Threshold Resummation for Dijet Cross Sections

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Abstract

We construct dijet differential cross sections at large momentum transfer, in which threshold logarithms have been summed to all orders in perturbation theory. This extends previous work on heavy quark production, by treating collinear singularities associated with hard, massless partons in the final state. The resummed corrections enable us to define, in the sense of factorization, the underlying color exchange mechanism. The influence of color exchange on the resummed cross section is contained in the eigenvalues and eigenvectors of an anomalous dimension matrix, which describes the factorization of coherent soft gluons from the hard scattering. The precise formulas depend on the partonic scattering angles and energies, as well as on the method used to define the jets in the final state. For cone dijets at fixed invariant mass, we find leading logarithmic corrections that, like those in the Drell-Yan process, are positive, and which grow with increasing dijet invariant mass. Other choices of dijet cross section can give, however, qualitatively different behavior, even at leading logarithm.
1 Introduction

Factorized cross sections in perturbative QCD separate universal parton densities from process-specific factors that describe the perturbative hard scattering of the partons [1]. These factors, or hard-scattering functions, may include singular distributions, defined by their integrals with smooth functions, such as the parton densities relevant to the process. To be specific, for the production of a system of mass $M$, through the collision of partons of invariant mass squared $\hat{s}$, the hard-scattering function will contain, at $n$th order, terms as singular as $(\alpha_s^n/n!)[\ln^{2n-1}(1 - z)/(1 - z)]_+$, with $z \equiv M^2/\hat{s}$. The limit $z \to 1$ is the edge of partonic phase space, or partonic threshold, at which no energy remains for QCD radiation. These singular contributions often increase the cross section. The classic case is the production of Drell-Yan pairs, with $M = Q$ the invariant mass of the electroweak vector boson produced by the annihilation of a quark pair. The possibility of controlling such singular distributions to all orders in perturbation theory was recognized early [2], and the full analysis, including nonleading logarithms was carried out [3, 4]. The formalism is closely related to the even earlier resummation of logarithms in the transverse momenta of the pairs [5]. Finally, it was noted that the resummation in logarithms of $1 - z$ can be performed, not only for the fully inclusive Drell-Yan cross section $d\sigma/dQ^2$, but also for the cross section at fixed rapidity, $d\sigma/dQ^2dy$ [6]. We will have occasion to recall and employ the explicit forms of these resummations below.

In this paper, we shall derive threshold resummation for several physically-relevant dijet cross sections at large momentum transfer. We will find both similarities and differences compared to Drell-Yan and related cross sections, due to the color content of partons emerging from the hard scattering. One difference is that in jet production the hard scattering is itself a QCD subprocess, and involves color exchange. Another is that the kinematics at threshold depend upon the method used to define the event. We shall see that both of these features of jet production influence higher-order corrections in an essential way. The broad outlines of our reasoning, however, follow the derivation of resummation from factorization, as discussed in Ref. [7].

When a hard process is mediated by QCD, rather than color-singlet electroweak annihilation, the situation becomes, not surprisingly, more complex. Nevertheless, it is not difficult to recognize that the leading threshold logarithms associated with the initial state are the same for QCD processes as for Drell-Yan at each order in perturbation theory. This observation was made the basis for the first estimates of the effects of resummed perturbation theory in heavy quark production [8, 9, 10]. The extension to nonleading logarithmic corrections has been carried out recently for heavy quarks [11, 12, 13]. At this level, intriguing effects begin to show up, in which the color structure of the underlying hard scattering influences the pattern of soft radiation near threshold. These results are related to the long-standing program of developing observable consequences of QCD coherence, in which properties of hadronic final states reflect color flow in partonic subprocesses [14].

For dijet (as opposed to heavy quark) production, threshold singularities also differ from the Drell-Yan case because of the presence of the final-state jets, of momentum $p_i$, $i = 1, 2$. When the cross section is evaluated at fixed dijet total invariant mass,
\[(p_1 + p_2)^2\], these differences turn out to be at the level of next-to-leading logarithm in the singular distributions. As we shall show, however, when the cross section is evaluated at fixed \((p_1 \cdot p_2)\), new leading-logarithmic contributions arise, which are negative. Depending on the partonic subprocess, these new contributions may even overcome the enhancements from initial state interactions, and produce an overall suppression of the cross section, relative to lowest order.

The resummation of singular distributions at partonic threshold is important because corrections very close to the edge of phase space may be numerically large, and taken at leading logarithm often (although not always) grow, rather than decrease, with the order \([11]\). The resummed cross section also requires integrals over the argument of the running coupling \([3, 4]\), indicating enhanced sensitivity to soft gluon effects, and suggesting the necessity of incorporating nonperturbative corrections. In addition, the resummation of threshold singularities in QCD hard scattering requires a general approach to the interplay of color exchange at short distances with the pattern of gluon radiation into the final state \([14]\). We shall see that beyond leading logarithms it is in general not possible to separate initial from final state emission, and that, in fact, the color structure of the hard scattering directly influences the flow of energy into the final state, not only at low order, but to all orders in perturbation theory.

As in Ref. \([12]\), we shall make strong use of the factorization properties of gauge theory cross sections at high energy. The basic complication in these arguments arises from the divergences of massless field theories in Minkowski space, in which the separation of long- and short-distance dynamics must be carried out relative to the light-like directions intrinsic to the problem. The principles and methods underlying the arguments below have been developed, for instance, in Refs. \([1, 15, 16]\). We shall try, however, to make the discussion relatively self-contained, and to explain our use of technical results as they arise.

We begin in Sec. 2 by identifying the cross sections, based on cone algorithms for jets, which we shall use as illustrative examples in this paper. In the following two sections, we go on to discuss the factorization properties of these cross sections (Sec. 3), and then to give interpretations for the various functions that appear in the factorized cross section in terms of field-theoretic matrix elements of nonlocal operators (Sec. 4). The pattern of these arguments is close to that used previously for resummation in heavy quark cross sections \([12]\), but now modified to treat the extra collinear divergences associated with the final-state jets. In Sec. 5, we organize the singular behavior of each of the component functions of the factorization formula. As in the case of heavy quarks, we identify an anomalous dimension matrix, which controls color-sensitive gluon radiation into the final state\(^1\). Combining these results, we give the resummed expressions for dijet cross sections in moment space. We observe that the form of our resummed cross section, Eq. (71), does not depend on the details of the jet-identification algorithm, and applies to any infrared-finite dijet cross section at large momentum transfer. We conclude with a short summary, which

\(^1\)In a forthcoming paper we will compute this matrix at one-loop order and discuss its diagonalization for the basic parton processes, including gluon-gluon scattering.
looks towards future work.

2 Cone-based Dijet Cross Sections

2.1 Definitions

In this section, we introduce dijet cross sections, describing the inclusive hadronic production of a pair of jets,

\[ h_A(p_A) + h_B(p_B) \rightarrow J_1(p_1) + J_2(p_2) + X(k) , \]  

at fixed rapidity interval,

\[ \Delta y = \frac{1}{2} \ln \left( \frac{p_1^+ p_2^-}{p_1^- p_2^+} \right) , \]  

and total rapidity,

\[ y_{JJ} = \frac{1}{2} \ln \left( \frac{p_1^+ + p_2^+}{p_1^- + p_2^-} \right) . \]  

We consider the situation represented in Fig. 1. We have two final-state jets, identified by cones of opening angles \( \delta_1 \) and \( \delta_2 \). The four-vector jet momenta \( p_i \) are defined as the total momenta flowing into the cones. In the limit of vanishing \( \delta_i \), the \( p_i^\mu \) approach light-like momenta.

To define the dijet cross sections, we must specify a large invariant, \( M_{JJ} \), which is held fixed. A natural choice is the dijet invariant mass,

\[ M_{JJ}^2 = (p_1 + p_2)^2 , \]  

but other specifications are possible. We shall see, in fact, that the nature of the resummed cross section depends critically on this choice. We shall illustrate this point with the alternate definition

\[ M_{JJ}^2 = 2 p_1 \cdot p_2 , \]  

the scalar product of the two jet momenta. In either case, large \( M_{JJ} \) at fixed \( \Delta y \) implies a large momentum transfer in the partonic subprocess.

As we integrate over allowable final states, the actual total momentum flowing into the cones may differ from a light-like vector by corrections proportional to the jet opening angle. To simplify our discussion, we shall assume that the cones are small enough so that all contributions proportional to \( \delta_i \ll 1 \) may be neglected, but at the same time large enough so that

\[ \alpha_s(Q) \ln \left[ \frac{1}{\delta_1} \right] \ll 1 , \]  

where \( Q \) is any of the hard scales of the cross section, typically the momentum transfer \([15, 16]\).
The introduction of cones removes all the final-state collinear singularities from the partonic cross section, which is then infrared safe, once the initial-state collinear singularities have been factored into universal parton distribution functions.

We assume that the jets are identified by an algorithm that samples phase space for sets of particles flowing into cones of size $\delta$. In this search, the jet direction, and hence the center of the cone, is defined by the total momentum of the corresponding set of particles. The details of the jet identification algorithm are otherwise not important to our arguments. Nevertheless, it may be helpful to have a definite procedure in mind.

To be specific, we define our cone jet algorithms as follows. Consider a final state $|X\rangle = |\{q_i\}_{X}\rangle$, consisting of the set $X \equiv \{q_i\}_{X}$ of particles with momenta $q_i^\mu$. For any subset of $X$, $x \subset X$, we define first a unit vector,

$$\hat{n}_x \equiv \frac{\sum_{j \in x} q_j^\mu}{\sqrt{\sum_{j \in x} q_j^\mu}}. \quad (7)$$

Given $\hat{n}_x$, we define a new set, $x'(\hat{n}_x, \delta)$, consisting of those particles flowing into a cone of half-angle $\delta$, centered about $\hat{n}_x$,

$$x' \equiv \left\{ q_k : \frac{\hat{q}_k \cdot \hat{n}_x}{|\hat{q}_k|} \geq \cos \delta \right\}. \quad (8)$$

In these terms, a jet is defined to be any set of particles, $x \subset X$, for which

$$x = x'$$

i.e., for which the particles within the cone defined by the set constitute the set exactly.

Of course, with this definition, a general final state may have many jets. To define our dijet cross sections, we take $p_1^\mu$ and $p_2^\mu$ to be the pair of jets of highest energy in the sample, consistent with some restrictions on their directions, implicit in our choices of ranges for $y_{JJ}$ and $\Delta y$ above. These algorithms are insensitive to emission of zero-momentum lines and/or rearrangements of momenta among collinear-moving particles. We may therefore expect them to be infrared safe [15, 17]. Of course, as noted in [15], the cancellation of virtual and real corrections fails when the final state reaches the edge of the region of phase space that defines a cone-jet cross section. (For a recent discussion of such momentum configurations in event shapes, see [18].) This happens, for example, when a finite-energy line $q_l \in x$, reaches the boundary of the cone about $\hat{n}_x$, i.e., when

$$\frac{\hat{q}_l \cdot \hat{n}_x}{|\hat{q}_l|} = \cos \delta. \quad (10)$$

Such singularities, however, are restricted to a lower-dimensional surface in phase space, and thus do not spoil the finiteness of the inclusive jet cross section [15].

