Accelerated self-constricted electron streams in plasma

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The mechanism is described of radial oscillations in a neutralized cylindrical electron stream in an accelerating electric field.

The analysis is based on the two-fluid model of plasma. Analytical expressions for small amplitude oscillations and numerical solutions for large amplitudes are derived. It is found, when electron-positive ion collisions are taken into account, that for dense streams in low electric fields the radial oscillations (pinch oscillations) can destroy the streaming character of the electron flow and thus prevent its acceleration.

INTRODUCTION

The phenomena exhibited by self-focused, magnetically confined streams are of interest not only to a mathematical physicist, but increasingly so in astrophysics (Alfvén 1950), in thermonuclear research (Pease 1957) and in the development of new guiding fields for particle accelerators (Budker 1956).

FIGURE 1. Configuration in the velocity space for $v_{\text{drift}} > \sqrt{(kT/m)}$.

The simplest and best understood configuration of a self-focused plasma stream is the stationary state of a one-dimensional current flow (Bennett 1934, 1955). Before such a state can be physically realized the current flow may exhibit many transient phenomena.

In the first phase the plasma particles are accelerated, in the next the plasma stream constricts owing to the growing magnetic forces. This constriction may excite a series of radial oscillations in the stream. Owing to these and other types of oscillations the stream may disperse or degenerate into turbulent motion before a stationary state can be attained. Analyses of some of these transients have been published (Leontovich 1956; Allen 1957) based on the assumption that the drift velocities of the particles are small compared with their mean thermal speeds.

This assumption will be dropped in this paper which will concentrate on problems in which the streaming motion predominates over the thermal motion. Only in such a plasma may one hope to obtain pinch diameters of the order of a fraction of a millimetre. The situation can be represented by a diagram in the velocity space shown in figure 1. Such a state of affairs is physically possible only if collisions between particles belonging to different streams are not of primary importance to the processes studied.

Our analysis will be concerned with a rotationally symmetrical plasma to which an axial electric field is applied and which is contained within a conducting cylindrical tube of radius R (figure 2).

Our main interest will be the contraction and radial oscillations of the plasma stream. In the second part of this paper we shall study the effect of collisions on the lateral dispersion of the contracted plasma. In the third part a transition to the

FIGURE 2

steady state will be described and we shall derive some criteria on the acceleration of the electron gas component of an ordinary plasma to relativistic drift velocities. The instability of the stream to kinking will not be considered (cf. Kruskal & Schwarzschild 1954).

1. EQUATIONS OF MOTION OF A PLASMA COLUMN

We shall use the two-fluid model of plasma and restrict ourselves first to the case of fully ionized hydrogen of such low density that electron-proton collisions may be neglected. This assumption will be justified *a posteriori*. The equations of motion can be written for the electron gas and the proton gas separately

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \text{grad } \mathbf{v} = -\frac{e}{mc}(c\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) - \frac{1}{n_e m} \text{grad } p_e, \quad (1)$$

$$\frac{\partial \mathbf{w}}{\partial t} + \mathbf{w} \cdot \text{grad } \mathbf{w} = \frac{e}{Mc}(c\mathbf{E} + \mathbf{w} \wedge \mathbf{B}) - \frac{1}{n_p M} \text{grad } p_p. \quad (2)$$

These are non-relativistic equations; relativistic corrections can be included in the simplest way by writing the equations in the rest system of the electrons. This will not be done here as it is not required in our problem.

The vector equations (1) and (2) can be written in cylindrical co-ordinates and for a current flow parallel to the z -axis as follows:

$$\frac{\partial v_z}{\partial t} + \frac{\partial v_z}{\partial r} v_r = -\frac{e}{mc} (cE_0 + cE' + v_r B), \quad (1a)$$

$$\frac{\partial v_r}{\partial t} + \frac{\partial v_r}{\partial r} v_r = -\frac{e}{mc} (cE_r - v_z B) - \frac{1}{n_e m} \frac{\partial p_e}{\partial r}, \quad (1b)$$

$$\frac{\partial w_z}{\partial t} + \frac{\partial w_z}{\partial r} w_r = \frac{e}{Mc} (cE_0 + cE' + w_r B), \quad (2a)$$

$$\frac{\partial w_r}{\partial t} + \frac{\partial w_r}{\partial r} w_r = \frac{e}{Mc} (cE_r - w_z B) - \frac{1}{m_p M} \frac{\partial p_p}{\partial r}, \quad (2b)$$

where v_z , v_r and w_z , w_r are the z - and r -components of $\mathbf{v}$, $\mathbf{w}$. E_0 is the applied electric field, E' is the self-induced electric field generated by $\partial B/\partial t$ and E_r is the electric field due to a radial electron-proton space-charge separation.

We shall now invoke the assumption of charge neutrality of plasma and let

$$v_r = w_r = w, \quad n_e = n_p = n.$$

This assumption results in neglecting radial oscillations of the electron gas with respect to the protons. This is justifiable as the processes of plasma contraction are slow compared with electron oscillations.

