ANALYTIC PROPERTIES OF THE SCHRÖDINGER AMPLITUDE

AT A FIXED ANGLE *)

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S U M M A R Y

The scattering from a given class of potentials including exchange forces is considered. For a fixed angle the scattering amplitude can be analytically continued into the complex energy plane with the exception of the usual cut along the positive real axis and certain poles and cuts on the negative real axis depending on the nature of the potential. Our results are a partial confirmation of the fixed angle dispersion relations recently proposed within the framework of field theory ⁸⁾.

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1. Introduction.

The study of the analytic properties of the scattering amplitude for a potential presents an interesting complement to the corresponding studies being carried out in field theory and recently this problem has been looked at in detail. Khuri ¹⁾ has shown that dispersion relations for fixed momentum transfer exist for a very general class of central potentials, the upper limit on the momentum transfer depending on the rate of decrease of the potential at infinity. Jost ²⁾ has shown that the Lehmann ellipses are also free from singularities in potential theory, i.e. if one investigates the analytic behaviour in the momentum transfer for fixed energies, there are certain regions free of singularities. Hamilton ³⁾ has recently extended Khuri's results to include tensor forces and at the same time has indicated the difficulties encountered in trying to include exchange forces.

What one would like is to be able to give the analytic properties of the scattering amplitude in all the variables and perhaps combine the results into a relation similar to the one proposed by Mandelstam ⁴⁾. In this paper we attempt something far less ambitious. We investigate the analytic properties of the scattering amplitude in the complex energy plane, while keeping the angle of scattering fixed at some physical value, for the class of potentials which may be written in the form ^{*})

$$V(r) = \int_{\mu}^{\infty} \rho_D(\sigma) \frac{e^{-\sigma r}}{r} d\sigma + P_M \int_{\mu}^{\infty} \rho_E(\sigma) \frac{e^{-\sigma r}}{r} d\sigma \quad (1.1)$$

with

$$\int_{\mu}^{\infty} [|\rho_D(\sigma)| + |\rho_E(\sigma)|] d\sigma < \mu \quad (1.2)$$

^{*}) Potentials of finite range do not belong to this class.

Here P_M is the Majorana exchange operator. The second condition implies that there are no bound states and allows one to use the Born series as a solution to the Schrödinger equation. (This is no great restriction since it is probably true that what holds for the Born series holds for Fredholm theory, the only difference being the poles appearing on the negative energy axis from the bound states in the latter.)

The investigation is carried out in four steps. First we study the scattering from a simple Yukawa potential

$$V(r) = \lambda \frac{e^{-\mu r}}{r} \quad (1.3)$$

$$|\lambda| < \mu \quad (1.4)$$

We shall show that the scattering amplitude may be extended analytically from the real energy axis to any finite point in the complex energy plane with a cut from 0 to ∞ , a pole at $^{*)} K^2 = -\frac{\mu^2}{4 \sin^2 \frac{\theta}{2}}$ and another cut from $K^2 = -\frac{\mu^2}{4 \sin^2 \frac{\theta}{2}}$ to $-\infty$.

The next step is to introduce an exchange potential writing

$$V(r) = \lambda_D \frac{e^{-\mu r}}{r} + \lambda_E \frac{e^{-\mu r}}{r} P_M \quad (1.5)$$

$$|\lambda_D| + |\lambda_E| < \mu \quad (1.6)$$

With this potential we find the same analytic properties except we now have two poles, one at $K^2 = -\frac{\mu^2}{4 \sin^2 \frac{\theta}{2}}$ and one at $K^2 = -\frac{\mu^2}{4 \cos^2 \frac{\theta}{2}}$ and the cut on the negative axis starts at

$$K^2 = -\mu^2 \min \left(\frac{1}{\sin^2 \frac{\theta}{2}}, \frac{1}{\cos^2 \frac{\theta}{2}} \right)$$

*) We take $\hbar = 2M_r = 1$ where M_r is the reduced mass of the two body system, therefore $K^2 = E$. We denote the angle of scattering by θ .