5
2.2 Factorization

The standard factorized form of a hard dijet cross section is

\[ \frac{d\sigma_{h_A h_B \rightarrow J_1 J_2}(S, \delta_1, \delta_2)}{dM_{JJ}^2 dy_{JJ} d\Delta y} = \frac{1}{S^2} \sum_{f_A, f_B = u, d, g} \int dx_A d\phi_{f/h_A}(x_A, \mu^2) \phi_{f/h_B}(x_B, \mu^2) \]

\[ \times H_{f_A f_B}\left(\frac{p_i \cdot p_j}{\mu^2}, \alpha_s(\mu^2), \delta_1, \delta_2\right), \tag{11} \]

where the explicit factor of $1/S^2$ is introduced to make $H_{f_A f_B}$ dimensionless. The incoming partons carry fractions $x_A$ and $x_B$ of the momenta of the incoming hadrons $h_A$ and $h_B$, respectively. These integrals are weighted by the nonperturbative but experimentally measurable parton distribution functions (densities), $\phi_{f/h}$. The factorization scale $\mu$ separates the long-distance physics described by the parton distributions from the short-distance hard scattering. The first argument, $p_i \cdot p_j/\mu^2$, of the hard-scattering function, $H_{f_A f_B}$, represents all large invariants formed from the momenta of the incoming partons and final-state jets, $p_i, p_j = x_A p_A, x_B p_B, p_1, p_2, i \neq j$.

The perturbative factors $H_{f_A f_B}$ are smooth functions only away from the edges of partonic phase space. The threshold for the partonic subprocess is conveniently parameterized by the variable $z$,

\[ z = \frac{M_{JJ}^2}{x_A x_B S} \equiv \frac{M_{JJ}^2}{\hat{s}}, \tag{12} \]

where $S = (p_A + p_B)^2$ is the overall center of mass energy squared, and $\hat{s} = x_A x_B S$ is the corresponding quantity for the subprocess. At $z = 1$ (partonic threshold) there is just enough partonic energy to produce the observed final state, with no additional radiation. In general, $H$ includes distributions with respect to $1 - z$, with singularities at $n$th order in $\alpha_s$ of the type

\[ \left[ \ln^m(1 - z) \right]_{+}, \quad m \leq 2n - 1, \tag{13} \]

defined, as usual, by their integrals with any smooth functions $F(z)$,

\[ \int_y^1 dz \left[ \ln^m(1 - z) \right]_{+} F(z) = \int_y^1 dz \left[ \ln^m(1 - z) \right]_{+} \left[ F(z) - F(1) \right] \]

\[ - F(1) \int_0^y dz \left[ \ln^m(1 - z) \right]_{+}. \tag{14} \]

All distributions of this sort have been resummed for Drell-Yan and more recently for heavy quark pair production cross sections at leading and nonleading logarithms [3, 4, 12]. In the following, we extend this analysis to dijet cross sections.

We can simplify Eq. (11), using the observation of [6], which applies to QCD as well as electroweak cross sections. According to Ref. [6], in computing the leading power $(1/(1 - z))$ of $H$ for $z \to 1$, we may treat the total rapidity $y_{JJ}$ of the
produced heavy system (the dijets) as a constant, equal to its value at threshold, 
\[ y_{\text{thresh}} = (1/2) \ln(x_A/x_B) \],

and factorize initial state collinear divergences into the light-cone distribution functions \( \phi_{f/f} \), expanded in \( \alpha_s \). The remainders give the perturbative expansion for the infrared-safe hard scattering function [1].
Singular distributions of the sort (13) are now conveniently organized by taking a Mellin transform of the rapidity-integrated partonic cross section (17) with respect to \( \tau \), Eq. (16),

\[
\int_0^1 d\tau \tau^{N-1} \int d y_{J J} S^2 \frac{d\sigma_{f_A f_B \rightarrow J_1 J_2}(S, \delta_1, \delta_2)}{d M_{J J}^2 d y_{J J} d \Delta y} = \sum_f \hat{\sigma}_{\bar{f}_A/f_A}(N + 1, \mu^2, \epsilon) \hat{\sigma}_{f_B/f_B}(N + 1, \mu^2, \epsilon) \\
\times \hat{\sigma}(N, M_{J J}/\mu, \alpha_s(\mu^2), \delta_1, \delta_2) ,
\]

(18)

with \( \hat{\sigma}(N) = \int_0^1 dz \, z^{N-1} \hat{\sigma}(z) \), and \( \hat{\sigma}(N + 1) = \int_0^1 dx \, x^N \hat{\phi}(x) \). To reduce clutter in the notation, we have denoted the set of \( 2 \rightarrow 2 \) partonic reactions \( f_A + f_B \rightarrow f_1 + f_2 \), collectively by \( f \). Divergent distributions in \( 1 - z \) produce powers of \( \ln(N) \), according to

\[
\int_0^1 dz \, z^{N-1} \left[ \frac{\ln^m(1 - z)}{1 - z} \right]_+ = \frac{-1}{m + 1} \ln^{m+1} \frac{1}{N} + O \left( \ln^{m-1} N \right) .
\]

(19)

In the following, we shall resum logarithms of \( N \), from which the singular distributions of \( \hat{\sigma} \) may be reconstructed by inverting the transform. In the limit of large \( N \), neglecting terms that decay as \( 1/N \), we may replace \( N + 1 \) by \( N \) in the arguments of the distribution functions \( \hat{\phi} \).

3 Refactorization for Leading Regions

We are now ready to construct a new, “refactorized”, expression for the partonic cross section, which generates all singular distributions of the form of Eq. (13) at partonic threshold. It will include, besides functions for the jet pair, a function that describes the soft, but still perturbative, radiation outside the cones, and which responds to the color flow in the hard scattering. The basic steps in the factorization process are quite similar to those already undertaken for Drell-Yan and heavy quark cross sections [3, 12], but the kinematics of jet production is sufficiently subtle to warrant a description of how the general treatment must be modified to be applicable in this case.

3.1 Leading regions and phase space

We consider the purely partonic reaction in which flavors \( f_A, f_B \), of momenta \( p_A \) and \( p_B \) collide to produce the observed jets. The first step in writing a factorized form for this partonic cross section at threshold is to identify those regions of momentum space that contribute to the cross section at the (singular) leading power of \( 1/(1 - \tau) \) with \( \tau = M_{J J}^2 / S = M_{J J}^2 / (p_A + p_B)^2 \).

For a two-jet cross section these leading regions are illustrated in Fig. 2, which represents the cross section in terms of a cut diagram [1, 15, 16]. Leading power contributions arise from momentum configurations in which on-shell lines of finite
energy fall into one of four “jet” subdiagrams. Two of these, labelled $\psi_{fA/fA}$ and $\psi_{fA/fB}$ in the figure, represent the perturbative evolution of the incoming partons, and the motion of their fragments into the final state. Logarithms arise from regions in momentum space where internal lines of the diagrams approach the mass shell at the subspaces illustrated by the figure. In Fig. 2 and below, we shall work in a general axial (or temporal), $n \cdot A = 0, n^2 \neq 0$, gauge. This simplifies the analysis somewhat, by insuring that collinear logarithms occur only internally to the jets [1, 15, 16]. The soft subdiagram, $S$, generates only a single infrared logarithm per loop in this class of gauges. We emphasize, however, that our final results will be gauge invariant, and that it is possible to reexpress all of our arguments in covariant gauges.

Returning to Fig. 2, the $\psi$’s give rise to a pair of active partons, whose scattering initiates the jet event. The two “short-distance functions” $h$ and $h^*$, to which the active partons connect, contain the effects of off-shell partons at the hard scatterings in the amplitude and its complex conjugate. The two remaining jets, $J_1$ and $J_2$, represent the fragmentation of the partons emerging from the hard scatterings into the observed jets. Finally, the function $S$ represents the soft radiation, coupling incoming and outgoing hard partons.

The leading regions pictured in Fig. 2 are identified by means of the analyticity and power counting techniques described in Refs. [1, 15, 16]. Compared to the leading regions of the Drell-Yan process, which they generalize, they differ by the presence of the observed final-state jets, to which soft radiation may couple, and which require a sum over color structures for the hard scatterings. To anticipate, the total parton cross section may be factorized into functions $\psi_i, J_i, h$ and $S$ corresponding to these quanta. We shall have more to say below about how and why this may be done, and how to define the singular functions explicitly as vacuum expectation values of nonlocal, composite operators in QCD.

We shall follow the procedure developed for the Drell-Yan process in Ref. [3] by constructing the jet and soft functions in such a way that they absorb all singular dependence on $1 - \tau$. This can be done in a straightforward manner, by matching the phase space of partons included in the $\psi$’s, $J$’s and $S$ with the phase space of the underlying process. It is easiest to carry out this analysis in the center of mass (c.m.) frame of the incoming partons.

Consider first exact threshold for the two-jet process. At partonic threshold in the c.m. frame, the jets $J_1$ and $J_2$ carry equal and opposite-moving momenta, $p_1$ and $p_2$,

$$p_1^u + p_2^u = (M_{JJ}, \vec{0}) = p_A^u + p_B^u.$$  (20)

Now suppose we emit soft radiation of total momentum $k$ outside the jet cones. The total squared invariant mass necessary for this radiation in the partonic process is

$$S \equiv (p_A + p_B)^2 = (p_1 + p_2 + k)^2 = (p_1 + p_2)^2 + 2M_{JJ}k_0 + O(M_{JJ}^2(1 - \tau)^2),$$  (21)

where we have used Eq. (20). In terms of the variable $\tau$, Eq. (16), we then have,

$$1 - \tau = \frac{2k_0}{M_{JJ}} + O((1 - \tau)^2) \quad \left(M_{JJ}^2 = (p_1 + p_2)^2\right)$$

9
\[ 1 - \tau = \frac{p_1^2 + p_2^2}{M_{JJ}^2} + \frac{2k_0}{M_{JJ}} + \mathcal{O}((1 - \tau)^2) \quad (M_{JJ}^2 = 2p_1 \cdot p_2). \]  

(22)

Thus near threshold, when \( M_{JJ}^2 = (p_1 + p_2)^2 \), the total invariant mass of the two jets, the variable \( 1 - \tau \) is given by the total c.m. energy radiated outside the cones, up to corrections suppressed by a power of \( 1 - \tau \), which we have agreed to neglect. On the other hand, for \( M_{JJ}^2 = 2p_1 \cdot p_2 \), the variable \( 1 - \tau \) receives contributions from the jet masses as well. This is a crucial difference, and leads to different behavior near threshold at the leading logarithmic level. The reason for this difference is that partonic threshold in the variable \( M_{JJ}^2 = 2p_1 \cdot p_2 \) is surprisingly more restrictive than in the variable \( M_{JJ}^2 = (p_1 + p_2)^2 \). In the latter case, since we sum over all particles and momenta emitted into the cones, there is already considerable phase space available for the production of a pair of jets of the specified invariant mass \( (M_{JJ}) \), total rapidity and scattering angles. The analogs of these thresholds in \( e^+e^- \) annihilation are jet configurations in which all energy is emitted into two opposite-moving jet cones, with no soft radiation in the intervening directions. The sum over all such states has infrared, but not collinear divergences, as the latter cancel in the sum over allowed states. Then, when an infrared energy resolution \( \delta E \) is introduced, and soft radiation outside the cone is summed over up to \( \delta E \), we find one logarithm of \( \delta E \) per loop [16].