The radial electric field can now be eliminated from equations (1b) and (2b).

Also, owing to

$$\frac{m}{M} \ll 1 \quad \text{and} \quad \frac{w_z}{v_z} \ll 1,$$

equation (2a) is of little importance. It thus follows that only two equations need be used to describe the motion of the volume element of our plasma column. These are (putting $v_z = v$):

$$\frac{\partial v}{\partial t} + w \frac{\partial v}{\partial r} = -\frac{eE_0}{m} - \frac{eE'}{m} - \frac{e}{m} \frac{w}{c} B, \quad (3)$$

$$\frac{\partial w}{\partial t} + w \frac{\partial w}{\partial r} = \frac{e}{Mc} v B - \frac{1}{nM} \frac{\partial}{\partial r} (p_e + p_p). \quad (4)$$

The fields E' and B can be expressed by means of a vector potential A having an axial component only

$$E' = -\frac{1}{c} \frac{\partial A}{\partial t}, \quad (5)$$

$$B = -\frac{\partial A}{\partial r}. \quad (6)$$

From Maxwell's equations one obtains the relation between A and the current density

$$-\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial A}{\partial r} \right) + \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = \frac{4\pi}{c} i. \quad (7)$$

To these must be added finally the equation of continuity

$$\frac{\partial n}{\partial t} + \text{div } n\mathbf{w} = 0. \quad (8)$$

Substituting (5) and (6) into (3) and (4), expanding (8) and putting $i = -nev$ we get the following system of equations:

$$\frac{\partial v}{\partial t} + w \frac{\partial v}{\partial r} = -\frac{e}{m} E_0 + \frac{e}{mc} \frac{\partial A}{\partial t} + \frac{e}{mc} w \frac{\partial A}{\partial r}, \quad (9)$$

$$\frac{\partial w}{\partial t} + w \frac{\partial w}{\partial r} = -\frac{e}{Mc} v \frac{\partial A}{\partial r} - \frac{1}{nM} \frac{\partial}{\partial r} (p_e + p_p), \quad (10)$$

$$-\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial A}{\partial r} \right) + \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = -\frac{4\pi}{c} nev, \quad (11)$$

$$\frac{\partial n}{\partial t} + w \frac{\partial n}{\partial r} + \frac{n}{r} \frac{\partial}{\partial r} (rw) = 0. \quad (12)$$

At this point the connexion of the pressures p_e and p_p with n , r and t must be introduced. Assuming Maxwellian distributions of energies in the electron gas and proton gas with temperatures T_e and T_p ,†

$$p_e + p_p = nk(T_e + T_p) = nkT. \quad (13)$$

In general T_e and T_p as well as n may be functions of the radius r . However, in most cases of interest the conduction and convection of heat in the cross-section of the plasma column is sufficiently intense to assume that

$$\frac{\partial T_e}{\partial r} \simeq \frac{\partial T_p}{\partial r} \simeq 0. \quad (13a)$$

It will be shown later (§ 3) that for $v/c \ll 1$ the radiation losses from a self-confined plasma and the collision heating are small compared with the heat generated by pinch compression and, therefore, one may regard the contraction of the plasma column as an adiabatic compression. The question of the number μ of degrees of freedom into which this compression can feed energy is studied in appendix 1 and it is shown that for a not too short compression cycle $\mu = 3$. Using these considerations it can be shown that T obeys the following differential equation

$$\frac{\partial T}{\partial t} = \frac{4\pi}{3} \frac{T}{N} \int_0^R n \frac{\partial}{\partial r} (rw) dr, \quad (14)$$

where N is the number of electrons per cm length of the plasma column.

In principle, using equations (9) to (12) and (14), one can determine the dependent variables v , w , n , A and T as functions of the independent variables r and t . Before

† Provided the electron stream consists of electrons for which $v \gg \sqrt{(2kT_e/m)}$ the $e-e$ and $p-p$ collisions will be much more frequent than the $e-p$ collisions. As the $e-e$ and $p-p$ events tend to generate separate Maxwellian distributions, whereas $e-p$ events tend to destroy them, the prevalence of the former events provides for nearly Maxwellian distributions in both the proton fluid and the electron fluid. In the following we shall assume $T_e \gg T_p$.

attempting to derive such a general solution we shall find the solution corresponding to a steady state, i.e. a case for which

$$\frac{\partial}{\partial t} = 0, \quad E_0 = 0, \quad w = 0. \quad (15)$$

The only equations that are not identically satisfied are equations (10) and (11), which become

$$\frac{\partial A}{\partial r} = \frac{kT}{(ev/c)} \frac{-\partial n/\partial r}{n}, \quad (16)$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial A}{\partial r} \right) = \frac{4\pi}{c} nev. \quad (17)$$

The solution of these equations corresponding to $v = \text{const.}$ is

$$n = \frac{n_0}{\{1 + (r^2/r_0^2)\}^2}, \quad (18)$$

where

$$\frac{e^2}{mc^2} \pi r_0^2 n_0 = \frac{kT}{\frac{1}{2}mv^2}. \quad (19)$$

Equation (19) can be also written as

$$\frac{e^2}{mc^2} N = \nu = \frac{kT}{\frac{1}{2}mv^2}, \quad (20)$$

where N is the electron line density.

