \text{out})}_n = D^{(\text{out})}_n e^{ik_n \xi e^{-i\omega \eta}}, \quad (22) $$

where we denote

$$ \beta = 1 - a^2_0, \quad (23) $$

and $\phi^{(+, \text{out})}_n$ and $\phi^{(-, \text{out})}_n$ are positive and negative frequency solutions (the “out-modes”) in the out-region, respectively. The out-modes (22) form the basis in Hilbert space $H$ with the scalar product (18) in the out-region reads

$$ (\phi_n, \phi_n') = -i \int_0^L d\xi \left\{ \left( \frac{\partial \phi_n^*}{\partial \eta} - \frac{\partial \phi_n}{\partial \eta} \right) \eta = \text{const} \right\} \quad (24) $$

The out-modes are orthonormal provided

$$ D^{(\text{out})}_n = \frac{\beta}{4\pi |\alpha|} \delta_{\alpha \beta}, \quad (25) $$

Now let us solve the Eq.(15) in a general case. Noting that the metric (1) is invariant under translations in the
The solution of the equation $v$ roughly speaking(!), neglecting terms with $\Omega$ is no WKB approximation. Remind that the last means, roughly speaking(1), neglecting terms with $\Omega$, $\Omega''$, ... in the solution of the equation $v'' + \Omega^2(v) v = 0$. Indeed, we could now neglect in (31) the terms $a'' a (1-a^2)$ which are fulfilled. That is we shall only consider the modes $\xi$-direction and taking into account the periodic or antiperiodic conditions (13), (14) we may find solutions in the following form:

$$\phi_n(\eta, \xi) = u_n(\eta)e^{ik_n \xi}. \tag{26}$$

Substituting the expression (26) into the wave equation (15) we obtain the equation for $u_n(\eta)$:

$$(1-a^2)u'' + 2a(i_kn-a')u' -ik_n(a_kn-a')u = 0. \tag{27}$$

Introduce a new function $v(\eta)$ by the relation

$$u(\eta) = v(\eta) \exp \left(-\int_0^\eta a(i_kn-a')d\tilde{\eta}\right). \tag{28}$$

Note that in the in-region the relation (28) has the asymptotical form

$$u(\eta) = v(\eta), \tag{29}$$

whereas in the out-region it reads

$$u(\eta) = v(\eta)e^{-ik_n a \beta^{-1} \eta}. \tag{30}$$

After substituting Eq.(28) into Eq.(27) we obtain the second-order differential equation for $v(\eta)$ in so-called normal form $v'' + \Omega^2(\eta) v = 0$:

$$v'' + \frac{k_n^2 + a^2 + a''a(1-a^2)}{(1-a^2)^2} v = 0. \tag{31}$$

Further we shall solve this equation only for modes for which the conditions

$$k_n^2 \gg a^2, \quad k_n^2 \gg a''a(1-a^2) \tag{32}$$

are fulfilled. That is we shall only consider the modes whose wave length is much less than a typical scale of variation of the function $a(\eta)$. Taking into account the conditions (32) we could now neglect in (31) the terms $a''^2$ and $a''a(1-a^2)$ and rewrite

$$v'' + \frac{k_n^2 v}{(1-a^2)^2} = 0. \tag{33}$$

Now let us restrict our consideration by the special form of the function $a(\eta)$:

$$a^2(\eta) = \frac{1}{2}a_0^2(1 + \tanh \gamma \eta), \tag{34}$$

where $\gamma$ is a parameter. It is easy to see that the function $a(\eta)$ defined by Eq.(34) possesses the necessary asymptotical behavior (2). The quantity $T = (2\gamma)^{-1}$ gives the typical time of variation of the function $a(\eta)$ from one asymptotical value to another one. The conditions (32) reduce now to

$$k_n^2 \gg \frac{4}{27}a_0^2. \tag{35}$$

Substituting the expression (34) into Eq.(33) we could rewrite the equation (33) as