For the case \( M_{JJ}^2 = 2p_1 \cdot p_2 \), Eq. (22) shows that \( \tau = 1 \) requires vanishing jet invariant mass in addition to energy flow only into the cones. Many fewer final states then contribute at threshold, specifically only those final states which consist of massless jets (at the cones’ centers, according to the construction of Sec. 2.1). It is not surprising that large corrections arise as a result. Because they reflect a restriction in phase space, the additional corrections in this case are negative, corresponding to incomplete cancellation of collinear singularities.

The refactorized cross section is pictured in Fig. 3, where the double lines represent, as we shall see below, propagators associated with ordered exponentials of the gluon field in the jet directions. This decoupling of soft gluons from jet subprocesses is at the basis of proofs of factorization for single-particle inclusive cross sections [19] and of factorization for inclusive cross sections [20, 21]. We shall discuss the technical justification for the form of factorization proposed here in the next subsection. The physical basis of jet/soft factorization, however, is the inability of soft gluons to resolve the internal substructure of the jets. The effect of all attachments of soft gluons to the jets is summarized simply by gauge rotations on the partons that connect the jets to the hard scatterings [1, 5]. The gauge rotations may be represented by the ordered exponentials of the gauge fields that we shall encounter below.

Given the structure in Fig. 3, we may write the partonic cross section \( d\sigma_{f_A f_B \rightarrow J_1 J_2} \) in a preliminary factorized form, as the product of functions which describe the short- and long-distance processes in the scattering,
Here $H_{IL}$ includes the hard scatterings. The overall rapidity, $y_{JJ}$, is integrated over (although for $\tau$ near 1, it is always small in the c.m. system). At the same time, both the long- and short-distance factors depend upon the rapidity difference $\Delta y$, or equivalently, the c.m. scattering angle. The indices $I, L$ describe possible color structures at the hard vertices, for the scattering amplitude and for the complex conjugate amplitude. We may further factorize the fully short-distance function into contributions from the amplitude and its complex conjugate,

$$H_{IL}^{(f)} \left( \frac{M_{JJ}}{\mu}, \Delta y, \alpha_s(\mu^2) \right) = h_L^{*(f)} \left( \frac{M_{JJ}}{\mu}, \Delta y, \alpha_s(\mu^2) \right) h_I^{(f)} \left( \frac{M_{JJ}}{\mu}, \Delta y, \alpha_s(\mu^2) \right).$$

(24)

As $\tau \to 1$, only purely virtual loops can carry momenta of order $M_{JJ}^2$. These are the contributions kept in $H_{IL}$. The color structures $L$ and $I$, of course, depend implicitly on the flavor content of the hard scattering, denoted by $f$. For example, when the underlying partonic hard-scattering is $q\bar{q} \rightarrow q\bar{q}$, there are only two possible color structures, which may be chosen as singlet and octet in the $s$ or $t$ channel. Similarly, the color-dependent factors $h$ depend as well upon the spin of outgoing scattered lines. The dependence on spin, however, follows the same pattern as dependence on flavor, and we shall not exhibit it explicitly. We always assume unpolarized collisions.

It is also useful to factorize the long-distance part of the cross section, $K_{LI}$, separating partons that are collinear to the incoming quarks from those that are soft and “central” in rapidity. Referring to Eq. (22), the sum of the weights associated with the functions is $(1 - \tau)$, so that the convolution is of the form

$$K_{LI}^{(f)} \left( \frac{M_{JJ}}{\mu}, (1 - \tau)M_{JJ}, \Delta y, \alpha_s(\mu^2), \delta_1, \delta_2 \right)$$

$$= \int_0^1 dw_A \, dw_B \, dw_1 \, dw_2 \, dw_S \, \delta(1 - \tau - w_A - w_B - w_1 - w_2 - w_S)$$

$$\times \psi_{fA/fA} \left( w_A, \frac{M_{JJ}}{\mu}, \alpha_s(\mu^2), \epsilon \right) \psi_{fB/fB} \left( w_B, \frac{M_{JJ}}{\mu}, \alpha_s(\mu^2), \epsilon \right)$$

$$\times S_{LI}^{(f)} \left( \frac{w_S M_{JJ}}{\mu}, \Delta y, \alpha_s(\mu^2) \right)$$

$$\times J^{(f_1)} \left( w_1, \frac{M_{JJ}}{\mu}, \alpha_s(\mu^2), \delta_1 \right) J^{(f_2)} \left( w_2, \frac{M_{JJ}}{\mu}, \alpha_s(\mu^2), \delta_2 \right) + O(1 - \tau).$$

(25)

Dependence on the gauge vector $n^\mu$ is present in each factor, but has been suppressed. The five functions $\psi_A, \psi_B, J^{(f_1)}, J^{(f_2)}$ and $S_{LI}^{(f)}$ are all convoluted together in terms of the weights that each function contributes to the final state. This is possible because the weights identified in Eq. (22) above are additive in the particles of the final state, and because the factorization implies that there is no interference between final-state particles associated with the different functions. Let $k^0_S$ be the energy of particles.
emitted by the soft function $S$ outside the cones, and similarly for the four other functions. Then the lack of interference implies that

$$k^0 = k_A^0 + k_B^0 + k_1^0 + k_2^0 + k_S^0,$$

(26)

with $k^0$ the energy in Eq. (22). The complete contributions of each of the functions to the weights are given by

$$w_A = \frac{2k_A^0}{M_J J},$$

$$w_B = \frac{2k_B^0}{M_J J},$$

$$w_1 = \frac{p_1^2 + 2M_J J k_1^0}{M_J J},$$

$$w_2 = \frac{p_2^2 + 2M_J J k_2^0}{M_J J},$$

$$w_S = \frac{2k_S^0}{M_J J},$$

(27)

for $M_J^2 = 2p_1 \cdot p_2$, and by

$$w_A = \frac{2k_A^0}{M_J J},$$

$$w_B = \frac{2k_B^0}{M_J J},$$

$$w_1 = \frac{2k_1^0}{M_J J},$$

$$w_2 = \frac{2k_2^0}{M_J J},$$

$$w_S = \frac{2k_S^0}{M_J J},$$

(28)

for $M_J^2 = (p_1 + p_2)^2$. Note that the product $\psi_A \psi_B J_1 J_2 S$ of jet and soft functions behaves as $(w_A w_B w_1 w_2 w_S)^{-1}$ when $1 - \tau$ vanishes [3, 7], which is how logarithmic enhancements arise. We will be able to neglect contributions to $p_i^2$ except from $J^{(f_i)}$ because we neglect corrections proportional to $\delta_i$. Finally, the dependence on the jet opening angles is included entirely in the jet functions, up to corrections proportional to the $\delta_i$. Although it is possible to do so, we shall not attempt a resummation in the opening angles (recall our assumption, Eq. (6)). Before exploring the consequences of these factorized expressions, let us discuss their justification.

3.2 Relation to factorization for inclusive processes

The details of the proof of the factorization of soft quanta from a jet depend on whether the jet in question is “initial-state”, as the $\psi_i$ in Fig. 3, or “final-state”, as the $J_i$. Factorization is somewhat simpler in the latter case, and was discussed in Ref. [19] in the context of single-particle inclusive annihilation, and in Ref. [5] in the context of transverse momentum distributions. The arguments are essentially identical in this case. The factorization of soft interactions from initial-state jets is an essential ingredient in proofs of factorization for inclusive processes, and was extensively discussed in Refs. [20, 21] in this connection.

In factorization proofs for inclusive cross sections, the goal is to show that soft interactions cancel, but the factorization of soft quanta from jets is an important subsidiary result in this demonstration [21]. It is necessary, however, to distinguish
the factorization of soft quanta from partonic jets in Eq. (25), in which \( z \) is fixed near unity (near threshold), from their factorization in the inclusive case, where \( z \) is freely integrated over. Let us recall a few of the relevant arguments of Ref. [20, 21], as they apply here. First, the difference between initial-state and final-state jets lies in the singularity structures of the jets, in terms of the momentum components of soft quanta. For final-state jets, all poles are in the same half-plane, corresponding to final-state interactions only. This simplifies the analysis considerably. Indeed, a short argument based on contour deformation allows us to use the eikonal, or “soft”, approximation for soft gluon momenta within jets, and to replace all the details of the jets’ interactions with soft gluons by ordered exponentials [1, 5].

For initial-state jets, however, contour deformation is severely restricted, on a graph-by-graph basis, as poles from final-state and initial-state interactions conspire to pinch the momentum integration contours in regions for which the eikonal approximation fails [19]. It is, in general, only after the sum over final states that final-state interactions cancel, the pinches disappear, and soft quanta may be factored. When \( z \) is fixed near unity, or when moments are taken, final states are not all summed over with the same weight, and we may not, in general, factor soft quanta from jets in the same manner as in Ref. [21], without leaving over apparently nonfactoring remainders. In our case, however, such remainders are simply absorbed into the definition of the soft function \( S_{LI} \), through Eq. (25).

The only properties of \( S_{LI} \) that we shall need are that it is infrared safe and that it has at most a single, infrared logarithm per loop in the limit \( w_S \to 0 \). Its infrared safety follows from the universality of collinear singularities, which have been absorbed into the \( \psi_i \)’s. Its single-logarithmic dependence on \( w_S \) requires further explanation, however.

Double logarithms require both collinear and infrared enhancements. Collinear logarithms must arise from momentum configurations in which lines in \( S_{LI} \) that are parallel to the initial-state jets are much more energetic than soft lines that connect the two jets. But this requires that the soft lines are much softer in energy than \( w_S M_{JJ} \), since the total, and hence the jet energy is bounded by this quantity. We may then apply the reasoning of Ref. [21] to any such momentum configuration to show that such a region cancels, until the soft gluons become energetic enough so that they may no longer be ignored kinematically compared to lines that are parallel to the incoming momenta. But in this region, we cannot generate collinear logs, and we expect at most one logarithm per loop. We shall review the explicit forms of the soft and jet functions in the following section. First, however, we discuss the role of moments.

