Unitary Irreducible Representations of the Lorentz Group

CHRISTIAN FRONSDAL
CERN, Geneva, Switzerland
(Received September 9, 1958)

The unitary irreducible representations of the inhomogeneous, proper Lorentz group are determined, using a prescription given by Wigner, with special emphasis on the case of zero rest mass. The principal results are: (a) the construction of one-component representations for the case of zero mass and discrete spin; (b) the existence of a Foldy-Wouthuysen transformation for zero mass and spin $\frac{1}{2}$; (c) the construction of "position operators" for zero mass and spins $\frac{1}{2}, 1$; (d) the complete synthesis of the Dirac, Majorana, and Maxwell one-particle theories.

INTRODUCTION

WIGNER¹ has given a classification of all unitary, irreducible representations of the inhomogeneous proper Lorentz group, together with a prescription for their explicit construction. While it is well known that the representations employed in present field theories have their natural place in Wigner's classification, it appears that this fact has always been established *a fortiori*. It was thought to be of a certain interest, especially pedagogically, to determine the unitary, irreducible representations directly using Wigner's prescription, although no results of practical use could be foreseen. In particular, we were curious to see what the one-component representations for zero mass and arbitrary spin look like. This program is carried out in the first three sections.

Foldy² has carried out a partial synthesis of covariant particle equations, limited to the case of nonvanishing mass. His starting point is a set of unitary representations of the Lorentz group, which in turn were found by an analysis of the very theories to be synthesized. By means of the results referred to above it is possible to give a synthesis which is complete both in the respect of including the important case of zero mass, and in having a natural starting point. This is reported in Secs. IV to VI.

The synthesis, in the case of nonvanishing mass, follows the pattern of Foldy's work, and makes use of the Foldy-Wouthuysen (F-W) transformation. A similar transformation is found for the case of zero mass. This gives occasion to extend the discussion of Foldy and Wouthuysen,³ concerning what they call the mean position operator, to the case of zero mass. It is found that such a position operator exists for the neutrino, and more surprisingly, for the Maxwell field (Sec. VII).

Only infinitesimal transformations are discussed. We shall use the term Lorentz rotation for the Lorentz transformations connecting reference frames with relative velocity but the same orientation of the space axes. For a representation of a Lorentz rotation with infinitesimal velocity θ we write $D[1+\theta\cdot\mathbf{L}]$. Similarly we

write $D[1+\theta\cdot\mathbf{R}]$ for a representation of a real rotation through an infinitesimal angle θ .

I. SUMMARY OF WIGNER'S PRESCRIPTION

The following is a very much abbreviated derivation of the unitary, irreducible representations of the proper, inhomogeneous Lorentz group.

In an irreducible representation, the energy-momentum four-vector $p_i = (\mathbf{p}, E)$ varies over the set of all real numbers consistent with the condition

$$E = \omega, \quad \text{or} \quad E = -\omega, \quad (1)$$

where

$$\omega = +(\mathbf{p}^2 + m^2)^{1/2}.$$

Choose a particular set of 4 real numbers a_i , representing one of the possible set of values of p_i , and define the "little group" as that subgroup of the group of homogeneous proper Lorentz transformations which leaves a_i invariant. More precisely, since a_i is a fixed set of numbers, and not a 4 vector, the little group leaves p_i invariant if it happens to have the initial value a_i .

Next choose, for every p_i , a particular Lorentz transformation $\alpha(p)$ which transforms a four-vector y_i with initial value a_i into the four-vector p_i . Given any homogeneous proper Lorentz transformation $y \rightarrow \Lambda y$, the transformation

$$y \rightarrow \Lambda' y \equiv \alpha^{-1}(p) \Lambda \alpha(\Lambda^{-1}p) y \quad (2)$$

belongs to the little group. It is in fact obvious that if $y_i = a_i$, then $(\Lambda' y)_i = a_i$ as well. It may further be proved that, given any representation of the little group, a representation of the whole group is obtained by the identification

$$D[\Lambda] \varphi(p) = D[\Lambda'] \varphi(\Lambda^{-1}p). \quad (3)$$

For details and proofs the reader is referred to the original paper by Wigner.¹ The precise meaning of (3) will become more clear through its application in the following sections.

II. UNITARY REPRESENTATIONS FOR NONZERO MASS

A convenient choice for a_i is in this case

$$a_i = (0, 0, 0, m),$$

where m must have the same sign as p_0 . The little group

¹ E. P. Wigner, Ann. Math. **40**, 149 (1939).

² L. L. Foldy, Phys. Rev. **102**, 568 (1956).

³ L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

is the real rotation group. The transformation $\alpha(p)$ is the Lorentz rotation from the rest system to the system of momentum $\mathbf{p}$:

$$\begin{aligned}\alpha(p)\mathbf{y} &= \mathbf{y} + \left(\frac{E}{m} - 1\right) \frac{\mathbf{p} \cdot \mathbf{y}}{p^2} \mathbf{p} + \frac{y_0}{m} \mathbf{p}, \\ \alpha(p)y_0 &= -y_0 + \frac{\mathbf{y} \cdot \mathbf{p}}{m}, \\ \alpha^{-1}(p)\mathbf{y} &= \mathbf{y} + \left(\frac{E}{m} - 1\right) \frac{\mathbf{p} \cdot \mathbf{y}}{p^2} \mathbf{p} - \frac{y_0}{m} \mathbf{p}, \\ \alpha^{-1}(p)y_0 &= -y_0 - \frac{\mathbf{y} \cdot \mathbf{p}}{m}.\end{aligned}\quad (4)$$

The transformations $\Lambda'y$, as defined by (2) are readily calculated for infinitesimal Λ , with the following result

$$\Lambda'y = \mathbf{y} + \boldsymbol{\theta} \times \mathbf{y}, \quad (5a)$$

if

$$\Lambda\mathbf{y} = \mathbf{y} + \boldsymbol{\theta} \times \mathbf{y},$$

and

$$\Lambda'y = \mathbf{y} - \frac{\boldsymbol{\theta} \times \mathbf{p}}{E+m} \times \mathbf{y}, \quad (5b)$$

if

$$\Lambda\mathbf{y} = \mathbf{y} + y_0 \boldsymbol{\theta}.$$

In either case Λ' leaves y_0 invariant.