The final steps consist of generalizing the potential, initially to

$$V(r) = \int_{\mu}^{\infty} \rho_0(\sigma) \frac{e^{-\sigma r}}{r} d\sigma \quad (1.7)$$

and afterwards to the form (1.1). In this case the singularities along the negative axis depend upon the form of the $\rho(\sigma)$'s, the analytic properties elsewhere remaining unchanged. However, in no case is there a singularity on the negative real axis between 0 and $-\frac{\mu^2}{4} \min\left(\frac{1}{\sin^2 \frac{\theta}{2}}, \frac{1}{\cos^2 \frac{\theta}{2}}\right)$.

2. Equations for the Scattering Amplitude.

We start from the Schrödinger equation

$$\left(\nabla^2 + K^2 - \lambda \frac{e^{-\mu r}}{r}\right) \psi = 0 \quad (2.1)$$

and write it as an integral equation with the Green's function containing the boundary condition of outgoing waves at infinity.

$$\psi(\vec{x}) = e^{i\vec{K} \cdot \vec{x}} - \lambda \int G(\vec{x}, \vec{y}) \frac{e^{-\mu y}}{y} \psi(\vec{y}) d\vec{y} \quad (2.2)$$

$$G(\vec{x}, \vec{y}) = \lim_{\varepsilon \rightarrow 0} \frac{1}{(2\pi)^3} \int d\vec{q} \frac{e^{i\vec{q} \cdot (\vec{x} - \vec{y})}}{q^2 - K^2 - i\varepsilon} = \frac{1}{4\pi} \frac{e^{iK|\vec{x} - \vec{y}|}}{|\vec{x} - \vec{y}|} \quad (2.3)$$

The scattering amplitude is then

$$f(K, \theta) = -\frac{\lambda}{4\pi} \int_0^{\infty} e^{-i\vec{K}' \cdot \vec{y}} \frac{e^{-\mu y}}{y} \psi(\vec{y}) d\vec{y} \quad (2.4)$$

where $\vec{K}'$ is the outgoing momentum. Since we are only interested in elastic scattering we have $\vec{K}' = K \hat{n}_{\vec{K}}$, where $\hat{n}_{\vec{K}}$ is a unit vector in the direction of scattering.

The integral equation for $\psi(\vec{x})$ can now be iterated and when put into the expression for $f(K, \theta)$ (2.4) we get the Born series ⁵⁾. The result is

$$f(K, \theta) = -\frac{\lambda}{4\pi} \left[\frac{4\pi}{\mu^2 + (\vec{K}' - \vec{K})^2} + \sum_{n=1}^{\infty} \lambda^n f_n \right] \quad (2.5)$$

The first term is just the Fourier transform of the potential and

$$f_n^C = \left(-\frac{1}{4\pi}\right)^n \int d\vec{y} d\vec{z} d\vec{x}_1 \dots d\vec{x}_{n-1} e^{-i\vec{K}' \cdot \vec{y}} \frac{e^{-\mu y}}{y} \frac{e^{iK|\vec{y}-\vec{x}_1|}}{|\vec{y}-\vec{x}_1|} \frac{e^{-\mu x_1}}{x_1} \dots \frac{e^{iK|\vec{x}_{n-1}-\vec{z}|}}{|\vec{x}_{n-1}-\vec{z}|} \frac{e^{-\mu z}}{z} e^{i\vec{K} \cdot \vec{z}} \quad (2.6)$$

Where f_n^C indicates the coordinate representation.