3.3 Moments of the partonic cross section

From Eqs. (23) and (25), we find for the full partonic cross section,

\[
\frac{d\sigma_{f_A f_B \rightarrow J_1 J_2}(S, \delta_1, \delta_2)}{dM_{JJ}^2 d\Delta y} = \frac{1}{S^2} \sum I \sum L \sum H_{IL}^{(F)} \left( \frac{M_{JJ}}{\mu}, \Delta y, \alpha_s(\mu^2) \right)
\]
\begin{align*}
& \times \int_0^1 dw_A \, dw_B \, dw_1 \, dw_2 \, dw_S \, \delta(1 - \tau - w_1 - w_2 - w_A - w_B - w_S) \\
& \times \psi_{f_A/f_A}(w_A, M_{JJ}/\mu, \alpha_s(\mu^2), \epsilon) \, \psi_{f_B/f_B}(w_B, M_{JJ}/\mu, \alpha_s(\mu^2), \epsilon) \\
& \times S^{(f)}(w_S M_{JJ}/\mu, \Delta y, \alpha_s(\mu^2)) \\
& \times J^{(f_1)}(w_1, M_{JJ}/\mu, \alpha_s(\mu^2), \delta_1) \, J^{(f_2)}(w_2, M_{JJ}/\mu, \alpha_s(\mu^2), \delta_2) \\
& + \mathcal{O}(1 - \tau), \quad (29)
\end{align*}

where, as above, \( \tau = M_{JJ}^2/S \). In moment space, this becomes

\begin{align*}
& \int_0^1 d\tau \, \tau^{-1} S^2 \, \frac{d\sigma_{f_AF_B \rightarrow J_JJ}(S, \delta_1, \delta_2)}{dM_{JJ}^2 d\Delta y} = \sum_i \sum_{IL} H^{(f)}_{iL}(M_{JJ}/\mu, \Delta y, \alpha_s(\mu^2)) \\
& \times \tilde{\psi}_{f_A/f_A}(N, M_{JJ}/\mu, \alpha_s(\mu^2), \epsilon) \, \tilde{\psi}_{f_B/f_B}(N, M_{JJ}/\mu, \alpha_s(\mu^2), \epsilon) \\
& \times \tilde{S}^{(f)}(M_{JJ}/\mu N, \Delta y, \alpha_s(\mu^2)) \\
& \times \tilde{j}^{(f_1)}(N, M_{JJ}/\mu, \alpha_s(\mu^2), \delta_1) \, \tilde{j}^{(f_2)}(N, M_{JJ}/\mu, \alpha_s(\mu^2), \delta_2) \\
& + \mathcal{O}(1/N). \quad (30)
\end{align*}

Comparing Eqs. (18) and (30) we derive for the Mellin transform of the hard scattering function the “refactorized” expression, accurate to \( O(1/N) \),

\begin{align*}
\tilde{\sigma}_f(N) = & \left[ \frac{\tilde{\psi}_{f_A/f_A}(N, M_{JJ}/\mu, \epsilon) \tilde{\psi}_{f_B/f_B}(N, M_{JJ}/\mu, \epsilon)}{\phi_{f_A/f_A}(N, \mu^2, \epsilon) \phi_{f_B/f_B}(N, \mu^2, \epsilon)} \right] \\
\times & \sum_{IL} H^{(f)}_{iL}(M_{JJ}/\mu, \Delta y, \alpha_s(\mu^2)) \, \tilde{S}^{(f)}(M_{JJ}/\mu N, \Delta y, \alpha_s(\mu^2)) \\
\times & \tilde{j}^{(f_1)}(N, M_{JJ}/\mu, \alpha_s(\mu^2), \delta_1) \, \tilde{j}^{(f_2)}(N, M_{JJ}/\mu, \alpha_s(\mu^2), \delta_2). \quad (31)
\end{align*}

The first factor is “universal” between electroweak and QCD-induced hard processes, and was computed first with \( f_A = q \) for the Drell-Yan cross section [3].

To organize both the \( \mu \)- and \( N \)-dependences of the jet and soft functions, we develop definitions for them in terms of matrix elements in the next section. We will then go on to show how the renormalization of these matrix elements leads to the resummation of logarithms of \( N \).

4 Soft and Jet Functions

We now turn to the explicit forms of the jet and soft functions that result from factorization, in terms of matrix elements of composite operators. It is the renormalization properties of these operators that will lead to resummation of threshold singularities, in the next section.
4.1 Center of mass distribution

We begin with the center of mass parton distribution functions $\psi_{f/f}$ [3], appearing in the “refactorized” expression, Eq. (29), where they absorb long-distance contributions of the initial-state jets, while respecting the overall phase space restrictions near partonic threshold. The functions $\psi$ differ from standard light-cone parton distributions by being defined at fixed energy, rather than light-like momentum fraction. They were introduced for the inclusive Drell-Yan cross section, and applied as well to heavy quark production [12]. For completeness, we may define them by analogy to light-cone parton distributions via the matrix elements,

$$
\psi_{q/q}(x, 2p_0/\mu, \epsilon) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} dy_0 \ e^{-ixp_0 y_0} \langle q(p)|\bar{q}(y_0, \bar{0}) \frac{1}{2} v \cdot \gamma q(0)|q(p)\rangle
$$

$$
\psi_{\bar{q}/q}(x, 2p_0/\mu, \epsilon) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} dy_0 \ e^{-ixp_0 y_0} \langle \bar{q}(p)|\bar{q}(y_0, 0)\frac{1}{2} v \cdot \gamma q(0)|q(p)\rangle 
$$

$$
\psi_{g/q}(x, 2p_0/\mu, \epsilon) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} dy_0 \ e^{-ixp_0 y_0} \langle g(p)|v_\mu F_{\mu\nu}^\pm(y_0, 0) v_\nu F^{\nu\pm}(0)|g(p)\rangle,
$$

where the matrix element is evaluated in $n\cdot A = 0$ gauge in the partonic c.m. frame (in Ref. [3], $A_0 = 0$ gauge was chosen). The vector $v$ is light-like in the opposite direction from $p^i$, so that for $\bar{p}$ in the $\pm 3$ direction, $v \cdot \gamma = \gamma^\pm$. In the antiquark distribution, the combination $\bar{q}\bar{q}$ is treated as a Dirac matrix to define the trace. Charge conjugation invariance implies that $\psi_{q/q} = \psi_{\bar{q}/\bar{q}}$. The moments of these distributions may be factorized into a product of moments of the light-cone parton distribution $\phi$, defined in any scheme, times an infrared safe function [3]. The argument $\epsilon$ in $\psi$ represents the universal collinear singularities that $\psi$ absorbs in the factorized cross section, Eq. (29).

4.2 Matrix elements for final-state jets

As the second step in the construction of the functions into which the cross section factorizes, we treat the final-state jets $J^{(f)}$. In axial gauges, all collinear logarithms are generated by the imaginary parts of two-point functions. In the factorized cross section, Eq. (29), the final-state jets are linked to the other functions through the weight convolution. The only direct dependence on the momentum of final-state particles associated with $J^{(f)}$ is in the hard-scattering function $H_{fL}^{(f)}$, which depends on the total momenta emitted into the cones through $\Delta_y$ and $M_{JJ}$. In $H_{fL}$, the final-state (and initial-state) jet momenta are approximated by light-like momenta $p_i^{(0)} = \beta_i M_{JJ}/\sqrt{2}$, with $\beta_i$ a light-like velocity vector in the direction of the jet. The vector $p_i^{(0)}$ is determined by the three-momentum of particles flowing into the jet cone at partonic threshold. In constructing the jet function, we will therefore sum over the complete phase space of the particles associated with $J^{(f)}$, subject to fixed total spatial momentum of particles within the cone, and fixed weight $w_i$.

To be specific, let $\ell_i$ be the total momentum of the final-state particles of $J^{(f)}$, $p_i$ be the total momentum of particles of $J^{(f)}$ emitted into cone $i$, and $k_i$ the total
momentum of the particles of $J^{(f_i)}$ emitted outside cone $i$, so that $\ell_i = p_i + k_i$. The sum over phase space includes the integral over all $k_i$, and over the invariant mass $p_i^2$, at fixed values of $\vec{p}_i^{(0)}$, defined as above. The remaining phase space is that of two light-like vectors $\vec{p}_i^{(0)}$, one for each of the two jets. These vectors are fixed in our differential cross section for $M_{JJ}$ and $\Delta y$, Eq. (29). Overall factors associated with this two-particle phase space are implicitly absorbed into the definition of the $H_{IL}$.

Consider, for example, an outgoing quark. Near threshold, the jet function is the cut quark two-point Green function, summed over final states at fixed $w_i$ and $\vec{p}_i^{(0)}$. This two-point function depends, in general, on $\vec{p}_i^{(0)}$, and on the gauge vector, $n^\mu$. It may thus be defined as

$$J^{(f)}_{\beta\alpha,ba} \left( \vec{p}_i^{(0)}, w_i, M_{JJ}, \mu, \alpha_s(\mu^2), \delta_i \right) = \sum_\xi 2|\vec{p}_i^{(0)}| (2\pi)^3 \delta^3 \left( \vec{p}_i^{(0)} - \vec{p}_\xi \right) \times \delta \left( w_i - \langle w(\xi, \delta_i) \rangle \right) \langle 0| f_{\beta,\mu}(0) |\xi| f_{\alpha,a}(0) |0 \rangle,$$

where $f_{\beta,b}$ is the field of flavor $f$, with Dirac and color indices $\beta$ and $b$, respectively. As usual, we suppress dependence on $n^\mu$. The factor $2|\vec{p}_i^{(0)}| (2\pi)^3 \delta^3 \left( \vec{p}_i^{(0)} - \vec{p}_\xi \right)$ in Eq. (33) matches the normalization of $p_i^{(0)}$ phase space, and fixes $\vec{p}_\xi$, the spatial momentum of all particles in state $\xi$ that flow into the cone. The sum is over all states $\xi$ with the specified jet momentum, consistent with a contribution $w_i$ to the overall weight.

In our examples, the weight is given by either the first or second line in Eq. (22). With either definition, as $w_i$ vanishes, the momenta of all particles outside the cone are forced to vanish. In the case $M_{JJ}^2 = 2p_1 \cdot p_2$, for example, final-state particles from $J^{(f_i)}$ contribute through the term $p_i^2/2M_{JJ}^2$ if they are within the cone, and through $2k_i^0/M_{JJ}$ if they are outside the cone, as in Eq. (27). The function $w(\xi, \delta_i) = p_i^2/M_{JJ}^2 + 2k_i^0/M_{JJ}$ is simply the sum of these two contributions, where $k_i^0$ is the total energy of final state particles in $\xi$ that flow outside the cone $\delta_i$. In our approximation, we may neglect corrections to $J^{(f_i)}$ that are proportional to $\delta_i$, the jet invariant mass, $p_i^2$, and $w_i$.

Near partonic threshold, the invariant mass of jet $i$ is given by

$$p_i^2 \simeq 2p_i^{(0)} \cdot p_i = \sqrt{2M_{JJ}} \beta_i \cdot p_i,$$

where again $\beta_i$ is a light-like vector in the jet direction. In a frame where $\vec{p_i}$ is in the +3 direction, $\beta_i \cdot p_i$ is the minus (opposite-moving) light-cone component of the total momentum of the particles emitted into the cone. (Recall that by the jet definition of Sec. 2.1, $\vec{p_i,T} = 0$; see Eq. (7).)

Another consequence of our approximations is that for a quark jet the only Dirac matrix structure that we need to retain is proportional to $\gamma \cdot p_i^{(0)}$,

$$J^{(f_i)}_{\beta\alpha,ba} \left( \vec{p}_i^{(0)}, w_i, M_{JJ}, \mu, \alpha_s(\mu^2), \delta_i \right) = \left( \gamma \cdot p_i^{(0)} \right)_{\beta\alpha} \delta_{ba} J^{(f_i)} \left( w_i, \frac{M_{JJ}}{\mu}, \alpha_s(\mu^2), \delta_i \right),$$

(35)
where the function \( J^{(f)} \) without indices is a scalar distribution in \( w_i \). This is the function in the factorized cross section Eq. (29). Corrections to (35) are suppressed by a power of \( w_i \). Similar considerations apply to gluon jets.

### 4.3 The eikonal cross section

Having identified matrix elements that describe initial-state and final-state jets, it only remains to give an explicit construction for the soft functions \( S^{(i)} \) in Eq. (29), in terms of matrix elements of composite operators. We shall proceed in two steps.