Any unitary irreducible representation of the real 3-dimensional rotation group is equivalent to a representation of the form

$$D[1 + \boldsymbol{\theta} \cdot \mathbf{R}]\varphi(\mathbf{p}) = (1 + \boldsymbol{\theta} \cdot \mathbf{s}/2i)\varphi(\mathbf{p} - \boldsymbol{\theta} \times \mathbf{p}),$$

where $\mathbf{s}/2$ are the spin matrices appropriate for spin s , and have the dimension $2s+1$. In the case of spin $\frac{1}{2}$, $\mathbf{s}$ are the Pauli spin matrices.

The content of (3) and (5) is that the Lorentz rotation with velocity $\boldsymbol{\theta}$ is to be represented by the same operator as the real rotation through the angle

$$-\boldsymbol{\theta} \times \mathbf{p}/(E+m).$$

Hence (3) and (5) yield the following representation of the proper homogeneous Lorentz group⁴

$$D[1 + \boldsymbol{\theta} \cdot \mathbf{R}]\varphi(\mathbf{p}) = (1 + \boldsymbol{\theta} \cdot \mathbf{s}/2i)\varphi(\mathbf{p} - \boldsymbol{\theta} \times \mathbf{p}), \quad (6a)$$

$$D[1 + \boldsymbol{\theta} \cdot \mathbf{L}]\varphi(\mathbf{p}) = \left(1 + \frac{1}{2}i \frac{\boldsymbol{\theta} \times \mathbf{p}}{E+m} \cdot \mathbf{s}\right)\varphi(\mathbf{p} - E\boldsymbol{\theta}). \quad (6b)$$

It should perhaps be emphasized that (5) does not describe the transformation of y_i under Lorentz transformations. The point is that we have established a

⁴ The dependence of $\varphi(\mathbf{p}, E)$ on E is suppressed whenever convenient.

homomorphism between the two sets of transformations

$$y \rightarrow \Lambda y, \quad p \rightarrow \Lambda p; \quad (7a)$$

and

$$y \rightarrow \Lambda' y, \quad p \rightarrow \Lambda p. \quad (7b)$$

To see this one may ascertain that the commutation relations for (7b) are satisfied in the following form

$$\begin{aligned}\left[\left(-\frac{\boldsymbol{\theta}' \times \mathbf{p}}{E+m} \times -E\boldsymbol{\theta}' \cdot \frac{d}{d\mathbf{p}} \right), \left(-\frac{\boldsymbol{\theta} \times \mathbf{p}}{E+m} \times -E\boldsymbol{\theta} \cdot \frac{d}{d\mathbf{p}} \right) \right] y \\ = - \left(\boldsymbol{\theta}'' \times -\boldsymbol{\theta}'' \times \mathbf{p} \cdot \frac{\partial}{\partial \mathbf{p}} \right) y,\end{aligned}$$

where

$$\boldsymbol{\theta}'' = \boldsymbol{\theta}' \times \boldsymbol{\theta}, \quad \frac{d}{d\mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} + \frac{\mathbf{p}}{E} \frac{\partial}{\partial E}.$$

The commutation relations for (6) may be verified in a similar way; a detailed discussion of the form of (6) is, however, postponed to a later section (Sec. IX).

III. UNITARY ONE COMPONENT REPRESENTATIONS FOR ZERO MASS

In this case the following choice for a_i is convenient

$$a_i = (\mathbf{a}, \pm 1), \quad a^2 = 1, \quad (8)$$

where again the sign of a_0 must be the sign of p_0 . The little group consists of three transformations, of which one is a real rotation with $\mathbf{a}$ as axis. The other two transformations are represented by the identity in those representations which correspond to discrete spin.⁵ We shall not be interested in representations with continuous spin. When calculating Λ' for this case, we may therefore retain only terms which represent rotations with $\mathbf{a}$ as axis, dropping terms which represent either Lorentz rotations or real rotations with axes perpendicular to $\mathbf{a}$.

The following choice of transformations $\alpha(p)$ is probably not the most convenient one, since the calculations turned out to be rather lengthy:

$$\begin{aligned}\alpha(p)\mathbf{y} &= \mathbf{a} \cdot \mathbf{p} \mathbf{y}_1 + (\mathbf{a} \times \mathbf{p}) \times \mathbf{y} + \mathbf{y}_2 |\mathbf{p}|, \\ \alpha^{-1}(p)\mathbf{y} &= (1/p^2) [\mathbf{a} \cdot \mathbf{p} \mathbf{y}_1 - (\mathbf{a} \times \mathbf{p}) \times \mathbf{y} + \mathbf{y}_2 |\mathbf{p}|].\end{aligned}\quad (9)$$

Here $\mathbf{y}$ has been decomposed into a part $\mathbf{y}_1$ in the $\mathbf{a}, \mathbf{p}$ plane, and a part $\mathbf{y}_2$ normal to it.⁶

The results for $\Lambda'y$ are

$$\Lambda'y = \mathbf{y} + \frac{\boldsymbol{\theta} \cdot (\mathbf{a} + \mathbf{p}/|\mathbf{p}|)}{\mathbf{a} \cdot (\mathbf{a} + \mathbf{p}/|\mathbf{p}|)} \mathbf{y}, \quad (10a)$$

if

$$\Lambda\mathbf{y} = \mathbf{y} + \boldsymbol{\theta} \times \mathbf{y},$$

⁵ The little group is isomorphic to the group of rotations and translations in two-dimensional Euclidean space. The spin is discrete only if the "momentum" vanishes.

⁶ For (9) to be a Lorentz rotation of y_i , y_0 has to undergo a simultaneous transformation. This part of the transformation $\alpha(p)$ does not enter the calculations, however.

and

$$\Lambda' \mathbf{y} = \mathbf{y} + \frac{E}{|\mathbf{p}|} \frac{\boldsymbol{\theta} \cdot (\mathbf{a} \times \mathbf{p} / |\mathbf{p}|)}{\mathbf{a} \cdot (\mathbf{a} + \mathbf{p} / |\mathbf{p}|)}, \quad (10b)$$

if

$$\Lambda \mathbf{y} = \mathbf{y} + \boldsymbol{\theta} y_0.$$

To all intents and purposes, the little group is the single-parameter group of real rotations with $\mathbf{a}$ as axis. The unitary, irreducible, one- or two-valued representations of this group are

$$\varphi(\mathbf{p}) \rightarrow (1 - is\boldsymbol{\theta}) \varphi(\mathbf{p} - \boldsymbol{\theta} \mathbf{a} \times \mathbf{p}),$$

where s is the spin, and can take the values $0, \pm \frac{1}{2}, \pm 1, \dots$.