We could also have iterated the equation for ψ by using the first form of $G(\vec{x}, \vec{y})$ given above. This leads to an expression for f_n in a momentum representation which we denote by f_n^K

$$f_n^K = \lim_{\varepsilon \rightarrow 0} 4\pi \left(-\frac{1}{2\pi^2}\right)^n \int d\vec{K}_1 \dots d\vec{K}_n \frac{1}{\mu^2 + (\vec{K}' - \vec{K}_1)^2} \frac{1}{(K_1^2 - K^2 - i\varepsilon)} \frac{1}{\mu^2 + (\vec{K}_1 - \vec{K}_2)^2} \dots \frac{1}{K_n^2 - K^2 - i\varepsilon} \frac{1}{\mu^2 + (\vec{K} - \vec{K}_n)^2} \quad (2.7)$$

We shall need one more form of f_n and that is the one obtained from f_n^K by parametrizing the integrand and doing the momentum integrations. This parametrization is carried out in detail in Appendix I, together with the proof of the properties stated below.

$$f_n^P = \lim_{\varepsilon \rightarrow 0} A_n \int_0^1 dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{2n-1}} dx_{2n} \frac{1}{(a'_1 a'_2 \dots a'_n)^{3/2}} \left(\frac{1}{D_n}\right)^{\frac{n}{2}+1}$$

$$\begin{aligned} D_n &= \alpha(x_1, \dots, x_{2n}) K^2 + \beta(x_1, \dots, x_{2n}) K'^2 + \gamma(x_1, \dots, x_{2n}) \vec{K} \cdot \vec{K}' + \mu^2 x_n - i\varepsilon(1-x_n) \\ &= K^2(\alpha + \beta + \gamma \cos \theta) + \mu^2 x_n - i\varepsilon(1-x_n) \\ &= -K^2 F_\theta(x) + \mu^2 x_n - i\varepsilon(1-x_n) \end{aligned}$$

A_n is a numerical constant, the a_i are complicated functions of the differences of the x 's but are all positive in the region of integration, and α , β and γ are all real. Since we are to take the limit as $\varepsilon \rightarrow 0$ and since for $X_n=0$, $F_\theta \neq 0$, we can just as well write (for positive K^2)

$$\begin{aligned} D_n &= -(K^2 + i\varepsilon) F_\theta(x) + \mu^2 X_n \\ &= -K^2 F_\theta(x) + \mu^2 X_n \end{aligned}$$

where we now consider K^2 to be complex and take the limit as it approaches the positive real axis from above.

Since the x integration is over a finite region and the denominator of the integrand can never vanish for K^2 complex, f_n^P is an analytic function of K^2 in the whole complex energy plane with the exception of the real axis ^{5) *)}. It is shown in the appendix that on the real axis there is a cut extending from $K^2=0$ to ∞ and from $K^2 = -\frac{\mu^2}{4\sin^2\frac{\theta}{2}}(n+1)^2$. It is also shown there that $f_n^P \rightarrow 0$ as $|K^2| \rightarrow \infty$.

In the region where f_n^C exists (this region lies in the upper half K plane and includes the real axis) we shall show in the next section that it is an analytic function of K .

f_n^C is also continuous onto the real axis. Since the same values there are taken by the boundary values of the analytic function $f_n^P(K)$, we can conclude that f_n^C and f_n^P represent the same analytic function in the common region of existence. That is, if there is an analytic function which takes the (continuous) values of the scattering amplitude as boundary values, then that analytic function is unique ⁷⁾.

*) We shall frequently go from the K to the K^2 plane and vice versa. We use $K = \sqrt{K^2}$. Confining K to the upper $\frac{1}{2}K$ plane means K^2 covers the whole K^2 plane with a cut from 0 to ∞ . The strip $0 \leq \text{Im} K \leq b$ is transformed into a parabola opening on the real K^2 axis and intersecting the K^2 axis at $K^2 = -b$.