The plasma column is evidently a very sharply defined structure, one-half of its mass being concentrated within the cylinder of radius r_0 (and 90 % of its mass located within $r = 3r_0$).

This result and equation (19) can also be obtained by considering the Boltzmann distributions of electrons and protons in their respective rest systems, due to each other's electromagnetic fields (Bennett 1934).

In order to solve equations (9) to (12) and (14), let us now assume that, in spite of contraction processes, the plasma column does not depart appreciably from the sharply defined cylindrical structure obtained from the steady state analysis. This is so if the contraction is slow compared with the energy relaxation time t_D in the electron rest-system as then the departure from thermodynamic equilibrium is negligible (appendix 1). If this condition is not satisfied the density distribution will not remain functionally the same as that of equation (18), but even then it is plausible to define some plasma boundary equivalent to the r_0 -cylinder as the forces due to the self-magnetic field always tend to preserve a sharp density distribution.

Let us then describe the motion of a small volume element located at $r = r_0$. The equations (9) to (11) become

$$\frac{\partial v_b}{\partial t} = -\frac{e}{m} E_0 + \frac{e}{mc} \frac{\partial A}{\partial t} + \frac{e}{mc} w_b \frac{\partial A}{\partial r}, \quad (21)$$

$$\frac{\partial w_b}{\partial t} = -\frac{e}{Mc} v_b \frac{\partial A}{\partial r} + \frac{2k(T_e + T_p)}{Mr_0}, \quad (22)$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial A}{\partial r} \right) - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = \frac{4\pi}{c} nev_b, \quad (23)$$

where the subscript b denotes the quantities at the $r = r_0$ boundary.

This description is appropriate if the plasma reflexion skin depth δ is larger than the radius r_0 ,† and one may also retain the assumption that $v = v_b$ is not a function of radius.

If one assumes the Bennett's density distribution, the equation of continuity is satisfied only if

$$w = (r/r_0) w_b. \quad (24)$$

Evidently this does not correspond to a physically realizable model and one expects that for $r > r_0$ the Bennett distribution breaks down. This does not represent an appreciable error owing to the sharp fall in density in the region of $r > r_0$.

Equation (14) can be simplified if one makes an approximation, which is consistent with the previous assumptions, that

$$2\pi \int_0^R n \frac{\partial}{\partial r} (rw) dr = (4\pi w_b/r_0) \int_0^R nr dr = 2w_b N/r_0 = 2N \frac{\dot{r}_0}{r_0}.$$

Then
$$\frac{T}{T_1} = \left(\frac{r_{10}}{r_0} \right)^{\frac{4}{3}}, \quad (25)$$

where r_{10} is the initial value of r_0 and $T_1 = T_{e1} + T_{p1}$ is the initial temperature of the plasma.

The vector potential A follows from equations (16) and (17) and it is

$$A = -\frac{e}{c} N v \ln \frac{1 + R^2/r_0^2}{1 + r^2/r_0^2}. \quad (26)$$

From this one may derive both $\partial A_b/\partial t$ and $\partial A_b/\partial r$.

These are

$$\frac{\partial A_b}{\partial t} = -\frac{e}{c} N v \left\{ \frac{w_b}{r_0} \frac{1 - R^2/r_0^2}{1 + R^2/r_0^2} + \frac{1}{v} \frac{dv}{dt} \left[\ln \left(1 + \frac{R^2}{r_0^2} \right) - \ln 2 \right] \right\}, \quad (27)$$

$$\frac{\partial A_b}{\partial r} = -\frac{e}{c} N v \frac{1}{r_0}. \quad (28)$$

The vector potential A derived in this way does not satisfy equation (23), because of the term $(1/c^2) \partial^2 A_b/\partial t^2$ which has not been taken into account in the steady state calculations. However, in our case this term can be safely neglected. Then the potential A satisfies equation (23), as equation (23) is now the same as equation (17).

When one substitutes equations (25), (27), (28) into (21) and (22) one obtains (writing now d/dt for $\partial/\partial t$ and r for r_0)

$$\frac{dv}{dt} \left\{ 1 + \nu \ln \frac{1}{2} \left(1 + \frac{R^2}{r^2} \right) \right\} = -\frac{e}{m} E_0 + \nu \frac{vw}{r} \frac{2R^2/r}{1 + R^2/r^2}, \quad (29)$$

$$\frac{dw}{dt} = -\frac{m}{M} \frac{\nu v^2}{r} + \frac{2kT_1 r_{10}^{\frac{4}{3}}}{M} r^{-\frac{7}{3}}. \quad (30)$$

For large R/r_0 one has

$$\nu \ln \frac{1}{2} \left(1 + \frac{R^2}{r_0^2} \right) \sim 2\nu \ln \sqrt{\frac{1}{2}} \left(\frac{R}{r_0} \right) = 2\nu \left(\ln \frac{R}{r_0} - 0.35 \right). \quad (31)$$

† As $\delta = c/\omega_p = 5.35 \times 10^5 n^{-\frac{1}{2}}$ one obtains from the condition $\delta/r_0 > 1$ a criterion $\nu \ll 2.4$.