The content of (3) is now that any Lorentz transformation may be represented by the same operator as a real rotation with $\mathbf{a}$ as axis, through an angle given by (10), *viz*

$$D[1 + \boldsymbol{\theta} \cdot \mathbf{R}] \varphi(\mathbf{p})$$

$$= \left(1 + \frac{s \boldsymbol{\theta} \cdot (\mathbf{a} + \mathbf{p} / |\mathbf{p}|)}{i \mathbf{a} \cdot (\mathbf{a} + \mathbf{p} / |\mathbf{p}|)} \right) \varphi(\mathbf{p} - \boldsymbol{\theta} \times \mathbf{p}), \quad (11a)$$

$$D[1 + \boldsymbol{\theta} \cdot \mathbf{L}] \varphi(\mathbf{p})$$

$$= \left(1 + \frac{s E}{i |\mathbf{p}|} \frac{\boldsymbol{\theta} \cdot (\mathbf{a} \times \mathbf{p} / |\mathbf{p}|)}{\mathbf{a} \cdot (\mathbf{a} + \mathbf{p} / |\mathbf{p}|)} \right) \varphi(\mathbf{p} - E \boldsymbol{\theta}). \quad (11b)$$

Because, as emphasized above, a_i is not a vector but a fixed set of 4 numbers, (11) is not covariant. The same is of course true of the representations for nonzero mass. Just as in the latter case one must construct a covariant physical interpretation of the theory, so we must in the present case construct a physical interpretation which is independent of the choice of $\mathbf{a}$. That this situation is not so unusual as one might think, is seen by remembering that a similarly arbitrary feature exists in conventional theories. Thus, in Dirac theory the representation of the γ matrices is arbitrary to a certain extent. In that theory also, one must construct a physical interpretation which is insensitive to this arbitrariness. Later it will be seen that the freedom of choice of $\mathbf{a}$ corresponds exactly to the freedom of choice of representation of the Pauli spin matrices in the Majorana theory.

IV. CANONICAL REPRESENTATION

In conventional theories (Dirac, Majorana, Fierz-Pauli) of half-odd integer spin fields, the wave-function is given different names according to whether the energy is positive or negative. This is particularly evident in the canonical representation invented by Case.⁷ From our point of view, which closely parallels that of Foldy,² this doubling of the representation space is a necessary first step in the derivation of first order wave equations. The

wave functions discussed in the two preceding sections satisfy, of course, only the second order Klein-Gordon equation.

(a) Nonzero Mass

We agree to give $\varphi(\mathbf{p})$ the new name ψ_1 if the energy is positive, and ψ_2 if it is negative. This may be written

$$\psi = \begin{bmatrix} \frac{E+\omega}{2\omega} \varphi \\ -\frac{E-\omega}{2\omega} \varphi \end{bmatrix}. \quad (12)$$

It is seen that ψ satisfies

$$E\psi = \beta \omega \psi, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (13)$$

The matrix β is $2(2s+1)$ -dimensional. The behavior of ψ under Lorentz transformations is essentially given by (6). No generality is lost by choosing the same representation of the $\mathbf{s}$ matrices for either sign of the energy. However, we must distinguish four possibilities concerning the sign of m . The commutation relations are satisfied in each case. Either

$$m = \pm |m|, \quad (14)$$

or

$$m = \pm (E/\omega) |m|. \quad (15)$$

When deriving (4) by Wigner's prescription it was necessary to take the sign of $m = a_0$ equal to the sign of E . This is not a good argument in favor of (15), however, as either of the 4 cases can be obtained by a slight generalization of that prescription. The strength of (15), however, (with either sign) is that it permits the construction of a Wigner time-reversal operator, while (14) does not.⁸ Adopting (15), the properties of $\psi(\mathbf{p})$ can be summarized as follows:

$$E\psi = \beta \omega \psi, \quad (16a)$$

$$D[1 + \boldsymbol{\theta} \cdot \mathbf{R}] \psi(\mathbf{p}) = (1 - \frac{1}{2} i \boldsymbol{\theta} \cdot \mathbf{s}) \psi(\mathbf{p} - \boldsymbol{\theta} \times \mathbf{p}), \quad (16b)$$

$$D[1 + \boldsymbol{\theta} \cdot \mathbf{L}] \psi(\mathbf{p}) = \left(1 + \frac{1}{2} i \beta \frac{\boldsymbol{\theta} \times \mathbf{p}}{\omega \pm |m|} \cdot \mathbf{s} \right) \psi(\mathbf{p} - E \boldsymbol{\theta}). \quad (16c)$$

These are essentially the representations which Foldy² obtained by applying a Foldy-Wouthuysen transformation to the Dirac representation. Since we can find no basis for distinguishing between the two signs in (16c), a physical interpretation of the theory must be independent of a particular choice.

⁷ K. M. Case, Phys. Rev. **95**, 1323 (1954).

⁸ See Sec. VII.

(b) Zero Mass

The procedure is similar to the case of nonzero mass. Define

$$\psi = \begin{pmatrix} \frac{E+|\mathbf{p}|}{2|\mathbf{p}|} \varphi \\ -\frac{E-|\mathbf{p}|}{2|\mathbf{p}|} \varphi \end{pmatrix}, \quad (17)$$

and write the analog of (13) in the form

$$E\psi = \mathbf{b} \cdot \boldsymbol{\sigma} |\mathbf{p}| \psi, \quad \mathbf{b} \cdot \boldsymbol{\sigma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (18)$$