We start by representing the coupling of soft gluons to the partons involved in the hard scattering, of flavors \( f_A, f_B, f_1 \) and \( f_2 \), by ordered exponentials, also known as Wilson lines. We denote the resulting nonlocal operator as describes the emission of soft radiation, due to both incoming and outgoing hard partons. We shall introduce the notation

\[
\Phi^{(f)}_{\beta}(\lambda_2, \lambda_1; x) = P \exp \left( -ig \int_{\lambda_1}^{\lambda_2} d\eta \beta \cdot A^{(f)}(\eta \beta + x) \right),
\]

with \( A^{(f)} \) the gauge field, represented as a matrix in the representation of flavor \( f_i \), \( i = A, B, 1 \) or 2, of the gauge group \( SU(3) \), and \( \beta \) the velocity four-vector of the parton whose interactions with soft gluons are being approximated. The operator \( P \) orders group products in the same sense as the ordering in the integration variable \( \lambda \), with the \( A_i \)'s with lower values of \( \lambda \) to the right. The difference in \( \Phi^{(q)}_{\beta} \) and \( \Phi^{(q)}_{\beta} \) is in the matrices \( A^{(q)} = A_q^\dagger (\lambda_q/2) \) and \( A^{(q)} = A_q^\dagger (\lambda_q^*/2) \), with \( \lambda_q \) the Gell-Mann matrices. Note that this notation differs slightly from that in Ref. [12].

As we have observed above, these ordered exponentials summarize not only the coupling of soft gluons to a single quark or hard gluon line, but also to an entire jet connected to the hard scattering by such a parton line [1, 5, 19]. By connecting these Wilson lines at a local vertex, we construct an eikonal nonlocal operator, which describes the emission of soft radiation, due to both incoming and outgoing hard partons. We denote the resulting nonlocal operator as \( w^{(f)}_I \),

\[
w^{(f)}_I(x)_{\{c_{b}\}} = \sum_{\{d_{b}\}} \Phi^{(f_j)}_{\beta_{d_{b}}} (\infty, 0; x)_{c_{b}, d_{b}} \Phi^{(f_i)}_{\beta_{c_{b}}} (\infty, 0; x)_{c_{b}, d_{b}} \times \left( c_1^{(f)} \right)_{d_{b}, d_{b}, d_{b}} \Phi^{(f_{i})}_{\beta_{d_{b}}} (0, -\infty; x)_{d_{b}, d_{b}, d_{b}} \Phi^{(f_{b})}_{\beta_{d_{b}}} (0, -\infty; x)_{d_{b}, d_{b}, d_{b}},
\]

with the \( \beta_i \) the four-velocities of the Wilson lines that represent the initial- and final-state jets. The color tensor \( (c_1^{(f)})_{d_{b}, d_{b}, d_{b}} \) describes the couplings of the ordered exponentials with each other in color space.

We use the operator \( w_I \) to define a dimensionless “eikonal cross section”, describing the emission of gluons by the ordered exponentials,

\[
\sigma^{(teik)}_{L_i} \left( \frac{wM_{fj}}{\mu}, \alpha_s(\mu^2), \epsilon \right) = \sum_{\xi} \delta \left( w - w(\xi, \delta_i) \right)
\times \langle 0 | T \left[ \left( w_I^{(f)} (0) \right)_{\{b_{i}\}} \right] | \xi \rangle \langle \xi | T \left[ w_I^{(f)} (0)_{\{b_{i}\}} \right] | 0 \rangle,
\]

where \( \xi \) designates a set of intermediate states, whose contributions to the weight are given by \( w(\xi, \delta_i) \). Recalling from Eq. (34) that the final-state jet masses are linear in
the momenta of particles emitted into the cones, we may identify the contribution to
the total weight from state $\xi$ as
$$w(\xi, \delta_i) = \frac{\sqrt{2}(\beta_1 \cdot k_1' + \beta_2 \cdot k_2') + 2k'^0}{M_{JJ}}$$
(39)
for $M_{JJ}^2 = 2p_1 \cdot p_2$, where $k_i'$ is the momentum emitted into the $i$th jet cone, while $k'^0$ is the energy emitted outside the cones. Similarly, for $M_{JJ}^2 = (p_1 + p_2)^2$, we have
$$w(\xi, \delta_i) = \frac{2k'^0}{M_{JJ}}.$$  
(40)
Defined in this fashion, the eikonal cross section contains both collinear and ultraviolet
divergences, which will be treated by factorization and renormalization below.

Setting aside the ultraviolet divergences for the moment, we note that factorization
implies that this cross section is a good picture of the emission of soft radiation as
a result of the hard scattering, at least for soft quanta that are outside the cones
of the final-state jets, and not collinear to the incoming partons. Inside the cones,
or in the directions of the incoming partons, the collinear divergences of the eikonal
cross section are qualitatively the same as those in the full partonic cross section,
but quantitatively different in general. The differences are due to the original eikonal
approximation, necessary to factorize soft emission from the jets.

4.4 The soft function

To extract the infrared safe soft function $S_{LI}^{(i)}$ from the eikonal cross section Eq. (38),
we separate collinear and infrared divergences in the eikonal cross section. Contributions
from collinear quanta are by construction incorporated in the functions $\psi_i$ and $J_i$ in Eq. (29). To include these regions in $S_{LI}^{(i)}$ would be double counting.

We avoid this double counting by eliminating collinear singularities associated
with the initial-state and final-state jets from $S_{LI}^{(i)}$. The observation that makes this
procedure possible is that the eikonal cross section may be factored into initial-state
and final-state jets, and a left-over “reduced” soft function, $S$, in the same manner as
the full partonic cross section. This, “eikonal”, factorization is illustrated in Fig. 4.
The result is a convolution in the weights of final-state partons associated with each
of these functions. By analogy to Eqs. (27) and (28), these weights are given by
$$w'_A = \frac{2k'_A^0}{M_{JJ}} \quad \quad \quad w'_B = \frac{2k'_B^0}{M_{JJ}}$$
$$w'_1 = \frac{\sqrt{2}\beta_1 \cdot k'_1 + 2k'_1^0}{M_{JJ}} \quad \quad \quad w'_2 = \frac{\sqrt{2}\beta_2 \cdot k'_2 + 2k'_2^0}{M_{JJ}}$$
$$w'_S = \frac{2k'_S^0}{M_{JJ}}$$
(41)
for $M_{JJ}^2 = 2p_1 \cdot p_2$, and by
$$w'_A = \frac{2k'_A^0}{M_{JJ}} \quad \quad \quad w'_B = \frac{2k'_B^0}{M_{JJ}}$$
\[ w'_1 = \frac{2k'_1}{M_{JJ}} \quad \quad \quad w'_2 = \frac{2k'_2}{M_{JJ}} \]
\[ w'_S = \frac{2k'_S}{M_{JJ}}, \quad (42) \]

for \( M_{JJ}^2 = (p_1 + p_2)^2 \), where we have used Eq. (34) to parameterize the contribution of the eikonal jet functions to the corresponding jet invariant masses. The primes simply indicate that these variables refer to a convolution for the eikonal, rather than the full, cross section.

The soft function \( S_{LI}^{(f)} \) found by factoring \( \sigma^{(f,\text{eik})} \) is exactly the same soft function found from the factorization of the full partonic cross section, Eqs. (23)-(25) above, because the soft radiation is insensitive to the internal structure of the jets and the hard scattering. Thus we have, by analogy to Eq. (25) and (29),
\[
\sigma_{LI}^{(f,\text{eik})}\left(\frac{w M_{JJ}}{\mu}, \Delta y, \alpha_s(\mu^2), \epsilon\right) = \\
\int_0^1 dw_A dw_B dw_1 dw_2 dw'_S \delta(w - w'_1 - w'_2 - w'_A - w'_B - w'_S) \\
\times \prod_{c=A,B} j_{IN}^{(f)}\left(\frac{w'_c M_{JJ}}{\mu}, \alpha_s(\mu^2), \epsilon\right) \prod_{d=1,2} j_{OUT}^{(f)}\left(\frac{w'_d M_{JJ}}{\mu}, \alpha_s(\mu^2), \delta_d\right) \\
\times S_{LI}^{(f)}\left(\frac{w'_S M_{JJ}}{\mu}, \Delta y, \alpha_s(\mu^2)\right). \quad (43)\]

\( j_{IN}^{(f)} \) and \( j_{OUT}^{(f)} \) are initial-state and final-state jet eikonal functions, respectively, which summarize the dynamics of gluons collinear to the Wilson lines of the eikonal cross section. They can be given specific operator definitions.

Consider first the eikonal distributions for the initial-state jets, \( i = A, B \). The phase space for the initial-state eikonal jets is defined by the total energy that they emit into the final state (see Eqs. (41) and (42)). As a result, their definitions are similar to those for the full center-of-mass distributions, Eq. (32),
\[
\mathcal{J}_{IN}^{(f)}\left(\frac{w'M_{JJ}}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \\
\frac{M_{JJ}}{2\pi} \int_\infty^{-\infty} dy_0 \ e^{-iw'_M_{JJ}y_0} \\
\times \langle 0 | \text{Tr}\left\{ \hat{T}[\Phi^{(f)}_{\beta_i}(0,-\infty;0)]\hat{T}[\Phi^{(f)}_{\beta_i}(0,-\infty;0)] \right\} | 0 \rangle, \quad (44)\]

with \( y' = (y_0, \vec{0}) \) a vector at the spatial origin. As in Eq. (32), \( \epsilon \) in the arguments of \( \mathcal{J}_{IN}^{(f)} \) denotes collinear singularities.

Similarly, the collinear dynamics of the eikonal final-state jets are summarized by matrix elements analogous to Eq. (33),
\[
\mathcal{J}_{OUT}^{(f)}\left(\frac{w'M_{JJ}}{\mu}, \alpha_s(\mu^2), \delta_i\right) = \\
\sum_{\xi} \delta(w'_i - w(\xi, \delta_i)) \\
\times \langle 0 | \text{Tr}\left\{ \hat{T}[\Phi^{(f)}_{\beta_i}(\infty,0;0)]\hat{T}[\Phi^{(f)}_{\beta_i}(\infty,0;0)] \right\} | 0 \rangle, \quad (45)\]
with \( i = 1, 2 \). Here again \( w(\xi, \delta_i) \) is given by either Eq. (41) or (42).

We construct (moments of) the soft function by dividing the moments of the eikonal cross section (43) by the product of moments of the eikonal jets, Eqs. (44) and (45),

\[
\tilde{S}^{(f)}_{LI} \left( \frac{M_{JJ}}{N\mu}, \Delta y, \alpha_s(\mu^2) \right) = \frac{\tilde{\sigma}^{(f,eik)}_{LI} \left( \frac{M_{LL}}{N\mu}, \Delta y, \alpha_s(\mu^2), \epsilon \right)}{\tilde{\sigma}^{(f,n)}_{IN} \left( \frac{M_{LL}}{N\mu}, \alpha_s(\mu^2), \epsilon \right)} \frac{1}{\tilde{\sigma}^{(f)}_{OUT} \left( \frac{M_{LL}}{N\mu}, \alpha_s(\mu^2), \delta_1 \right)} \frac{\tilde{\sigma}^{(f)}_{OUT} \left( \frac{M_{LL}}{N\mu}, \alpha_s(\mu^2), \delta_2 \right)}{\tilde{\sigma}^{(f)}_{OUT} \left( \frac{M_{LL}}{N\mu}, \alpha_s(\mu^2) \right)}.
\]

(46)

This simply corresponds to the standard factorization procedure in axial gauge [1]. At one loop, all diagrams that are two-particle reducible by cutting incoming or outgoing eikonal lines with the same four-velocity \( \beta_i \) in the amplitude and its complex conjugate are eliminated from the soft function, which is then free of collinear divergences.