Let us define a mass $m^* = m\{1 + 2\nu[\ln(R/r_0) - 0.35]\}$ which is the effective longitudinal mass of the electrons in the stream.

Provided $\nu \leq 1$ and the changes in r_0 are not larger than two orders of magnitude one may assume that

$$\frac{m^*}{m} \sim \text{const.} \sim \left\{1 + \nu \left(\ln \frac{R}{r_{0m}} - 0.35\right)\right\} = \eta^{-1}, \quad (32)$$

where r_{0m} is a suitable mean value for r_0 .

The final form of equations (29) and (30) is therefore

$$\frac{dv}{dt} = -\frac{e}{m^*} E_0 + 2\eta\nu \frac{vw}{r}, \quad (33)$$

$$\frac{dw}{dt} = -\frac{m}{M} \nu \frac{v^2}{r} + \alpha c^2 r^{-\frac{7}{3}}, \quad (34)$$

where $\alpha = 2kT_1 r_{10}^{\frac{4}{3}} / Mc^2$.

Small-amplitude solution

Equations (33) and (34) can be solved analytically for small oscillations of the quantities v , w and r . Let us put

$$v = v_0 + v_1; \quad w = w_0 + w_1; \quad r = r_0 + r_1; \quad E_0 = 0.$$

Neglecting all second-order quantities, equations (33) and (34) become

$$\dot{v}_1 = 2\eta\nu v_0 \frac{\dot{r}_1}{r_0}, \quad (35)$$

$$\ddot{r}_1 = -\frac{m}{M} \nu \frac{v_0^2}{r_0} \left(\frac{2v_1}{v_0} - \frac{r_1}{r_0}\right) - \frac{7}{3} \frac{\alpha c^2 r_1}{r_0^{\frac{10}{3}}}. \quad (36)$$

If the quantities v_1 and r_1 change harmonically with time, i.e. as $e^{i\omega t}$, one obtains from equations (35) and (36) a characteristic equation for ω

$$\omega = \left\{ \frac{m}{M} \left(\frac{v_0}{r_0} \right)^2 (4\eta\nu - 1) \nu + \frac{7}{3} \frac{\alpha c^2}{r_0^{\frac{10}{3}}} \right\}^{\frac{1}{2}}. \quad (37)$$

From the original equation (34) one gets for the steady state

$$\alpha c^2 r_0^{-\frac{10}{3}} = \frac{m}{M} \nu \frac{v_0^2}{r_0^2}. \quad (38)$$

Substituting this into equation (37) one obtains for the frequency of small amplitude oscillations of the plasma channel

$$\omega = 2 \frac{v_0}{r_0} \nu \sqrt{\left\{ \frac{m}{M} \left(\eta + \frac{4}{3} \right) \right\}}. \quad (39)$$

Another interesting result of the small amplitude analysis concerns the change in the amplitude of r_1 as a result of changing v_0 . Let us consider equation (36).

In view of equation (39) one may write (36) as

$$\ddot{r}_1 = 4 \left(\frac{v_0^2}{r_0^2} \nu^2 \frac{m}{M} \left(\eta + \frac{4}{3} \right) \right) r_1. \quad (40)$$

This is the equation of a harmonic oscillator. The energy of this oscillator is

$$W = \frac{1}{2} M r_1^2 \omega^2. \quad (41)$$

If v_0 and r_0 are changing slowly compared with the plasma column contractions, i.e. if

$$\frac{\partial v_0}{\partial t} \ll v_1 \omega, \quad \frac{\partial r_0}{\partial t} \ll r_1 \omega,$$

then one may apply the adiabatic theorem

$$W/\omega = \text{invariant}. \quad (42)$$

FIGURE 3. $E = 3 \text{ V/cm}$; $r_1 = 3 \text{ cm}$; $T_1 = 2 \times 10^4 \text{ °K}$; $\eta = 3.5$.

Using (41), (42) and (39) one gets

$$r_1^2 = \text{const. } r_0/v_0. \quad (43)$$

The result of equation (43) is only a qualitative one because in the basic equations of plasma motion we have neglected the effect of viscosity. It can be shown that the viscous forces cause a stronger damping of the pinch oscillations (appendix 2) than that predicted by equation (43).

Numerical solution

It is evident that the small amplitude analysis cannot be applied to the development of the initial contractions as these are expected to involve changes of radius r_0 of at least one order of magnitude.