In this case the matrix $\mathbf{b} \cdot \boldsymbol{\sigma}$ is a 2-dimensional matrix. Although the behavior of ψ under Lorentz transformations is essentially given by (11), the ambiguity is greater than in the case of nonzero mass. Again one finds, however, that when the existence of a Wigner time reversal operator is invoked, the arbitrariness is reduced to that of a sign, as in the case of nonzero mass. The transformations are (the double signs are coupled)

$$D[1 + \boldsymbol{\theta} \cdot \mathbf{R}] \psi(\mathbf{p}) = \left(1 \pm \frac{|\mathbf{s}|}{i} \frac{\boldsymbol{\theta} \cdot (\mathbf{a} + \mathbf{p}/|\mathbf{p}|)}{\mathbf{a} \cdot (\mathbf{a} + \mathbf{p}/|\mathbf{p}|)} \right) \psi(\mathbf{p} - \boldsymbol{\theta} \times \mathbf{p}), \quad (19a)$$

$$D[1 + \boldsymbol{\theta} \cdot \mathbf{L}] \psi(\mathbf{p}) = \left(1 \pm \frac{|\mathbf{s}|}{i} \frac{\boldsymbol{\theta} \cdot (\mathbf{a} \times \mathbf{p}/|\mathbf{p}|)}{\mathbf{a} \cdot (\mathbf{a} + \mathbf{p}/|\mathbf{p}|)} \right) \psi(\mathbf{p} - E\boldsymbol{\theta}). \quad (19b)$$

As in the case of the arbitrary sign in (16), one would here seem to have occasion to remark that a physical interpretation of the theory must not depend on the sign in (19). The analogy between the appearance of the double signs in (16) and (19) is, as we have seen, the closest possible. Some further considerations on this are given below (Sec. IX).

V. THE F-W TRANSFORMATION

Up to this point we have treated the general case of arbitrary spin. In the following this could have been achieved only by sacrificing the simplicity of the presentation. For this reason, and because Nature apparently refuses to yield particles of spin higher than 1, we shall limit all subsequent discussion to the spins 0, $\frac{1}{2}$, and 1. In this section attention is focused on the case of spin $\frac{1}{2}$.

(a) Nonzero Mass

Define a new wave-function $\chi(\mathbf{p})$ by the Foldy-Wouthuysen transformation

$$\chi(\mathbf{p}) = \exp[-\frac{1}{2}\lambda\beta\boldsymbol{\alpha} \cdot \mathbf{p}/|\mathbf{p}|] \psi(\mathbf{p}), \quad (20)$$

where the α matrices are defined in terms of the 2-

dimensional σ matrices as follows:

$$\alpha = \begin{pmatrix} 0 & -i\sigma \\ i\sigma & 0 \end{pmatrix}.$$

In (20), λ is essentially the Lorentz angle between $\mathbf{p}_i$ and $\mathbf{a}_i$:

$$\cos\lambda = \pm |\mathbf{m}|/\omega, \quad \sin\lambda = \mathbf{p}/\omega. \quad (21)$$

An equivalent form of (20) is⁹

$$\chi(\mathbf{p}) = \frac{\omega \pm |\mathbf{m}| - \beta \boldsymbol{\alpha} \cdot \mathbf{p}}{\{2\omega(\omega \pm |\mathbf{m}|)\}^{\frac{1}{2}}} \psi(\mathbf{p}). \quad (22)$$

The wave equation for χ is obtained immediately by transforming (16a)

$$E\chi = (\boldsymbol{\alpha} \cdot \mathbf{p} \pm |\mathbf{m}|\beta)\chi, \quad (23a)$$

and the transformations of χ are

$$D[1 + \boldsymbol{\theta} \cdot \mathbf{R}] \chi(\mathbf{p}) = (1 - \frac{1}{2}i\boldsymbol{\theta} \cdot \boldsymbol{\sigma}) \chi(\mathbf{p} - \boldsymbol{\theta} \times \mathbf{p}), \quad (23b)$$

$$D[1 + \boldsymbol{\theta} \cdot \mathbf{L}] \chi(\mathbf{p}) = \left(1 + \frac{1}{2}\boldsymbol{\theta} \cdot \boldsymbol{\alpha} - \frac{\boldsymbol{\theta} \cdot \mathbf{p}}{E} \right) \chi(\mathbf{p} - E\boldsymbol{\theta}). \quad (23c)$$

In order to obtain (23c) it was necessary to make use of (23a). In the process the unitarity of the transformation was lost. Unitarity can be restored in an infinite variety of ways. For example, by virtue of (23a), one may write

$$D[1 + \boldsymbol{\theta} \cdot \mathbf{L}] \chi(\mathbf{p}) = \left(1 + \frac{1}{2}\boldsymbol{\theta} \cdot \boldsymbol{\alpha} \frac{\boldsymbol{\alpha} \cdot \mathbf{p} \pm |\mathbf{m}|\beta}{E} - \frac{\boldsymbol{\theta} \cdot \mathbf{p}}{E} \right) \chi(\mathbf{p} - E\boldsymbol{\theta}), \quad (24)$$

which is unitary. This equivalence of (23c) and (24) can, of course, be established in the absence of interactions only.

The last term in (23c) can be eliminated by multiplying $\chi(\mathbf{p})$ by $\omega^{\frac{1}{2}}$. One then obtains the defining properties of the Dirac theory. We have thus carried out a complete synthesis from the construction of unitary irreducible representations of the Lorentz group.

(b) Zero Mass

The Foldy-Wouthuysen transformation appropriate for the case of vanishing mass cannot be obtained as the limit of (20) as $|\mathbf{m}| \rightarrow 0$. It is, nevertheless, of a very similar form. Define¹⁰

$$\chi(\mathbf{p}) = \exp[\pm \frac{1}{2}i\lambda S] \psi(\mathbf{p}), \quad (25)$$

where

$$S = (\mathbf{p} \times \mathbf{b}) \cdot \boldsymbol{\sigma} / |\mathbf{p} \times \mathbf{b}|, \quad (26)$$

⁹ The double signs of (16), (21), and the equations that follow are coupled, as are all subsequent double signs in the discussion of fields with mass.