Both the eikonal distributions Eq. (44) and (45), and the soft function Eq. (46), are as yet unrenormalized. Eikonal lines, and vertices made from their products, may be renormalized in the usual multiplicative manner [22, 23, 24, 25, 26]. This renormalization will play an important role in resummation below.

5 Resummation

We are now ready to use the renormalization properties of the jet and soft functions to organize \( N \)-dependence in the factorized cross section in moment space, Eq. (31).

5.1 Renormalization of the soft function

The soft function \( \tilde{S}^{(f)}_{LI} \), as emphasized above, requires renormalization as a composite operator. This renormalization is a direct consequence of the factorization we have discussed. The ultraviolet divergences of \( \tilde{S}^{(f)}_{LI} \) arise when approximations appropriate for soft gluons are extended to all momenta. The renormalization scale for the soft function, then, acts as an effective cutoff, separating soft from hard gluons. Indeed, in the product \( H_{IL}S_{IL} \), ultraviolet divergences induced by the factorization cancel against each other by construction, since the original diagrams have no UV divergences beyond those taken into account by the usual renormalization of the theory [5, 3, 23, 7].

Because \( H \) and \( S \) occur in a product, they must renormalize multiplicatively, with separate renormalization factors for the amplitude and the complex conjugate [23, 7]

\[
H^{(f)(0)}_{IL} = \prod_{i=A,B,1,2} Z_i^{-1} \left( Z_S^{(f)-1} \right)_{IC} H_{CD} [(Z_S^{(f)})^{-1}]_{DL} \\
S^{(f)(0)}_{IL} = (Z_S^{(f)})_{LB} S_{BA} Z_S^{(f)}_{IA}.
\]

(47)
where $Z_i$ is the renormalization constant of the $i$th incoming parton (flavor $f_A \ldots f_2$) connecting to $H^{(f)}$, and $Z_{S_{CD}}^{(f)}$ is a matrix of renormalization constants, describing the renormalization of the soft function.

From Eq. (47), the soft function $S^{(f)}_{LI}$ satisfies the renormalization group equation

$$
\left( \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right) S^{(f)}_{LI} = -\left( \Gamma^{(f)} S \right)_{L} S^{(f)}_{LI} - S^{(f)}_{LA} \left( \Gamma^{(f)} S \right)_{AI},
$$

where we have introduced a soft anomalous dimension matrix, $\Gamma^{(f)} S$, which is computed directly from the UV divergences of the soft function. We can compute the matrix in a minimal subtraction renormalization scheme, taking $\epsilon = \epsilon_{UV} = 4 - D$, with $D$ the number of space-time dimensions. One-loop anomalous dimensions are then given by

$$
\Gamma^{(f)} S(g) = -\frac{g}{2} \frac{\partial}{\partial g} \text{Res}_{\epsilon \to 0} Z^{(f)} S(\mu, \epsilon). \tag{49}
$$

The determination and analysis of $\Gamma^{(f)} S$ has been carried out in the case of heavy quark production in [12]; its calculation for massless quarks [23] and gluons at one loop will be the subject of a companion paper. In any case, the solution to Eq. (48) takes the form

$$
\text{Tr} \left\{ H^{(f)} \left( \frac{M_{JJ}}{\mu}, \Delta y, \alpha_s(\mu^2) \right) \tilde{S}^{(f)} \left( \frac{M_{JJ}}{N\mu}, \Delta y, \alpha_s(\mu^2) \right) \right\} = \text{Tr} \left\{ H^{(f)} \left( \frac{M_{JJ}}{\mu}, \Delta y, \alpha_s(\mu^2) \right) \times \tilde{P} \exp \left[ \int_{\mu}^{M_{JJ}/N} \frac{d\mu'}{\mu'} \Gamma^{(f)} S (\epsilon_s(\mu'^2)) \right] \times \tilde{S}^{(f)} \left( 1, \Delta y, \alpha_s \left( \frac{M_{JJ}^2}{N^2} \right) \right) \right. \\
\left. \times P \exp \left[ \int_{\mu}^{M_{JJ}/N} \frac{d\mu'}{\mu'} \Gamma^{(f)} S (\epsilon_s(\mu'^2)) \right] \right\}, \tag{50}
$$

where the trace is taken in the space of color structures. The symbols $P$ and $\tilde{P}$ refer to path-ordering in the same sense as the integration variable $\mu'$ and against it, respectively (for example, $P$ orders $\Gamma^{(f)} S (\epsilon_s(\mu^2))$ to the far right and $\Gamma^{(f)} S (\epsilon_s(M_{JJ}^2/N^2))$ to the far left). As usual, at leading logarithmic accuracy in $\mu$, we can simplify this result to a sum of exponentials, by choosing a basis in which the matrix $\Gamma^{(f)} S$ is diagonal.

### 5.2 Initial-state jets

Starting from Eq. (31), we recall the resummed expression for the prefactor ($\psi/\phi$) in Eq. (31), the exponentiation of whose $N$-dependence follows, as usual, from its factorization properties [3, 7]. The general expression for moments of the ratios of
the functions $\psi$ and $\phi$, evaluated at the common scale $\mu = M_{JJ}$, is [12]

$$
\left[ \frac{\tilde{\psi}_{f/f}(N,1,\epsilon)}{\tilde{\phi}_{f/f}(N,M_{JJ}^2,\epsilon)} \right] = R(f) \left( \alpha_s(M_{JJ}^2) \right) \exp \left[ E_f(N,M_{JJ}) \right],
$$

where

$$
E_f(N,M_{JJ}) = - \int_0^1 dz \frac{z^{N-1} - 1}{1 - z} \left\{ \int \frac{(1-z)^{m_S}}{\lambda} A(f) \left[ \alpha_s(\lambda M_{JJ}^2) \right] \right. 
+ B(f) \left[ \alpha_s((1-z)^{m_S} M_{JJ}^2) \right] + \frac{1}{2} \nu(f) \left[ \alpha_s((1-z)^2 M_{JJ}^2) \right] \biggr\}. \tag{52}
$$

The parameter $m_S$ and the resummed coefficients $B(f)$ depend on the factorization scheme, that is, on the definition of $\phi_{f/f}$. The results are rather different for DIS and $\overline{\text{MS}}$ schemes, in particular [4, 7]. This difference must be compensated for by differences in the parton distributions themselves. With DIS ($\overline{\text{MS}}$) factorization schemes, we have $m_S = 1(0)$ in Eq. (52). In Eq. (51), $R(f)(\alpha_s)$ is an $N$-independent function of the coupling, which can be normalized to unity at zeroth order.

The $A(f), B(f)$ and $\nu(f)$ are finite functions of their arguments. To reach next-to-leading order accuracy in $\ln(N)$, we need [4, 27]

$$
A(f)(\alpha_s) = C_f \left( \frac{\alpha_s}{\pi} + \frac{1}{2} K \left( \frac{\alpha_s}{\pi} \right)^2 \right), \tag{53}
$$

with $C_f = C_F$ ($C_A$) for an incoming quark (gluon), and with $K$ given by [28]

$$
K = C_A \left( \frac{67}{18} - \frac{\pi^2}{6} \right) - \frac{5}{9} n_f, \tag{54}
$$

where $n_f$ is the number of quark flavors. $B(f)$ is given for quarks in the DIS scheme by

$$
B_{(q)}(\alpha_s) = - \frac{3}{4} C_F \frac{\alpha_s}{\pi}, \tag{55}
$$

while it vanishes in the $\overline{\text{MS}}$ scheme for quarks and gluons. (The DIS scheme is normally only applied to quarks, although extended definitions for gluons are possible [29].) Finally, the lowest-order approximation to $\nu(f)$, which is scheme-independent, is [12]

$$
\nu(f) = 2 C_f \frac{\alpha_s}{\pi}. \tag{56}
$$

The results just discussed are useful, but not quite adequate for our needs, because they assume that $\mu = M_{JJ}$, as would be appropriate for Drell-Yan cross sections. For jet cross sections, we generally need additional freedom to choose the factorization scale. To change the scale $\mu$, we need the renormalization group behavior of the parton distributions $\psi$ and $\phi$, whose ratio enters Eq. (31).

The center of mass distribution $\psi$ requires no overall renormalization as a composite operator [3]. From its definition, Eq. (32), we see that $\psi$, and each of its
moments, renormalizes multiplicatively, because it is the matrix element of a product of renormalized operators. As a result, it obeys the renormalization group equation
\[ \mu \frac{d\tilde{\psi}_{ff}(N, M_{JJ}/\mu, \epsilon)}{d\mu} = 2\gamma_f(\alpha_s(\mu^2)) \tilde{\psi}_{ff}(N, M_{JJ}/\mu, \epsilon), \tag{57} \]
with \( \gamma_f \) the anomalous dimension of the field of flavor \( f \), which is, of course, independent of \( N \).

The dependence of the light-cone distribution \( \tilde{\phi}_{ff} \) on the factorization scale \( \mu \) is slightly more complex than for \( \psi \), and depends on the factorization scheme that we choose. The simplest case is the \( \overline{\text{MS}} \) scheme. Each of the moments of the \( \overline{\text{MS}} \) version of \( \phi \) obeys a renormalization group equation with the anomalous dimension of the color-diagonal splitting function for that flavor,
\[ \mu \frac{d\tilde{\phi}_{ff}(N, \mu^2, \epsilon)}{d\mu} = 2\gamma_{ff}(N, \alpha_s(\mu^2)) \tilde{\phi}_{ff}(N, \mu^2, \epsilon). \tag{58} \]

For the \( \overline{\text{MS}} \) distribution, this relation holds by definition, because the distributions are defined as the matrix elements of operators on the light-cone, whose renormalization is described by the splitting functions \[30, 10, 27\]. Only the flavor-diagonal evolution survives in the large-\( N \) limit, because only color diagonal splitting functions are singular for \( x \to 1 \).