$$v'' + \frac{4k_n^2 v}{(2 - a_0^2(1 + \tanh \gamma \eta))^2} = 0. \tag{36}$$

A solution of this equation could be found in terms of hypergeometric series. In particular, to find a solution which will be positive frequency in the in-region we have to choose the solution of the equation (36) as follows:

$$v_n^{(in)}(\eta) = D_n^{(in)}(-z)^{-i\mu}(1-\beta z)^\mu F(q, r; s; \beta z), \tag{37}$$

where $D_n^{(in)}$ is defined by (20) and

$$z = e^{2\gamma \eta}, \quad \mu = \frac{\omega}{2\gamma}, \quad \beta = 1 - a_0^2$$

$$s = 1 - 2i\mu, \quad \sigma = \frac{1}{2} + \sqrt{\frac{1}{4} - \mu^2(1-\beta)^2}, \tag{38}$$

$$q = \sigma + i\mu, r = \sigma - i\mu, \frac{1}{\beta},$$

and $F(q, r; s; z)$ is a Gaussian hypergeometric function. Taking into account the relations (28) and (26) we can now write down the solution of the wave equation (15) as

$$f_n(\eta, \xi) = D_n^{(in)}e^{ik_n \xi}e^{-i\omega \eta} \exp \left(-\int_0^\eta \frac{a(i_kn-a')}{(1-a^2)^2}d\tilde{\eta}\right) \times (1-\beta z)^\mu F(q, r; s; \beta z) \tag{39}$$

Note that $F(q, r; s; 0) = 1$ for any $q$, $r$ and $s$. Now it is not difficult to see that this solution in the in-region, where $\eta \to -\infty$ or $\eta \to +\infty$, has the following asymptotical form: $f_n(\eta, \xi) \approx D_n^{(in)}e^{ik_n \xi}e^{-i\omega \eta}$, which coincides with the positive frequency in-modes $\phi_n^{(+, in)}$.

**B. Bogolubov coefficients**

Now let us remind some mathematical aspects for describing of the physical phenomenon of particle creation by a time-depend gravitational field.

So, we have obtained the set of solutions $\{f_n\}$ which are positive frequency (the “in-modes”) in the past. Let $\{F_n\}$ be positive frequency solutions (the “out-modes”) in the future. (We do not need to know an explicit form of these solutions. It will be enough to know their asymptotic properties in the out-region, i.e. $F_n \approx \phi_n^{(+, out)}$.) We may choose these two sets of solutions to be orthonormal, so that

---

*Let me emphasize that the approximation which we use is no WKB approximation. Remind that the last means, roughly speaking(!), neglecting terms with $\Omega$, $\Omega''$, ... in the solution of the equation $v'' + \Omega^2(\eta) v = 0$.}
\( (f_n, f_m^*) = (F_n, F_m^*) = \delta_{nm}, \)
\( (f_n^*, f_m^*) = (F_n^*, F_m^*) = -\delta_{nm}, \)
\( (f_n, f_m^*) = (F_n, F_m^*) = 0. \) (40)

The in-modes may be expanded in terms of the out-modes:
\[ f_n = \sum_m (\alpha_{nm} F_m + \beta_{nm} F_m^*). \] (41)

Inserting this expansion into the orthogonality relations, Eq. (40), leads to the conditions
\[ \sum_m (\alpha_{nm} \alpha_{n'm}^* - \beta_{nm} \beta_{n'm}^*) = \delta_{nm'}, \] (42)
and
\[ \sum_m (\alpha_{nm} \alpha_{n'm} - \beta_{nm} \beta_{n'm}) = 0. \] (43)

The field operator, \( \phi \), may be expanded in terms of either the \( \{f_n\} \) or the \( \{F_n\} \):
\[ \phi = \sum_n (\alpha_n f_n + a_n^\dagger f_n^*) = \sum_n (b_n F_n + b_n^\dagger F_n^*). \] (44)