¹⁰ The double signs of (19), (25), and all subsequent equations pertaining to the case of zero mass are coupled.

and λ is the azimuthal angle between $\mathbf{p}$ and $\mathbf{b}$:

$$\cos\lambda = \pm \mathbf{b} \cdot \mathbf{p} / |\mathbf{p}|, \quad \sin\lambda = |\mathbf{b} \times \mathbf{p}| / |\mathbf{p}|. \quad (27)$$

We may write (25) in a form similar to (22)

$$\chi(\mathbf{p}) = \frac{|\mathbf{p}| \pm \mathbf{p} \cdot \mathbf{b} \pm i(\mathbf{p} \times \mathbf{b}) \cdot \boldsymbol{\sigma}}{\{2|\mathbf{p}|(|\mathbf{p}| \pm \mathbf{b} \cdot \mathbf{p})\}^{\frac{1}{2}}}. \quad (28)$$

The wave equation and transformation properties for χ are found to be, if we take $\mathbf{b} = \pm \mathbf{a}$,

$$E\chi = \pm \boldsymbol{\sigma} \cdot \mathbf{p}\chi, \quad (29a)$$

$$D[1 + \boldsymbol{\theta} \cdot \mathbf{R}]\chi(\mathbf{p}) = (1 - \frac{1}{2}\boldsymbol{\theta} \cdot \boldsymbol{\sigma})\chi(\mathbf{p} - \boldsymbol{\theta} \times \mathbf{p}), \quad (29b)$$

$$D[1 + \boldsymbol{\theta} \cdot \mathbf{L}]\chi(\mathbf{p}) = (1 \pm \frac{1}{2}\boldsymbol{\theta} \cdot \boldsymbol{\sigma} - \boldsymbol{\theta} \cdot \mathbf{p}/2E)\chi(\mathbf{p} - E\boldsymbol{\theta}). \quad (29c)$$

The remarks made after (23) apply to (29c) as well. The simplest way to restore unitarity is to note that (in the absence of interaction) (29c) is equivalent to

$$D[1 + \boldsymbol{\theta} \cdot \mathbf{L}]\chi(\mathbf{p}) = (1 + i\boldsymbol{\theta} \times \mathbf{p} \cdot \boldsymbol{\sigma}/2E)\chi(\mathbf{p} - E\boldsymbol{\theta}). \quad (30)$$

It may be noted that, both in the case of nonzero mass and in that of vanishing mass, the transformations $\psi \rightarrow \chi$ are unitary.

Equations (29) are the defining equations of the Majorana theory¹¹ for zero mass [if the last term in (29c) is eliminated as before]. We can now justify the remarks made at the end of Sec. III. In fact, combining (18) with the identification $\mathbf{b} = \pm \mathbf{a}$ [which was made above in order to ensure the elimination of $\mathbf{a}$ and $\mathbf{b}$ from the transformations (29b) and (29c)], one has

$$\mathbf{a} \cdot \boldsymbol{\sigma} = \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (31)$$

which means that whatever the direction $\mathbf{a}$, we take such a representation of the matrix-vector $\boldsymbol{\sigma}$ that its projection along $\mathbf{a}$ is equal to the right-hand side of (31). Hence there is a one to one correspondence between the direction of $\mathbf{a}$ in the one-component theory and the representation of $\boldsymbol{\sigma}$ in the Majorana theory, and there is no basis for any prejudice against the appearance of $\mathbf{a}$ in the former.

VI. ANOTHER DOUBLING: THE MAXWELL FIELD

In the case of integral spin one may also consider the canonical form of the theory. This was done, in fact by Case,⁷ and by Foldy,² in the case of nonvanishing mass. However, the only interesting application is to the Maxwell theory, i.e., the case of zero mass and unit spin. The conventional formulation of that theory differs from the theories synthesized above in that different components of the wave function do not correspond to different signs of the energy, but only to different spin. It would require too much space to give the synthesis of the electromagnetic field in detail. The representation

of the Lorentz group on the electromagnetic potential is equivalent to

$$D = \{D(\mathbf{a}) \oplus D^*(\mathbf{a})\} \otimes \{D(-\mathbf{a}) \oplus D^*(-\mathbf{a})\}, \quad (32)$$

where $D(\mathbf{a})$ is given by (11) with $|s| = \frac{1}{2}$. If $\psi_{\mu\nu}$ is the wave function transforming according to (32) we have, explicitly

$$\begin{aligned} D[1 + \boldsymbol{\theta} \cdot \mathbf{R}]\psi_{\mu\nu}(\mathbf{p}) &= \left[\delta_{\mu\alpha}\delta_{\nu\beta} + \frac{1}{2i} \frac{\boldsymbol{\theta} \cdot (\mathbf{a} + \mathbf{p}/|\mathbf{p}|)}{\mathbf{a} \cdot (\mathbf{a} + \mathbf{p}/|\mathbf{p}|)} \right. \\ &\quad \left. + \frac{1}{2i} \frac{\delta_{\mu\alpha}\mathbf{a} \cdot \boldsymbol{\sigma}_{\beta\nu}}{-\mathbf{a} \cdot (-\mathbf{a} + \mathbf{p}/|\mathbf{p}|)} \right] \psi_{\alpha\beta}(\mathbf{p} - \boldsymbol{\theta} \times \mathbf{p}), \quad (33) \end{aligned}$$

and similarly for $D[1 + \boldsymbol{\theta} \cdot \mathbf{L}]$. The relation of A^i to $\psi_{\mu\nu}$ is given by

$$\mathbf{A} - (\mathbf{p}/E)\mathbf{A}^0 = \mathbf{B}, \quad (34)$$

$$\begin{aligned} B^i &= \sigma^i_{\nu\mu} \left[\frac{|\mathbf{p}| + \mathbf{p} \cdot \boldsymbol{\sigma} \mathbf{a} \cdot \boldsymbol{\sigma}}{\{2|\mathbf{p}|(|\mathbf{p}| + \mathbf{a} \cdot \mathbf{p})\}^{\frac{1}{2}}} \right]_{\mu\alpha} \\ &\quad \times \left[\frac{|\mathbf{p}| - \mathbf{a} \cdot \boldsymbol{\sigma} \mathbf{p} \cdot \boldsymbol{\sigma}}{\{2|\mathbf{p}|(|\mathbf{p}| - \mathbf{a} \cdot \mathbf{p})\}^{\frac{1}{2}}} \right]_{\beta\nu} \psi_{\alpha\beta}. \quad (35) \end{aligned}$$

The subsidiary conditions which eliminate the spin-zero parts of B^i are

$$\mathbf{p} \cdot \mathbf{B} = 0, \quad (36)$$

$$B^0 = 0 \quad (37)$$

In order to see that A^i as defined by (34) can be identified with the electromagnetic potential, one may verify that the transformations (33) follow for $\psi_{\mu\nu}$ if the conventional transformation properties are assumed for A^i . [The opposite is not true, since (34) cannot be solved for A^i .] The wave equation is, of course, implicitly assumed for $\psi_{\mu\nu}$. It follows that B^i satisfies the wave equation, and that A^i may be assumed to do likewise. The subsidiary condition (36), in terms of A^i , is the Lorentz condition, while (37) cannot be directly translated into a condition on A^i . Instead, since not A^i , but only the left-hand side of (34) is given by B^i , there follows that the addition of a 4 gradient to A^i does not change B^i . Hence, the subsidiary condition (37) corresponds to the requirement of gauge-invariance.