In order to gain insight into the processes leading to the development of the first pinch, equations (33) and (34) were solved numerically for values of E_0 , v , r_{10} and T_1 which are experimentally feasible. The functions $r(t)$ and $v(t)$ thus obtained are shown in figure 3.

The results show an important feature of the fast pinch, i.e. that the electron temperature reached when

$$r_0 = r_{\min.}$$

is much larger than that which could be attained in a quasi-steady pinch (i.e. where the inertia could be neglected).

2. DISPERSION OF THE ELECTRON STREAM DUE TO COLLISIONS

In the previous section the motion of the electrons was studied without taking the electron-proton collisions into account. However, there are situations during the contraction of a plasma column in which these collisions must not be neglected; in fact, the dispersion of the electron stream due to electron-proton collisions could lead to the loss of most of the electrons in the stream.

An electron passing through a proton-gas of density n in the direction z and with a velocity v will acquire a velocity $\Delta v_{\perp}$ at right angles to z due to its collisions with the protons. The scattering is (Spitzer 1956)

$$\langle (\Delta v_{\perp})^2 \rangle = \frac{8\pi e^4}{m^2 v} n \ln \Lambda \text{ (cm}^2/\text{s}^3\text{)}. \quad (44)$$

The rise in transversal temperature for an ensemble of electrons whose mean velocity is v is therefore

$$\dot{T}_e = \frac{\frac{1}{2}m\langle (\Delta v_{\perp})^2 \rangle}{k} = \frac{4\pi e^4}{m k v} n \ln \Lambda \sim \frac{1}{300} \frac{n}{\beta} \text{ (}^{\circ}\text{K/s)}, \quad (45)$$

where $\beta = v/c$.

Both (44) and (45) are valid for $v > \Delta v_{\perp}$ and $v > \sqrt{(kT_p/M)}$.

In absence of any other forces in the z -direction one may also write an equation for v . This is

$$m\dot{v} = -F, \quad (46)$$

where F is the viscous force. The expression for F follows from the conservation of energy

$$\frac{1}{2}m\langle (\Delta v_{\perp})^2 \rangle = mv\langle \Delta v \rangle. \quad (47)$$

Therefore
$$\langle \Delta v \rangle = \frac{4\pi e^4 n \ln \Lambda}{m^2 v^2}, \quad F = m\langle \Delta v \rangle. \quad (48)$$

This is the well-known 'coefficient of dynamical friction' (Chandrasekhar 1943).

The increase in temperature of our electron stream, due to collisions with the protons, is, from equation (45),

$$T = \frac{1}{300} \int_0^t \frac{n}{\beta} dt \text{ (}^{\circ}\text{K)}. \quad (49)$$

Evidently the greatest contribution comes from:

- (a) The initial stage of acceleration where β is very small.
- (b) The stages during which the plasma column is fully contracted and n is high and β small; in fact one would expect that the first full contraction will be most important in this respect.

It is beyond the scope of our analysis to treat the case (a). This is a problem belonging to the problems associated with the decoupling of the electron gas distribution from the positive ions in the velocity space (Gibson 1957). In order to accomplish this decoupling one must apply to the plasma an electric field

$$E_a > 4.5 \times 10^{-8} \frac{n}{T_e} \text{ (V/cm)}. \quad (50)$$

It can be shown that provided the acceleration field $E_0 > E_a$ the loss of particles from the plasma column due to the process (a) is negligible.

In order to appreciate the serious nature of the case (b) let us evaluate the integral (49), for constant n and β . One gets

$$T = \frac{Nt}{300\pi r^2 \beta} \text{ (}^\circ\text{K)}. \quad (51)$$

Let us compare this with the temperature T_c at which the plasma stream can still be confined if the stream velocity is $c\beta$. This is

$$T_c = \frac{1}{2} \frac{N_e^2 \beta^2}{k} \text{ (}^\circ\text{K)}. \quad (52)$$

Let us calculate the time in which $T = T_c$. Then the characteristic radius of the stream r_0 would have to increase by $(2)^{\frac{1}{2}}$ in order that the temperature of the electron gas be kept equal to T_c . This time is

$$t_c \simeq 0.8r^2\beta^3 \text{ (s)}. \quad (53)$$

From the graphs of $r(t)$ and $z(t)$ one observes (figure 3) that when

$$r = 0.1 \text{ cm}, \quad \beta = 10^{-2} \quad \text{for which} \quad t_c \simeq 0.008 \text{ } \mu\text{s}.$$

The duration of a characteristic contraction can be easily longer than $0.01 \text{ } \mu\text{s}$. It thus follows that the heating of the electron gas due to electron-proton collisions can be very serious even during one single contraction.

The effect of the friction force $F = m\langle\Delta v\rangle$ on the stream motion of the electrons can be represented by an additional term† on the right of equation (33).