3. Analyticity of $f(K, \theta)$.

In this section we shall prove that for

$$V(r) = \lambda \frac{e^{-\mu r}}{r} \quad (3.1)$$

$f(K, \theta)$ is an analytic function of K in a strip $0 \leq \text{Im } K < m\mu$, where m (an integer) can be made arbitrarily large, with a branch cut running from $K = \frac{i\mu}{\sin \theta/2}$ to $i\infty$ and a pole at $K = \frac{i\mu}{2 \sin \theta/2}$ coming from the Fourier transform of the potential. The problem is to show that the Born series is a sum of analytic functions in such a strip and that this sum is uniformly convergent there. Although f_n^P exists and is analytic for all complex K^2 , it is difficult to put an upper bound on its magnitude. What we shall do is show that f_n^C exists in the above strip provided that $n \geq 2m$, that it is analytic there, and that $\sum_{n=2m}^{\infty} \lambda^n f_n^C$ is uniformly convergent. This means we write

$$-4\pi \frac{f(K, \theta)}{\lambda} = \tilde{V} - \sum_{n=1}^{2m-1} \lambda^n f_n^P = \sum_{n=2m}^{\infty} \lambda^n f_n^C \quad (3.2)$$

and show that for $0 \leq \text{Im } K < m\mu$, the right hand side of this expression is an analytic function and therefore, $f(K, \theta)$ has the analytic properties of the finite sum standing on the left, that is, just those properties we set out to prove.

One can see from f_n^C that in our strip the Green's functions are well-behaved decreasing exponentials but that the plane wave factors standing at either end of the integrand may blow up. We use the fact that for a given K , if we have enough potentials in between these two end terms they can counteract this effect.

To prove our statement, we use the following theorem due to Titchmarsh ⁶⁾ and given by Khuri in the following form ¹⁾.

Theorem 1 :

If $\Phi(K, X)$ is analytic in K , regular in a certain region Γ of the complex K plane and if $\bar{\Phi}(K, X)$ is continuous in the closed region composed of Γ and its boundary B , then

$$f(K) = \int_{\Lambda} \bar{\Phi}(K, x) dx \quad (3.3)$$

is analytic in K , regular and uniformly bounded in Γ , provided there exists $\Psi(X)$ such that $|\bar{\Phi}(K, X)| \leq \Psi(X)$ for all K on B (and hence all K in Γ since $\bar{\Phi}(K, X)$ is analytic) and provided that

$$\int_{\Lambda} \Psi(x) dx < \infty \quad (3.4)$$

We wish to apply this theorem to f_n^C where the region Γ is the region enclosed by the real axis, the line $\text{Im } K = m\mu$ and by the two lines $\text{Re } K = \pm L$.

It is evident that the integrand of f_n^C satisfies the first three conditions and we shall show that for $n \gg 2m$ we can find a $\bar{\Psi}(X)$ satisfying the last condition. If we denote the integrand by J_n^C then, for $\text{Im } K = 0$

$$|J_n^C| \leq \frac{e^{-\mu y}}{y} \frac{1}{|\vec{y} - \vec{x}_1|} \frac{e^{-\mu x_1}}{x_1} \frac{1}{|\vec{x}_1 - \vec{x}_2|} \dots \frac{1}{|\vec{x}_{n-1} - \vec{z}|} \frac{e^{-\mu z}}{z} \equiv I_1^{(n)}. \quad (3.5)$$

For

$$\text{Im } K = m\mu \equiv b \quad (3.6)$$

we have

$$|J_n^C| \leq e^{by} \frac{e^{-\mu y}}{y} \frac{e^{-b|\vec{y} - \vec{x}_1|}}{|\vec{y} - \vec{x}_1|} \frac{e^{-\mu x_1}}{x_1} \dots \frac{e^{-b|\vec{x}_{n-1} - \vec{z}|}}{|\vec{x}_{n-1} - \vec{z}|} \frac{e^{-\mu z}}{z} e^{bz} \equiv I_2^{(n)} \quad (3.7)$$

Therefore, on both of these lines

$$|J_n^c| < I_1^{(n)} + I_2^{(n)} \quad (3.8)$$

Now on the two end lines $\operatorname{Re} K = \pm L$ we also have

$$|J_n^c| < I_1^{(n)} + I_2^{(n)} \quad (3.9)$$

(since $e^{\alpha X} < 1 + e^{\alpha X_0}$ for $0 \leq X \leq X_0$ and all real α). We can therefore let $L \rightarrow \infty$.