VII. THE REFLECTION OPERATORS

Construction of operators representing reflections in space and time are usually discussed from the point of view of the invariance of the wave equation. Satisfaction of the commutation relations with the operators representing proper Lorentz transformations is either ignored or taken for granted. In a representation where no wave equation exists (other than the Klein-Gordon equation, the invariance of which is trivial), the latter requirement is the guiding principle. The commutation rela-

¹¹ K. M. Case, Phys. Rev. **107**, 307 (1957).

tions are¹²

$$\begin{aligned} D[P]D[1+\theta\cdot\mathbf{R}] &= D[1+\theta\cdot\mathbf{R}]D[P], \\ D[P]D[1+\theta\cdot\mathbf{L}] &= D[1-\theta\cdot\mathbf{L}]D[P], \end{aligned} \quad (38)$$

and the same for $D[T]$.

The φ representation is characterized by (6) and (11). Time reversal has no meaning here, but for space-reflection we may take

$$D[P]: \varphi(\mathbf{p}, E) \rightarrow (E/\omega)\varphi(-\mathbf{p}, E), \quad (39)$$

for nonzero mass, and

$$\begin{aligned} D[P]: \varphi(\mathbf{p}, E) &\rightarrow \varphi^*(-\mathbf{p}, E), \\ \mathbf{a} &\rightarrow -\mathbf{a}, \end{aligned} \quad (40)$$

for zero mass.¹³ (The reversal of $\mathbf{p}$ is of course understood.)

The ψ representation allows the construction of an anti-unitary time reflection operator if we adopt (16) and (19). Then we have, for nonzero mass,

$$D[P]: \psi(\mathbf{p}, E) \rightarrow \psi(-\mathbf{p}, E), \quad (41a)$$

$$D[T]: \psi(\mathbf{p}, E) \rightarrow A\psi^*(\mathbf{p}, -E), \quad (41b)$$

where A is an operator defined by

$$A\mathbf{s} = -\mathbf{s}^*A, \quad A\beta = -\beta A. \quad (42)$$

Note that (41b) includes implicitly the reversal of the sign of a_0 , since we have derived (16) by means of (15b). Hence (41b) is quite analogous to (40). For zero mass, we have

$$\begin{aligned} D[P]: \psi(\mathbf{p}, E) &\rightarrow \psi^*(-\mathbf{p}, E), \\ \mathbf{a} &\rightarrow -\mathbf{a}, \end{aligned} \quad (43a)$$

$$D[T]: \psi(\mathbf{p}, E) \rightarrow A'\psi^*(\mathbf{p}, -E). \quad (43b)$$

If $\mathbf{a}$ is identified with $\mathbf{b}$, then

$$\begin{aligned} D[P]: \psi(\mathbf{p}, E) &\rightarrow A'\psi^*(-\mathbf{p}, E), \\ \mathbf{a} &\rightarrow -\mathbf{a}, \end{aligned} \quad (43c)$$

In (43b) and (43c), A' is defined by

$$A'\sigma = -\sigma^*A'. \quad (44)$$

The χ representation is characterized by (23) and (29). Equations (38), and those obtained from them by the replacement of P by T , are satisfied by the following identification, which can be obtained from (41) and (43) by means of the transformations (20) and (25). For nonzero mass, we have

$$\begin{aligned} D[P]: \chi(\mathbf{p}, E) &\rightarrow \beta\chi(-\mathbf{p}, E), \\ D[T]: \chi(\mathbf{p}, E) &\rightarrow A\chi^*(\mathbf{p}, -E). \end{aligned} \quad (45)$$

For zero mass, we have

$$\begin{aligned} D[P]: \chi(\mathbf{p}, E) &\rightarrow A'\chi^*(-\mathbf{p}, E), \\ D[T]: \chi(\mathbf{p}, E) &\rightarrow A'\chi^*(\mathbf{p}, -E). \end{aligned} \quad (46)$$

The latter transformations are those given by Case.¹¹

¹² See, for example, reference 2.

¹³ Concerning the appearance of the complex conjugate, see reference 11.

VIII. THE POSITION OPERATOR

Foldy and Wouthuysen³ discussed the definition of position for fields with mass, and found that the operator “ $\mathbf{x}$ ” in the canonical representation (i.e., the ψ -representation; we call this operator $\mathbf{x}_c$), when transformed into the Dirac representation (the χ -representation; this “ $\mathbf{x}$ ” is denoted $\mathbf{x}_D$), turns out to be identical with the position operator of Newton and Wigner.¹⁴ The latter is of interest as the only position operator which can be defined on positive- and negative-energy states separately. Applying the transformation (20), we find¹⁵

$$\mathbf{x}_c\chi = \left\{ \mathbf{x}_D + i\beta\frac{\boldsymbol{\alpha}}{2\omega} - i\frac{\beta\boldsymbol{\alpha}\cdot\mathbf{p}\mathbf{p} - \boldsymbol{\sigma}\times\mathbf{p}\omega}{2\omega^2(\omega\pm|m|)} \right\} \chi. \quad (47)$$

It is easily verified that $\mathbf{x}_c$ has the following properties. (a) Any two components of $\mathbf{x}_c$ commute. (b) It commutes with the positive-energy projection operator

$$\Lambda^+ = (1/2\omega)(\omega\pm\beta|m| + \boldsymbol{\alpha}\cdot\mathbf{p}).$$

(c) The time derivative of $\mathbf{x}_c$ is

$$\frac{d\mathbf{x}_c}{dt} = i[H, \mathbf{x}_c] = \frac{\mathbf{p}}{E}. \quad (48)$$