Thus we get a new set of equations

$$\dot{v} = -\frac{e\eta}{m} E_0 + 2\nu\eta \frac{vv}{r} - 9 \frac{e^2 c^2}{m} \frac{\eta\nu}{v^2 r^2}, \quad (54)$$

$$\dot{w} = -\frac{1}{2} \frac{m}{M} \frac{\nu v^2}{r} + \frac{k}{M} \frac{T}{r}. \quad (55)$$

† The friction force is evaluated at $r = r_0$, whereas the collision heating is averaged over the cross-section of the stream (see later, equation (58a)).

However, now the temperature variation is not simply that corresponding to adiabatic compression but includes also the Joule heat input. Using equations (25) and (45), one obtains

$$\dot{T} = -\frac{4}{3}T\frac{\dot{r}}{r} + 6\frac{e^2c^2}{k}\frac{\nu}{r^2v}. \quad (55a)$$

Numerical solutions of equations (54), (55) and (55a) were obtained for some typical values of E_0 , ν , r and T_1 . The functions $r(t)$ and $v(t)$ are plotted in figure 4. The tendency of the stream to blow up radially is especially evident with curves representing streams of large ν and small E_0 .

FIGURE 4. (a) $E_0 = 0.01$ ab. V/cm, $\nu = 1, \frac{1}{2}, \frac{1}{5}$; (b) $E_0 = 0.03$ ab. V/cm, $\nu = 1, \frac{1}{2}, \frac{1}{5}$.

3. THE ESTABLISHMENT OF A STEADY STATE

If the thermal energy of the electron gas could not be drained away, the electron temperature would rise owing to electron-proton collisions. As the rise in the velocity of the electron streams is limited due to relativistic effects, one could not hope, in such a case, to keep the stream confined.

In the type of toroidal plasma used in thermo-nuclear work the electron gas is cooled effectively by bremsstrahlen (Pease 1957). This cooling is not operative in the case where the directed drift velocity is much larger than the thermal velocity, i.e. in our type of plasma.

Here, the only radiation that can cool the electron stream is the cyclotron radiation. We shall now calculate under what conditions this cooling is large enough to dissipate the collision heating of the electron stream.

The heat input per electron is

$$\frac{1}{2}m\langle\Delta v_{\perp}^2\rangle = \frac{4\pi e^4}{mv_0} n \ln \Lambda \text{ (ergs/s)}. \quad (56)$$

The input per cm length of the column is

$$W_J = \frac{4\pi e^4}{mv_0} \ln \Lambda 2\pi n_0^2 \int_0^\infty \frac{r dr}{\{1 + (r^2/r_0^2)\}^4},$$

which is

$$W_J = \frac{2\pi}{3} \frac{e^4}{mv_0} \ln \Lambda \frac{N^2}{2\pi r_0^2} \text{ (ergs/cm s)} \quad (57)$$

and the average input per electron is

$$W_e = \frac{W_J}{N} = \frac{\ln \Lambda}{3} e^2 c \frac{\nu}{\beta_0 r_0^2} \text{ (ergs/s, electron)}. \quad (58)$$

In our case $\ln \Lambda \sim 18$ and therefore

$$W_{e1} = 6e^2 c \frac{\nu}{\beta_0 r_0^2} \text{ (ergs/s, electron)}. \quad (58a)$$

The power radiated by a particle in the self-magnetic field is

$$W = \frac{2}{3} (e^2/c^3) (\dot{v}_T)^2, \quad (59)$$

where

$$\dot{v}_T = (e/mc) v_T \wedge H \quad \text{and} \quad \frac{1}{2} m v_T^2 = k T_e. \quad (60)$$

Radiation from the plasma column per cm length is then (using also equation (20))

$$W_1 = 2\pi \frac{2}{3} \frac{e^4}{m^2 c^5} v_T^2 \int_0^\infty n H^2 r dr = \frac{1}{9} \frac{e^2 c}{r_0^2} \beta_0^4 \nu^3 N \text{ (ergs)}. \quad (61)$$

The average radiation output per electron is

$$W_{e2} = \frac{1}{9} e^2 c \frac{\nu^3}{r_0^2} \beta_0^4 \text{ (ergs/s, electron)}. \quad (62)$$

Comparing W_{e1} and W_{e2} it follows that a power balance cannot be established at non-relativistic velocities.

The validity of the formulae for W_{e1} and W_{e2} can be extended to the relativistic region. The corresponding expressions in the electron rest system are

$$W_{e1} = 6e^2 c \frac{\nu \gamma}{\beta_0 r_0^2}, \quad W_{e2} = \frac{1}{9} e^2 c \frac{\nu^3}{r_0^2} \beta_0^4 \gamma^3 \quad (63)$$

and the power balance equation $W_{e1} = W_{e2}$ gives

$$\gamma \nu = 7.35. \quad (64)$$

It is thus evident that one must accelerate the electron gas up to an energy of several MeV before an efficient radiation cooling sets in.† During this acceleration one must not heat the electron gas to a temperature at which the plasma column

† This cooling may be even less efficient than is suggested by equation (64) owing to internal reflexion of some of the cyclotron radiation within the plasma.

would expand so as to come into contact with the walls of the vessel. It has been pointed out that this danger is present mainly owing to the pinch oscillations of the column. If these oscillations can be considered to be damped out it can be shown that the electron stream may be accelerated to relativistic velocities, without expansion, by relatively small electric fields.