The \( a_n \) and \( a_n^\dagger \) are annihilation and creation operators, respectively, in the in-region, whereas the \( b_n \) and \( b_n^\dagger \) are the corresponding operators for the out-region. The in-vacuum state is defined by \( a_n |0\rangle_{in} = 0, \forall n \), and describes the situation when no particles are present initially. The out-vacuum state is defined by \( b_n |0\rangle_{out} = 0, \forall n \), and describes the situation when no particles are present at late times. Noting that \( a_n = (\phi, f_n) \) and \( b_n = (\phi, F_n) \), we may expand the two sets of creation and annihilation operator in terms of one another as
\[ a_n = \sum_m (\alpha_{nm} b_m - \beta_{nm} b_m^\dagger), \] (45)
or
\[ b_m = \sum_n (\alpha_{nm} a_n + \beta_{nm} a_n^\dagger). \] (46)

This is a Bogolubov transformation, and the \( \alpha_{nm} \) and \( \beta_{nm} \) are called the Bogolubov coefficients.

Let us assume that no particle were present before the gravitational field is turned on. In the Heisenberg approach \( |0\rangle_{in} \) is the state of the system for all time. However, the physical number operator which counts particles in the out-region is \( N_m = b_m b_m^\dagger \). Thus the number of particles created into mode \( m \) is
\[ \langle N_m \rangle = \langle m | b_m^\dagger b_m | m \rangle = \sum_n |\beta_{nm}|^2. \] (47)

To find the Bogolubov coefficients in our case we have to determine an asymptotical form of the in-modes in the out-region where \( \eta \to \infty \) or \( z \to -\infty \). With this aim we consider the analytical continuation of the hypergeometric function \( F(q,r; s; z) \) into the region of large values of \( |z| \) [13]:
\[ F(q,r; s; z) = \frac{\Gamma(s)\Gamma(r-q)}{\Gamma(r)\Gamma(s-q)}(-z)^{-q}F(q,1-s+q;1-r+q;\frac{1}{z}) \]
+ \( \frac{\Gamma(s)\Gamma(r-q)}{\Gamma(q)\Gamma(s-r)}(-z)^{-r}F(r,1-s+r;1-q+r;\frac{1}{z}). \) (48)

After substituting this expression into the Eq.(39) it is not difficult to see that the asymptotic of the in-modes in the out-region is
\[ f_n(\eta, \xi) \approx D_n^{\{out\}} e^{ik_n \xi} e^{-ik_n a_n} \beta^{-1} \eta \]
\times \left( (-\beta)^{-\mu \beta^{-1}(1-\beta)} \frac{\Gamma(s)\Gamma(r-q)}{\beta^{1/2}\Gamma(q)} e^{-i\omega \beta^{-1} \eta} \right)^{(-\beta)^{\mu \beta^{-1}(1+\beta)} \frac{\Gamma(s)\Gamma(q-r)}{\beta^{1/2}\Gamma(q)} e^{i\omega \beta^{-1} \eta} \right). \] (49)

Comparing the last expression with the asymptotical form of the out-modes, Eq.(22), we may write
\[ \phi_{in}(\eta, \xi) = A_n \phi_{in}^{(+\text{out})} + B_n \phi_{in}^{(+\text{out})} \]
\[ = A_n F_n + B_n F_n^*. \] (50)

where
\[ A_n = (-\beta)^{-\mu \beta^{-1}(1-\beta)} \frac{\Gamma(s)\Gamma(r-q)}{\beta^{1/2}\Gamma(q)} F(r,1-s+r;1-q+r;\frac{1}{z}), \] (51)
\[ B_n = (-\beta)^{\mu \beta^{-1}(1+\beta)} \frac{\Gamma(s)\Gamma(q-r)}{\beta^{1/2}\Gamma(q)} F(q,1-s+q;1-r+q;\frac{1}{z}). \] (52)