In contrast, the time derivative of $\mathbf{x}_D$ has the well-known paradoxical property that the eigenvalues of $d\mathbf{x}_c/dt$ are ± 1 . Because of (b), $\mathbf{x}_c$ can be defined on the positive-energy states. Using the relation

$$\boldsymbol{\alpha}\cdot\mathbf{p}\Lambda^+ = (\omega\mp|m|\beta)\Lambda^+,$$

we find

$$\begin{aligned} \mathbf{x}_c\Lambda^+ &= \Lambda^+\mathbf{x}_c\Lambda^+ \\ &= \Lambda^+(1+\beta)\left(\frac{\omega}{\omega\pm|m|}\right)^{\frac{1}{2}}\mathbf{x}_D\left(\frac{\omega}{\omega\pm|m|}\right)^{\frac{1}{2}}\Lambda^+, \end{aligned} \quad (49)$$

which is of the form given by Newton and Wigner.¹⁶

For mass zero the transformation (25) gives

$$\mathbf{x}_c\chi = \left\{ \mathbf{x}_D - \frac{\boldsymbol{\sigma}\times\mathbf{p}}{2p^2} \pm \frac{1}{2}\frac{\mathbf{b}\times\mathbf{p}}{|\mathbf{p}|\pm\mathbf{b}\cdot\mathbf{p}} \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{p^2} \right\} \chi. \quad (50)$$

It may be verified that (50) satisfies (a), (b), and (c), with

$$\Lambda^+ = (1/2p^2)(|\mathbf{p}| \pm \mathbf{p}\cdot\boldsymbol{\sigma}).$$

A form similar to (49) is

$$\begin{aligned} \mathbf{x}_c\Lambda^+ &= \Lambda^+\mathbf{x}_c\Lambda^+ \\ &= \Lambda^+(1+\mathbf{b}\cdot\boldsymbol{\sigma}) \\ &\quad \times \left(\frac{|\mathbf{p}|}{|\mathbf{p}| \pm \mathbf{b}\cdot\mathbf{p}} \right) \mathbf{x}_D \left(\frac{|\mathbf{p}|}{|\mathbf{p}| \pm \mathbf{b}\cdot\mathbf{p}} \right) \Lambda^+. \end{aligned} \quad (51)$$

¹⁴ T. D. Newton and E. P. Wigner, *Revs. Modern Phys.* **21**, 400 (1949).

¹⁵ This expression differs slightly from that given by Foldy and Wouthuysen. Its correctness is checked by verifying the properties listed directly below.

¹⁶ The same trivial transformation which eliminates the last term of (23c) makes the agreement with Newton and Wigner complete.

If one takes the limit of vanishing mass in (49), one obtains another position operator for zero mass. The difference between this and (51) is that the latter is the position operator in the two-component theory, while (49) is defined on 4 spinors.

The appearance of $\mathbf{b}$ in (51) shows that this does not transform like a vector under rotations. When considering the two cases of zero and nonzero mass, we have let the time-axis play a distinguished role. Had we not done so, we would have obtained space-time operators $\mathbf{x}_C, t_C$ which in either case would have involved a "four-vector" b_μ . The only difference between the two cases is that when the mass is not zero b_μ can be given 3 vanishing components. In either case, covariance is explicit for the transformations of the little group.

Next we write down the position operator for the electromagnetic field $\mathbf{B}$ introduced in the previous section. (In the following section it is shown how electrodynamics can be expressed in terms of $\mathbf{B}$ rather than A_i .) If $\mathbf{x}_C$ is the "x" operator in the ψ -representation, we have, in dyadic notation:

$$x_C^i \mathbf{B} = \Lambda \cdot \left\{ \frac{|\mathbf{p}| \mathbf{b}}{|\mathbf{p} \times \mathbf{b}|} x_D^i \frac{|\mathbf{p}| \mathbf{b}}{|\mathbf{p} \times \mathbf{b}|} + \frac{\mathbf{b} \times \mathbf{p}}{|\mathbf{b} \times \mathbf{p}|} x_D^i \frac{\mathbf{b} \times \mathbf{p}}{|\mathbf{b} \times \mathbf{p}|} \right\} \cdot \Lambda, \quad (52)$$

where $i=1, 2, 3$, and

$$\Lambda = 1 - \frac{\mathbf{p} \mathbf{p}}{p^2}. \quad (53)$$

The properties (a), (b), noted for the position operators discussed above, have their analogs in this case, too. (a) Different components of $\mathbf{x}_C$ commute. (b) $\mathbf{x}_C$ commutes with Λ . This means that if $\mathbf{B}$ satisfies the Lorentz condition, then so does $x_C^i \mathbf{B}$. In (52) we have written only that part of $\mathbf{x}_C$ which is effective when $\mathbf{p} \cdot \mathbf{B}$ is zero. Actually $\mathbf{x}_C$ can be defined on a state of definite transverse polarization.

The reason for our finding a position operator for the electromagnetic field, whereas Newton and Wigner showed that under their assumptions no such operator is possible, is that (52) does not explicitly transform like a vector.¹⁷ As discussed above, this is not a fundamental difference between the cases of zero and nonzero rest mass. The noncovariance of $\mathbf{x}_C$ in the latter case means that locality with respect to $\mathbf{x}_C$, even for fields with mass, depends on the reference frame. The usefulness of $\mathbf{x}_C$ in that case is simply connected with the fact that a nonrelativistic particle interpretation exists.

¹⁷ A note from Professor Wigner on this point is gratefully acknowledged.

IX. SUMMARY AND DISCUSSION OF RESULTS

Following the prescription of Wigner, we have constructed explicit unitary, irreducible representations of the inhomogeneous Lorentz group. These representations have the dimensionality $2s+1$ or 1, for nonzero and zero mass, respectively, and are given by (6) and (11). A rather unconventional feature of these representations is that the "spin part" fails to commute with the "orbital part," as is evident because the "spin" depends on the momentum. This is inseparably connected with the unitarity of the representation. In fact, if the "spin part" of the transformation were to commute with the orbital part, it would be a representation of the Lorentz group all by itself. But it is well known that no finite-dimensional, unitary representations of the Lorentz group exist. In the case of zero mass, for example, we have one-component but not one-dimensional representations. Both (6) and (11) are in fact infinite-dimensional, because of the presence of the operators p_i and $\partial/\partial p_i$, which have continuous spectra. In the χ representation (i.e., the Dirac or Majorana representation) the transformations do break down into a sum of two commuting parts, but the "spin part" is not unitary. This gives the well known difficulty in the Majorana theory, that no local c -number invariant exist.