Let us, therefore, find a field $E_0(t)$ which would maintain a steady pinch.

We demand therefore

$$\dot{r} = 0.$$

If one includes an average friction and average temperature rise into equations (33) and (34) one obtains†

$$\dot{v} = \frac{e}{m^*} E - 36\eta\nu \frac{e^2 c^2}{m} r_0^{-2} v^{-2}, \quad (65)$$

$$\dot{T} = 6 \frac{e^2 c^2}{k} \nu r_0^{-2} v^{-1}, \quad (66)$$

$$T = \frac{m}{2k} \nu v^2. \quad (67)$$

Differentiating equation (67) and comparing this with (66) one gets

$$\dot{v} v^2 = 6(e^2 c^2 / m r_0^2). \quad (68)$$

Substituting this into equation (65) one finds that E varies as

$$E = 6(ec^2 / \eta r_0^2) (1 + 6\eta\nu) v^{-2}. \quad (69)$$

From equation (68) one gets

$$v = \left[v_0^3 + 18 \frac{e^2 c^2}{m r_0^2} t \right]^{\frac{1}{3}} \quad (70)$$

and the corresponding E becomes

$$E = 6 \frac{e}{\eta r_0^2 \beta_0^2} (1 + 6\eta\nu) \left(1 + 18 \frac{e^2 t}{m c r_0^2 \beta_0^3} \right)^{-\frac{2}{3}} \text{ (elst. V/cm)}. \quad (71)$$

The energy of electrons in the stream is then given by

$$W = \int_0^t 300 E_0(t) v \, dt \text{ (eV)}. \quad (72)$$

The order of magnitude of time in which a relativistic stream is obtained follows directly from equation (70). Thus if one requires that finally

$$v/c \sim 1 \quad \text{and assumes that} \quad v \gg v_0$$

one obtains

$$t = 7.2 r_0^2 \text{ (s)}. \quad (73)$$

It is obvious that in order to avoid the long acceleration time one should try to accelerate a well constricted stream.

† This analysis is not applicable to relativistic streams.

However, such a contracted beam could be obtained only if

(a) one applies a field E_0 larger than the 'smooth field' E of equation (71);

(b) one is able to damp the pinch oscillations arising from $E_0 > E$.

The danger of pinch oscillations may be considerably reduced if E_0 is not much larger than E . The stream radius r_0 collapses slowly and the transverse kinetic energy per electron is kept much smaller than $\frac{1}{2}mv^2v$.

The application of $E_0 > E_s$ does not only generate a stream of small radius but also causes a steeper rise in β_0 than that predicted by equation (70). From some numerical examples it can be inferred (figure 3) that when the 'smooth acceleration' takes over the corresponding β_0 can be as large as $\frac{1}{3}$.

FIGURE 5. A recommended time dependence of E_0 .

CONCLUSIONS

According to the result of § 2, it is advisable to accelerate the electron gas initially by electric fields of the order of at least several V/cm so that it passes through the first contractions quickly and with as large velocity as possible.

It further appears that one could try to accelerate the electron stream in such a way as not to generate contractions. If such smooth stream acceleration is possible then one can hope to raise the electron energy to several MeV without the loss of plasma particles.

Once the stream is accelerated to several MeV the cyclotron radiation can dissipate the collision heat and the stream converges to the structure described by Budker.

These considerations suggest that the accelerating field E_0 should have a time-dependence as shown in figure 5.

Thus E_0 should be of the order of 10 V/cm during the first few contractions after which it should decrease as given by equation (71). The total acceleration time may then be of the order of 0.1 ms.

The development of the contractions may be also impeded if the stream flows parallel to an externally produced magnetic field. Such an arrangement would admit the application of strong electric fields and consequently the acceleration time could be probably decreased to an order of 1 μ s.†

† A report on this subject is now being prepared.

APPENDIX 1. THE NUMBER OF DEGREES OF FREEDOM IN PINCH COMPRESSIONS

The compression itself is two-dimensional as only the cross-section of the beam is changed, whilst its length (non-relativistic consideration) remains unaltered. The problem is whether the thermal energy due to the compression will be re-distributed into the degree of freedom corresponding to the non-compressed direction.

This depends on how long is the deflection-relaxation time t_D as compared with the period of the pinch compression. Using a formula for t_D given by Spitzer one has for electrons

$$t_D \simeq 0.023 T_e^{3/2} / n.$$

Initially $T_e \sim 10^4, \quad n \sim 3 \times 10^{11} \rightarrow t_D \sim 0.75 \times 10^{-7} \text{ s.}$

As the initial contraction takes generally a time several times longer than t_D one can assume that initially the equi-partition of heat energy into all the three degrees of freedom is effected. If this is so it can be shown (Linhart 1957) that

$$t_D = \text{const.}$$

during the whole compression process.