The factorization scale dependence now may be controlled, by the two evolution equations (57) and (58). The \( \overline{\text{MS}} \) scheme expression that includes the \( \mu \)-dependent prefactor in Eq. (31), thus generalizing Eq. (51), is
\[ \left[ \frac{\tilde{\psi}_{ff}(N, M_{JJ}/\mu, \epsilon)}{\tilde{\phi}_{ff}(N, \mu^2, \epsilon)} \right]_{\overline{\text{MS}}} = R_{(f)}(\alpha_s(\mu^2)) \exp \left[ E_{(f)}(N, M_{JJ}) \right] \]
\[ \times \exp \left\{ -2 \int_{\mu}^{M_{JJ}} \frac{d\mu'}{\mu'} \left[ \gamma_f(\alpha_s(\mu'^2)) - \gamma_{ff}(N, \alpha_s(\mu'^2)) \right] \right\}. \tag{59} \]

This expression enables us to change factorization scales in the resummed cross section. The corresponding factor for DIS scheme distributions is easily found by comparing the ratio in Eq. (51) in DIS and \( \overline{\text{MS}} \) schemes. The effect of the DIS scheme is to produce an extra factor in Eq. (59),
\[ \left[ \frac{\tilde{\psi}_{ff}(N, M_{JJ}/\mu, \epsilon)}{\tilde{\phi}_{ff}(N, \mu^2, \epsilon)} \right]_{\text{DIS}} = R_{(f)}(\alpha_s(\mu^2)) \exp \left[ E_{(f)}(N, M_{JJ}) \right] \]
\[ \times \exp \left\{ -2 \int_{\mu}^{M_{JJ}} \frac{d\mu'}{\mu'} \left[ \gamma_f(\alpha_s(\mu'^2)) - \gamma_{ff}(N, \alpha_s(\mu'^2)) \right] \right\} \]
\[ \times \exp \left[ \int_0^1 dz \left( \frac{z^{N-1} - 1}{1 - z} \right) \left\{ \int_{(1-z)}^1 \frac{d\lambda}{\lambda} A_{(f)}(\alpha_s(\lambda \mu^2)) \right\} \right]. \tag{60} \]
This difference may also be formulated in terms of a slightly modified evolution equation for DIS distributions, which takes into account their logarithmic moment dependence [31].

5.3 Final-state jets

The remaining factors in Eq. (31) are the final-state jet functions \( J^{(f)} \). The resummation of their \( N \)-dependence depends critically on the definition of the dijet cross sections. For illustrative purposes, we have identified the choices \( M_{JJ} = (p_1 + p_2)^2 \) and \( 2p_1 \cdot p_2 \). We begin with the latter.

The case \( M_{JJ} = 2p_1 \cdot p_2 \). In this case, the jet function contributes to the overall weight through the mass of the cone-jet, \( p^2_i \), and also through the total energy of particles emitted outside the cone. Thus, recalling Eq. (27), we may write \( J^{(f)} \) as an integral over these variables at fixed \( w_i \),

\[
J^{(f)} \left( w_i, \frac{M_{JJ}}{\mu}, \alpha_s(\mu^2), \delta_i \right) = \int dp_i^2 dk_i^0 \delta \left( w_i - \frac{p_i^2 + 2M_{JJ}k_i^0}{M_{JJ}^2} \right) 
\times I^{(f)} \left( \frac{M_{JJ}}{\mu}, \frac{p_i^2}{\mu^2}, \frac{k_i^0}{\mu}, \delta_i \right),
\]

(61)

with \( I^{(f)} \) a density in the invariant mass inside the cone, and energy outside. This makes its analysis a bit more complicated than the cases studied in Ref. [7].

In the limit of very small \( p_i^2 \), however, we may factorize soft gluons emitted at angles much larger than \( \sqrt{p_i^2/M_{JJ}^2} \) from a function that describes the dynamics of collinear partons. This factorization can be carried out by the procedure of Ref. [5], referred to in Sec. 3.2 above. As discussed above, couplings of soft gluons to the jet are first replaced by an eikonal, or “soft” approximation. Suppose the jet velocity is \( \beta_i \).

In the soft approximation, soft gluon momenta carried by jet lines are approximated by their “opposite-moving” component \( \beta_i \cdot q \), and similarly for soft gluon polarizations at the vertices where they are emitted by jet lines.

After the use of Ward identities, soft gluons factorize in the soft approximation [1, 5], and we derive a convolution of the form,

\[
I^{(f)} \left( \frac{M_{JJ}}{\mu}, \frac{p_i^2}{\mu^2}, \frac{k_i^0}{\mu}, \delta_i \right) = \int d(\beta_i \cdot q) \Sigma^{(f)} \left( \frac{M_{JJ}}{\mu}, \frac{p_i^2 - \sqrt{2M_{JJ}(\beta_i \cdot q)}}{\mu^2} \right) 
\times J^{(f)}_{\text{OUT}} \left( \frac{(\beta_i \cdot q)}{\mu}, \frac{k_i^0}{\mu} \right).
\]

(62)

Corrections to this relation are free of infrared divergences in the limit of small \( k_i^0 \), \( (\beta_i \cdot q) \), and hence do not contribute to the leading power for \( w_i \rightarrow 0 \). The function \( J^{(f)}_{\text{OUT}} \), which absorbs the soft gluon dynamics is the same eikonal jet function defined in Eq. (45) and shown in Fig. 4. The function \( \Sigma^{(f)} \) remains dependent on the jet’s longitudinal momentum, and is linked to \( J^{(f)}_{\text{OUT}} \) by a convolution in the small component \( (\beta_i \cdot q) \) of momentum emitted by that function into the cone. In the limit of small \( p_i^2 \), gluons emitted outside the cone are all associated with the eikonal function.
The most important feature of $\overline{\Sigma}^{(f_i)}$ for us is that it is independent of the cone size, up to corrections that vanish with $p_i^2$.

The convolution in Eq. (62) simplifies under moments of $J^{(f_i)}$, Eq. (61), with respect to $w_i$,

$$\int dw_i e^{-Nw_i} J^{(f_i)} \left( \frac{w_i}{\mu}, \frac{M_{JJ}}{\mu}, \alpha_s(\mu^2), \delta_i \right)$$

$$= \int d(\beta_i \cdot q) dk_i^0 e^{-N \left( \frac{\sqrt{2}(\beta_i \cdot q) + 2k_i^0}{M_{JJ}} \right)} j_{\text{OUT}}^{(f_i)} \left( \frac{(\beta_i \cdot q)}{\mu}, \frac{k_i^0}{\mu} \right)$$

$$\times \int dp_i^2 e^{-N \left( \frac{p_i^2}{M_{JJ}} \right)} \Sigma^{(f_i)} \left( \frac{M_{JJ}}{\mu}, \frac{p_i^2}{\mu^2} \right)$$

$$= \gamma^{(f_i)}_{\text{OUT}} \left( \frac{M_{JJ}}{N\mu} \right) \overline{\Sigma}^{(f_i)} \left( \frac{M_{JJ}}{\mu}, \frac{M_{JJ}^2}{N\mu^2} \right), \quad (63)$$

where in the second line, $p_i^2 = p_i^2 - \sqrt{2} M_{JJ} (\beta_i \cdot q)$. The moment integrals here and below all have lower limits at zero. Dependence on their upper limits is exponentially suppressed in $N$, and may be neglected. Our task now is to derive the $N$-dependence of the product $\gamma^{(f_i)}_{\text{OUT}} \overline{\Sigma}^{(f_i)}$.

To find the requisite $N$-dependence, we begin by observing that a very similar procedure may be applied to the full axial gauge two-point function, which we denote as $\Sigma^{(f_i)}_2$. Again, soft gluon emission away from the jet axis factorizes into a convolution form,

$$\Sigma^{(f_i)}_2 \left( \frac{M_{JJ}}{\mu}, \frac{p_i^2}{\mu^2} \right) = \int d(\beta_i \cdot q) \Sigma^{(f_i)} \left( \frac{M_{JJ}}{\mu}, \frac{p_i^2 - \sqrt{2} M_{JJ} (\beta_i \cdot q)}{\mu^2} \right) j_{\text{OUT}}^{(f_i)} \left( \frac{(\beta_i \cdot q)}{\mu} \right), \quad (64)$$

with $j^{(f_i)}$ the two-point eikonal function, with its eikonal line again in the $\beta_i$-direction, evaluated at fixed values of the $\beta_i \cdot q$ momentum component of emitted partons. Here $\overline{\Sigma}^{(f_i)}$ is the same function as in Eq. (62) for small $p_i^2$, because in this limit collinear gluons are emitted only at the center of the cone, far from its boundary. Just as for Eq. (63), $\Sigma^{(f_i)}_2$ factorizes into a product under moments, this time with respect to $p_i^2$,

$$\int dp_i^2 e^{-N \left( \frac{p_i^2}{M_{JJ}} \right)} \Sigma^{(f_i)}_2 \left( \frac{M_{JJ}}{\mu}, \frac{p_i^2}{\mu^2} \right) = \int d(\beta_i \cdot q) e^{-N \left( \frac{\sqrt{2}(\beta_i \cdot q)}{M_{JJ}} \right)} j_{\text{OUT}}^{(f_i)} \left( \frac{(\beta_i \cdot q)}{\mu} \right)$$

$$\times \int dp_i^2 e^{-N \left( \frac{p_i^2}{M_{JJ}} \right)} \Sigma^{(f_i)} \left( \frac{M_{JJ}}{\mu}, \frac{p_i^2}{\mu^2} \right)$$

$$= \gamma_{\text{OUT}}^{(f_i)} \left( \frac{M_{JJ}}{N\mu} \right) \overline{\Sigma}^{(f_i)} \left( \frac{M_{JJ}}{\mu}, \frac{M_{JJ}^2}{N\mu^2} \right). \quad (65)$$

Comparing Eqs. (63) and (65), we see that the moments of the cone-jet function $J^{(f_i)}$ with respect to $w_i$ are closely related to the moments of the full two-point function
\[ \Sigma_2 \text{ with respect to } p_i^2, \]
\[ \int dw_i e^{-Nw_i} J^{(f_i)}(w_i) = \left[ \frac{\gamma_i^{(f_i)}(\mu, N)}{\gamma_i^{(j_i)}(\mu, N)} \right] \int dp_i^2 e^{-N\left(\frac{p_i^2}{M_{JJ}^2}\right)} \Sigma_2^{(f_i)}(p_i^2). \] (66)

The \( N \)-dependence of the moments of the jet function are thus given by the moments of the full two-point function, times the ratio of moments of the two eikonal functions \( j_{\text{OUT}} \) and \( j_- \). We readily find their \( N \)-dependence as follows.

Moments of a cut two-point function with respect to its invariant mass were re-summed explicitly in Ref. [3]. The \( \ln(N) \) dependence exponentiates in analogy to Eqs. (51) and (52). Normalizing the two-point function to \( \delta(p_i^2/M_{JJ}^2) \) allows to order,
\[ \int dp_i^2 e^{-N\left(\frac{p_i^2}{M_{JJ}^2}\right)} \Sigma_2^{(f_i)}(\mu, p_i^2) = \exp\left[ E'_{(f_i)}(N, M_{JJ})\right], \] (67)
where
\[ E'_{(f)}(N, M_{JJ}) = \int_0^1 \frac{dz}{1-z} \left\{ \alpha_s((1-z)M_{JJ}^2) \right\} + B'_{(f)} \left\{ \alpha_s((1-z)M_{JJ}^2) \right\}. \] (68)

The prime on \( E' \) indicates the exponent for a final-state jet, while the function \( A_{(f)} \) is the same as in Eq. (53). The lowest order term in \( B'_{(f)} \) may be read off from the one-loop jet function. The results include a gauge dependence, which cancels against a corresponding dependence in the soft anomalous dimension matrix [23].

The other factor in Eq. (66), \( j_{\text{OUT}}(N)/j_-(N) \) actually affects the moments of the final-state jet only beyond next-to-leading logarithm in \( N \). This may be seen as follows. First, being the sum of fully eikonal diagrams, both \( j_{\text{OUT}} \) and \( j_- \) exponentiate in moment space, according to the general arguments of Ref. [32]. At the same time, double logarithmic contributions, which are associated with gluons collinear to the eikonal lines, match in \( j_{\text{OUT}} \) and \( j_- \), because the phase space for collinear gluons is the same in both functions. The ratio may thus include at most next-to-leading logarithms, associated with soft gluons emitted outside the cone. The contribution of next-to-leading logs in the exponential may then be determined from a one-loop calculation. As for the soft function above, single logarithms are determined by the renormalization of one-loop virtual diagrams, which are identical for \( j_{\text{OUT}} \) and \( j_- \). An explicit calculation therefore gives no logarithms at all in the ratio.