Our synthesis gives the representations of the Lorentz group on ψ or χ up to an equivalence based on the free-field wave equation. [See, for example, the remarks following (30).] This equivalence does, of course, break down when interactions are introduced. The criterion which seems to have been applied in selecting transformations under which the equations must be invariant in the presence of interaction, is that the spin and the orbital part of the transformation operators commute. The immediate consequences of this is the existence of local invariants, i.e., nonderivative couplings. Although this criterion applies successfully to the interactions of electrons with the Maxwell field, it is, perhaps, not quite certain that it applies to, say, the β -decay interaction. We should like to point out, moreover, that the above criterion is not *necessary* for electromagnetic interactions. To see this, we write down the Dirac equation

$$(\mathbf{p} - e\mathbf{A} + im)\chi = 0,$$

and make a gauge-transformation

$$\chi = \exp \left[\frac{e}{E} A^0 \right] \chi'. \quad (54)$$

Then, since

$$\mathbf{p}\chi = \exp \left[\frac{e}{E} A^0 \right] \left(\frac{e}{E} [\mathbf{p}, A^0] + \mathbf{p} \right) \chi',$$

we have

$$(\mathbf{p} - e\mathbf{B} + im)\chi' = 0, \quad (55)$$

where $\mathbf{B}$ is the field introduced in Sec. VI. It has the

properties:

$$B^0=0, \quad \mathbf{p} \cdot \mathbf{B}=0, \quad (56)$$

$$D[1+\boldsymbol{\theta} \cdot \mathbf{R}]\mathbf{B}(\mathbf{p})=(1+\boldsymbol{\theta} \times \mathbf{p})\mathbf{B}(\mathbf{p}-\boldsymbol{\theta} \times \mathbf{p}),$$

$$D[1+\boldsymbol{\theta} \cdot \mathbf{L}]\mathbf{B}(\mathbf{p})=\left(1-\frac{\mathbf{p}}{E} \cdot \boldsymbol{\theta}\right)\mathbf{B}(\mathbf{p}-E\boldsymbol{\theta}), \quad (57)$$

$$D[1+\boldsymbol{\theta} \cdot \mathbf{R}]B^0(\mathbf{p})=B^0(\mathbf{p}-\boldsymbol{\theta} \times \mathbf{p}),$$

$$D[1+\boldsymbol{\theta} \cdot \mathbf{L}]B^0(\mathbf{p})=B^0(\mathbf{p}-E\boldsymbol{\theta}).$$

Gauge-invariance is here explicit through the *invariant* condition $B^0=0$. However, to prove that the theory is really gauge-invariant, we must calculate the transformation properties of χ' , and see if these depend on $\mathbf{B}$ only. We find

$$\begin{aligned} D[1+\boldsymbol{\theta} \cdot \mathbf{R}]\chi'(\mathbf{p}) &= (1-\tfrac{1}{2}i\boldsymbol{\theta} \cdot \boldsymbol{\sigma})\chi'(\mathbf{p}-\boldsymbol{\theta} \times \mathbf{p}), \\ D[1+\boldsymbol{\theta} \cdot \mathbf{L}]\chi'(\mathbf{p}) &= [1+\tfrac{1}{2}\boldsymbol{\theta} \cdot \boldsymbol{\alpha} - (e/E)\boldsymbol{\theta} \cdot \mathbf{B}]\chi'(\mathbf{p}-E\boldsymbol{\theta}). \end{aligned} \quad (58)$$

The theory based on (55), (56) and the transformations (57), (58) is explicitly gauge-invariant, and completely equivalent to the usual formulation of the Dirac-Maxwell interaction. (The condition of gauge-invariance is here replaced by a new condition of relativistic invariance, which compels one to couple the $\mathbf{B}$ field in the particular combinations $\mathbf{p}-e\mathbf{B}$. The work of Capps,¹⁸ for example, can be made to apply to the new formulation simply by exchanging the expression "gauge-invariance" by "relativistic invariance.") The main point that we want to make here, is that the new formulation is very tractable and yet does not satisfy the criterion that the spin commute with the orbital angular momentum. [It may be, however, that the appearance of a nonlocal term in (57) is inseparably linked to the nonlinearity of the representation (58).]

Another question which is of interest in this connection, and which may turn out to give a clue concerning the ambiguity just referred to, is the following. It is well known that the c -number theory of interacting Dirac and Maxwell fields is equivalent to the Pauli

equation with an infinite series of corrective terms. The relativistic covariance of the latter is known only as a consequence of the covariance of the former. However, since the two-component Pauli spinors do allow a representation of the Lorentz group, it should be possible to demonstrate the covariance of the Pauli equation directly. Put another way, it should be possible to calculate the corrections to the Pauli equation without making appeal to the Dirac equation. (We expect, of course, ambiguities like that which corresponds to the possibility of adding a Pauli term to the Dirac equation). *A priori* we do not know, however, which if any of the several equivalent (equivalent, that is, in the absence of interaction) representations to use on the two-component spinor. The minimum requirement is that the commutation relations be satisfied identically, i.e., without reference to the Klein-Gordon equation. Such a representation is obtained from (16) by the substitution of $|m|$ by $+(E^2-p^2)^{1/2}$. As the criterion is necessarily different from that in the Dirac representation, an answer to this question might provide a clue to a more general criterion. Furthermore, if the proper representation can be found, and applied successfully to the calculation of corrective terms to the Pauli equation, the method might be applied to the calculation of recoil corrections to scattering amplitudes.

In Sec. IV we found that the ambiguity in the sign of the mass of massive particles, and of the sign of the spin (helicity) of mass-zero particles, have exactly the same origin. We expect that neither sign is observable, there is no indication in our analysis that one sign is more physical than the other. The question of mass-reversal invariance has been studied by several authors,¹⁹ and has been shown to lead to parity nonconservation in Fermi interactions. On the other hand, it is easy to see that, within the same framework, "helicity reversal invariance" leads to parity conservation. Hence we have the very perplexing situation that unobservability of the sign of the mass, and of the helicity of neutrinos, are incompatible requirements. One might say that a negative result of the celebrated parity experiments would have been no less curious than the actual one.

¹⁸ R. H. Capps, Phys. Rev. **99**, 926 (1955).

¹⁹ J. J. Sakurai, Nuovo cimento **7**, 649 (1958).