APPENDIX 2. DAMPING OF ACOUSTIC WAVES IN PLASMA DUE TO VISCOUS FORCES

Let us start with the hydrodynamic equation for a compressible viscous gas in which all the variables depend on the x -co-ordinate only. One has

$$\dot{v}_x = -\frac{1}{nM} \frac{\partial}{\partial x} p - \frac{2\eta}{nM} \frac{\partial^2 v_x}{\partial x^2}, \quad (\text{A } 1)$$

where $p = nkT = nkT_0(n/n_0)^{5/3}$ and η is the coefficient of viscosity.

Let us find a small amplitude solution of equation (A 1).

Let us put

$$n = n_0 + \nu_1, \quad v_x = v.$$

Neglecting second-order quantities and using the equation of continuity,

$$n_0 \frac{\partial v}{\partial x} = -\frac{\partial \nu_1}{\partial t}, \quad (\text{A } 2)$$

one gets
$$\dot{v} = -\frac{5}{3} \frac{kT_0}{\rho_0} \frac{\partial \nu_1}{\partial x} + \frac{2\eta}{n_0 \rho_0} \frac{\partial^2 \nu_1}{\partial x \partial t}, \quad (\text{A } 1a)$$

and differentiating equation (A 1a) with respect to x , one obtains

$$\frac{\partial^2 \nu_1}{\partial t^2} - \frac{5}{3} \frac{kT_0}{M} \frac{\partial^2 \nu_1}{\partial x^2} + \frac{2\eta}{\rho_0} \frac{\partial}{\partial t} \frac{\partial^2 \nu_1}{\partial x^2} = 0, \quad (\text{A } 3)$$

where $\rho_0 = n_0 M$.

The solution of equation (A 3) is of the type

$$\nu_1 = \nu e^{i\omega t + iKx}. \quad (\text{A } 4)$$

Let K be real. The corresponding characteristic equation is

$$K^2(A - i\omega B) = \omega^2, \quad A = \frac{5}{3} \frac{kT_0}{M}, \quad B = \frac{2\eta}{\rho_0}, \quad (\text{A } 5)$$

from which
$$\omega = -\frac{1}{2}iBK^2 + \sqrt{(\frac{1}{4}B^2K^4 + K^2A^2)}. \quad (\text{A } 6)$$

The damping is given by the imaginary part of ω . Thus the damping time-constant is

$$\tau = \frac{1}{I(\omega)} = \frac{n_0 M}{\eta K^2}. \quad (\text{A } 7)$$

The coefficient of viscosity η for an electron gas in the absence of magnetic fields is (Chapman & Cowling)

$$\eta_0 = \frac{1}{3}nm\lambda \sqrt{\frac{kT}{m}} = \frac{\sqrt{m}}{30e^4} (kT)^{\frac{5}{2}}. \quad (\text{A } 8)$$

In the presence of a magnetic field in which the radius of gyration is R_g or in a vessel whose dimensions are of the order of R_g one has to multiply equation (A 8) by a factor

$$[1 + (\lambda/R_g)^2]^{-1}.$$

If $\lambda/R_g > 1$, one can use the approximation

$$\eta = \eta_0 \left(\frac{R_g}{\lambda} \right)^2 = \frac{1}{3}nm \frac{\sqrt{(kT/m)}}{\lambda} R_g^2,$$

therefore

$$\eta = \frac{10}{3} \sqrt{m} e^4 n^2 R_g^2 (KT)^{-\frac{3}{2}}.$$

In our case $R_g \sim r_0$ the characteristic radius of the stream. Thus

$$\eta = 10^{-28} (N^2/r_0^2) T^{-\frac{3}{2}}. \quad (\text{A } 9)$$

Substituting (A 9) into (A 7) and writing also $K^{-2} \sim r_0^2$ one gets

$$\tau = 2.5 \frac{T^{\frac{3}{2}}}{N} r_0^2 (\text{s}). \quad (\text{A } 10)$$

If the compression of the electron gas is adiabatic (Linhart 1957)

$$T^{\frac{3}{2}} r_0^2 = \text{const.}$$

and one may substitute for T and r_0 the initial values of these quantities. Typical values are

$$T = 10^4, \quad r_0 = 2 \text{ cm.}$$

Then

$$\tau = 10^7/N (\text{s}) = 3.3\nu^{-1} (\mu\text{s}). \quad (\text{A } 10a)$$

Example: $N = 10^{12} \rightarrow \tau = 10 \mu\text{s}$.

It is evident, therefore, that the viscosity plays an important role in the damping of the pinch oscillations, at least initially, i.e. before the Joule heating raises appreciably T and r_0 . The importance of viscous damping becomes even more obvious when one takes into account the viscosity of the proton gas which is $\sqrt{(M/m)} = 43$ times higher than that of the electron gas.

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