In summary, up next-to-leading logarithm, \( j_{\text{OUT}}/j_- = 1 \), and the moments of the final-state jet in Eq. (66) with \( M_{JJ}^2 = 2p_1 \cdot p_2 \) are given by
\[ j^{(f_i)}\left(N, \frac{M_{JJ}}{\mu}, \alpha_s(\mu^2), \delta_i\right) = \int dw_i e^{-Nw_i} J^{(f_i)}(w_i) = \exp\left[E'_{(f_i)}(N, M_{JJ})\right], \] (69)
with \( E'_{(f)} \) defined as in Eq. (68), up to next-to-next-to-leading corrections in the function \( B'_{(f)} \).
An important feature of the exponent associated with the final-state jet is that it has the opposite overall sign from the exponent of Eq. (52), associated with the initial state. The leading logarithmic contributions of the final-state jets are negative, but are otherwise determined by the same functions $A_{(f)}(\alpha_s)$. Just as the leading-log initial-state contributions always act to enhance the cross section, those associated with the final state always suppress it, by an amount that depends on the partonic subprocess. Such singularities correspond to Sudakov suppression of scattering in the elastic limit [7, 23]. They are already present in explicit next-to-leading order calculations of single-jet inclusive cross sections [33]. It may be worth pointing out that for the specific choice of subprocess $q\bar{q}\to gg$, the net coefficient of the leading logarithm is negative in both DIS and \(\overline{\text{MS}}\) schemes (because $C_A > C_F/2$), in sharp contrast to the classic Drell-Yan calculation [3, 4].

The case $M_{jj}^2 = (p_1 + p_2)^2$. When we choose $M_{jj}$ as the total invariant mass of the two-jet system, the phase space at partonic threshold is, as mentioned in Sec. 3.1 above, changed significantly compared to the previous choice. In the present case, the masses of the dijets are not forced to zero at threshold (see Eq. (22)). In fact, the phase space at partonic threshold consists of all states in which each of the two jets carries total energy $M_{jj}/2$, with equal and opposite momenta. By dimensional considerations, the jet masses may increase to the order of $\delta^2 M_{jj}^2$. As a result of this larger phase space, collinear logarithms of $N$ cancel, leaving only infrared enhancements associated with soft emission outside the cone, which appear as single logarithms of $N$ per loop. The remnants of the dilogarithmic structure near threshold are terms of the form $\alpha_s^n (\ln \delta_i \ln N)^n$, which replace the leading $\alpha_s^n (\ln N)^2n$ terms of the previous case.

Although the jet functions with $M_{jj}^2 = (p_1 + p_2)^2$ involve fewer logs per loop than for $M_{jj}^2 = 2p_1 \cdot p_2$, the situation is in some sense more complicated than before. This is because of configurations in which the jets, although of minimum energy, are nevertheless massive. In these regions of phase space, soft emissions near the edge of the cone may resolve the flow of color within the cone [14], carried, for example, by fast partons separated by angles of order $\delta_i$. The couplings of soft gluons to such configurations may not be approximated by their coupling to a single eikonal line at the center of the cone. This approximation was accurate to all logs for $M_{jj}^2 = 2p_1 \cdot p_2$, because in that case the limit $N \to \infty$ forces the jet mass to zero, and hence the energetic partons to the center of the cone. Nevertheless, the approximation in which soft gluons are emitted by a single eikonal line at the center of the jet still captures the next-to-leading, $\alpha_s^n \ln^n N$, contributions. Any such eikonal cross section, with phase space that is symmetric in all particles, exponentiates [32, 34],

\[
\tilde{j}^{(f)}(N, M_{jj}, \alpha_s(\mu^2), \delta_i) = \exp[E_{(f)}'(N, M_{jj})] \\
E_{(f)}'(N, M_{jj}) = \exp \left[ \int_{\mu}^{M_{jj}/N} \frac{d\mu'}{\mu'} C_{(f)}'(\alpha_s(\mu'^2)) \right] ,
\]

where $C_{(f)}'(\alpha_s)$ is a finite perturbative series whose first term may be read off from a one-loop calculation. Beyond these next-to-leading logarithms, however, corrections
to Eq. (70) give a rather complicated sum of terms, each representing the coherent radiation due to a set of eikonal lines along arbitrary directions within the cone.

Again to avoid a proliferation of symbols, we use the same notation for the exponent here as for $M_{jj}^2 = 2p_1 \cdot p_2$, although the leading behavior in the two cases is quite different. For $M_{jj}^2 = (p_1 + p_2)^2$, the leading logarithmic behavior in moments (and therefore the leading distributions in momentum space) are not affected by the final-state jets, and thus retain the same (positive) contributions encountered in Drell-Yan cross sections.

We are now finally ready to assemble all the pieces necessary to write down a resummed cross section for dijet production.

5.4 The resummed cone-dijet cross sections

The characteristically nonabelian aspect of resummation in our case lies in the evolution of the soft functions. Recall that because the anomalous dimension matrix is not diagonal in general, solutions to the evolution equation (48) are ordered, rather than simple, exponentials. Substituting the solution (50) to Eq. (48) into the cross section in moment space, Eq. (31), and using Eqs. (51), (59) and (69) (or Eq. (70)), we find an expression that organizes all logarithms of $N$ at leading power in $N$, in $\overline{\text{MS}}$ factorization scheme,

$$
\tilde{\sigma}_f(N) = R_{(f)} \exp \left\{ \sum_{i=A,B} \left[ E_{(f)}(N, M_{JJ}) + \frac{2}{N} \int_{M_{JJ}/N} d\mu' \left[ \gamma_{f_i}(\alpha_s(\mu'^2)) - \gamma_{f_i}(N, \alpha_s(\mu'^2)) \right] \right] \right\}
\times \exp \left\{ \sum_{j=1,2} E'_{(f)}(N, M_{JJ}) \right\}
\times \text{Tr} \left\{ H^{(f)} \left( \frac{M_{JJ}}{\mu}, \Delta y, \alpha_s(\mu'^2) \right) \right\}
\times \tilde{P} \exp \left[ \int_{M_{JJ}/N} d\mu' \frac{d\mu'}{\mu'} \Gamma^{(f)}_S \left( \alpha_s(\mu'^2) \right) \right] \tilde{S}^{(f)}(1, \Delta y, \alpha_s \left( \frac{M_{JJ}^2}{N^2} \right))
\times P \exp \left[ \int_{M_{JJ}/N} d\mu' \frac{d\mu'}{\mu'} \Gamma^{(f)}_S \left( \alpha_s(\mu'^2) \right) \right],
$$

(71)

where as above the trace is taken in the space of color structures. For DIS scheme, the only change is the additional factor given in Eq. (60).

Without going into detail at this point, certain general features of the resummed cross section may be readily identified. The first exponential factors, $\exp[E_{(f)}(N)]$, serve, as in Drell-Yan cross sections, to enhance the cross section. The next factors, involving the anomalous dimensions of the parton fields and splitting functions, govern factorization scale dependence. The third factor, associated with final-state jets, always acts to suppress the cross section. We have noted already how the choice of jet algorithm can influence their size. The hard-scattering functions remain dependent on the scattering angle. The most interesting factors, however, are the ordered
exponentials of the soft anomalous dimension matrix. They depend on the scattering angle and they distinguish the roles of different color structures in the hard scattering through coherent gluon emission, linking initial- and final-state partons.

The logarithmic accuracy of the resummation depends upon the order to which the hard scatterings, and the anomalous dimensions, have been calculated. In a paper to follow this one, we shall compute the leading-order anomalous dimension matrices for each flavor combination in QCD. This will enable us to present explicit resummed dijet cross sections to next-to-leading order in \( \ln(N) \) [35].

5.5 Other jet algorithms and kinematics

Eq. (71) is typical of resummed jet cross sections, but its details depend on our use of cones to define the jets. Other choices, of course, are possible, and in cases, preferable. One of the disadvantages of the cone criterion is that it requires an extra parameter \( \delta_i \) for each jet. In addition, single-jet, rather than dijet cross sections are often more convenient experimentally. Single-jet inclusive cross sections have slightly different kinematic properties near threshold, compared to dijet, and therefore result in slightly different resummed expressions [12]. The extension of threshold resummation to cross sections with single-particle kinematics will be discussed in Ref. [36]. Nevertheless, it is clear that the result (71) is much more general than the cone algorithm that we have used. In fact, all that is necessary to derive such a resummed expression is a convolution form like Eq. (29), in which the weight is linear in the momenta of radiated gluons. Threshold logarithms in any such weight will be controlled by the same matrix of anomalous dimensions \( \Gamma_S \) that we have identified above. Differences between different algorithms will, in general, show up only in the functions \( B'_{(f)} \) for the final-state jets in Eq. (68).

6 Conclusion

In this paper, we have shown how to resum threshold logarithms in cone-based dijet cross sections. The method may be extended to other jet algorithms, and the explicit form of the resummed cross section will depend upon the details of the cross section chosen. In every case, however, the resummation depends upon the color structure of the hard scattering. As for the Drell-Yan cross section, the resummation of threshold singularities depends on the factorization scheme, although the color-dependent contribution does not. Our explicit result for the \( \overline{\text{MS}} \) factorization scheme is given in Eq. (71) above. It is, in principle, valid to all logarithmic accuracy, at leading power of \( N \) in moment space for the dijet cross section at fixed \( p_1 \cdot p_2 \), and to next-to-leading logarithm in \( N \) for fixed \( (p_1 + p_2)^2 \). Its inverse transform to momentum space therefore summarizes singular distributions in \( 1 - z \) to all logarithmic order in the former case and to next-to-leading order in the latter. As for heavy quark production, the general resummation can only be given in terms of ordered exponentials, due to mixing of color exchange tensors by soft gluon emission. In a forthcoming paper [35], we will derive the soft anomalous dimension matrix to one loop for the full range of flavor
scatterings that give rise to jet production, and give explicit expressions for dijet cross sections to leading order in the coupling, and to leading and next-to-leading logarithm in the moment variable.

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References


Figure 1: Configuration of cones in the partonic center of mass frame at partonic threshold.

Figure 2: Reduced diagram that represents the generic leading region for the dijet cross section.
Figure 3: Factorized form of the cross section into initial-state and final-state jet functions ($\psi$ and $J$, respectively) and the color-dependent soft function $S_{LI}^{(f)}$. At lowest order, $S_{LI}^{(f)}$ is given simply by the set of all diagrams in which a single gluon connects any two ordered eikonal lines moving in different directions. The construction of $S_{LI}^{(f)}$ beyond lowest order is discussed in Sec. 3.2.
Figure 4: Factorized eikonal cross section. With an appropriate choice of color normalization factor $1/N$, the soft function $S_{LI}^{(f)}$ is the same as in Fig. 3. The eikonal jet functions $j_{OUT}$ and $j_{IN}$ are defined in Eqs. (44) and (45), respectively.