As follows from Eq.(41), the coefficients \( A_n \) and \( B_n \) are related to the Bogolubov coefficients by \( \alpha_{nm} = A_n \delta_{nm}, \beta_{nm} = B_n \delta_{n-m} \). The number of particles created into mode \( n \) is now determined as
\[ \langle N_n \rangle = |B_n|^2. \] (53)

Using the Eq.(52) gives
\[ \langle N_n \rangle = \beta^{-1} \left| \frac{\Gamma(s)\Gamma(q-r)}{\Gamma(q)\Gamma(s-r)} \right|^2 = \beta^{-1} \left| \frac{\Gamma(s)\Gamma(q-r)}{\Gamma(q)\Gamma(s-r)} \right|^2 \] (54)
To carry out calculations in the above expression we shall use the following formulae [13]:
\[ |\Gamma(iy)|^2 = \frac{\pi}{y \sinh \pi y}, \quad |\Gamma(\frac{1}{2} + iy)|^2 = \frac{\pi}{\cosh \pi y}, \]
\[ |\Gamma(1 + iy)|^2 = \frac{\pi y}{\sinh \pi y}. \] (55)

By using this formulae and the relations (38) we may finally obtain the spectrum of created particles:
\[ \langle N_n \rangle = \frac{\cosh \frac{\pi w (1-\beta)}{\gamma \beta} + \cosh \pi \sqrt{\frac{(w(1-\beta))^2}{\gamma \beta^2} - 1}}{2 \sinh \frac{\pi w}{\gamma \beta} \sinh \frac{\pi w}{\gamma \beta}}. \] (56)
C. Particle creation near the chronology horizon

Now let us analyse how much particles of the massless scalar field is created at the moment of the chronology horizon forming. As was mentioned above, the value of the parameter \( a_0 \) determines either the time machine is formed or not. If \( a_0 < 1 \) then the chronology horizon is absent in the spacetime, but if \( a_0 = 1 \) then it appears in the infinitely far future. The expression (56) gives us the spectrum of particles created in the out-region. The spectrum depends on the parameter \( a_0 (\beta = 1 - a_0^2) \), and in the limit \( a_0 \to 1 \) the expression (56) will determine the number of particles created near the chronology horizon.

Going to the limit \( a_0 \to 1 \) in Eq.(56) gives

\[
\langle N_n \rangle = \frac{1}{\sinh \frac{\pi \omega}{\gamma}}.
\]

(57)

Thus we see that the number of particles created into mode \( n \) near the chronology horizon is finite. We may also conclude that the full number of particles \( N = \sum_n \langle N_n \rangle \) will be finite because the spectrum (57) is exponentially decreasing.

IV. CONCLUSION

Let us summarize. In this work we have constructed the model of spacetime in which, depending on the model’s parameter \( a_0 \), the chronology horizon could be formed; there are no causal pathologies if \( a_0 < 1 \), and the chronology horizon appears at the moment of time \( \eta_* < \infty \) if \( a_0 > 1 \). In the case \( a_0 = 1 \) closed lightlike curves are formed in the infinitely far future. The model represents a two-dimensional curved spacetime with the topology \( R^1 \times S^1 \) which is asymptotically flat in the past and in the future, but which is non-flat in the intermediate region. As a consequence, in this spacetime the creation of particles of quantized fields by the gravitational field is possible. We have studied the particle creation of a massless scalar field in the case \( a_0 < 1 \), i.e. in the case when the spacetime is globally hyperbolic and has no causal pathologies. As a result, the spectrum of created particles has been obtained (see Eq.(56)). In the limit \( a_0 \to 1 \) this spectrum gives the number of particles created into mode \( n \) near the chronology horizon (see Eq.(57)). The main result we have obtained is that the number of scalar particles created into each mode as well as the full number of particles remain finite at the moment of forming of the chronology horizon. This result might mean that the phenomenon of particle creation could not prevent the formation of time machine. However, to do the final conclusion one has to take into account a backreaction of created particles on a space-time metric.

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[4] The reader could find a review of various aspects of the time machine problem as well as a lot of references in an excellent monograph [5].