We now show that the integrals over I_1 and I_2 are finite. Take first

$$\int I_1^{(n)} = \frac{1}{(4\pi)^n} \int d\vec{y} d\vec{z} d\vec{x}_1 \dots d\vec{x}_{n-1} \frac{e^{-\mu y}}{y} \frac{1}{|\vec{y} - \vec{x}_1|} \frac{e^{-\mu x_1}}{x_1} \dots \frac{1}{|\vec{x}_{n-1} - \vec{z}|} \frac{e^{-\mu z}}{z} \quad (3.10)$$

From Appendix II we have

$$\int \frac{d\vec{z}}{|\vec{x} - \vec{z}|} f(z) \leq 4\pi \int_0^\infty z dz |f(z)| \quad (3.11)$$

and therefore, defining

$$M = \int_0^\infty y V(y) dy = \frac{1}{\mu} \quad (3.12)$$

$$N = \int_0^\infty y^2 V(y) dy = \frac{1}{\mu^2} \quad (3.13)$$

we find

$$\int I_1^{(n)} \leq 4\pi N M^n \quad (3.14)$$

For the second integral

$$\int I_2^n = \frac{1}{(4\pi)^n} \int d\vec{y} d\vec{z} d\vec{x}_1 \dots d\vec{x}_{n-1} e^{by} \frac{e^{-\mu y}}{y} \frac{e^{-b|\vec{y} - \vec{x}_1|}}{|\vec{y} - \vec{x}_1|} \dots \frac{e^{-b|\vec{x}_{n-1} - \vec{z}|}}{|\vec{x}_{n-1} - \vec{z}|} \frac{e^{-\mu z}}{z} e^{bz} \quad (3.15)$$

We start doing the integrations from the right noting (Appendix II)

$$\int d\vec{z} \frac{e^{-b|\vec{x}-\vec{z}|}}{|\vec{x}-\vec{z}|} \frac{e^{-\mu z}}{z} e^{bz} = \frac{4\pi}{\mu} e^{(b-\mu)x} \left[\frac{1 - e^{-(2b-\mu)x}}{(2b-\mu)x} \right] \quad (3.16)$$

Since

$$\frac{1 - e^{-y}}{y} \leq 1 \quad \text{for } y \geq 0 \quad (3.17)$$

we have

$$\int d\vec{z} \frac{e^{-b|\vec{x}-\vec{z}|}}{|\vec{x}-\vec{z}|} \frac{e^{-\mu z}}{z} e^{bz} \leq \frac{4\pi}{\mu} e^{(b-\mu)x} \quad (3.18)$$

for $2b-\mu \geq 0$. For the $\vec{x}_{n-1}$ integration we find

$$\begin{aligned} \int d\vec{x} \frac{e^{-b|\vec{y}-\vec{x}|}}{|\vec{y}-\vec{x}|} \frac{e^{-\mu x}}{x} \frac{4\pi}{\mu} e^{(b-\mu)x} &= \frac{4\pi}{\mu} \int d\vec{x} \frac{e^{-b|\vec{y}-\vec{x}|}}{|\vec{y}-\vec{x}|} \frac{e^{-2\mu x}}{x} e^{bx} \\ &\leq \frac{4\pi}{\mu} \cdot \frac{4\pi}{2\mu} e^{[b-2\mu]y} \end{aligned} \quad (3.19)$$

for $2b-2\mu \geq 0$. We repeat this process m times starting from the right and m times starting from the left. We obtain

$$\int I_2^{(n)} \leq \left(\frac{4\pi}{\mu} \right)^{2m} \left(\frac{1}{m!} \right)^2 \left(\frac{1}{4\pi} \right)^n \int d\vec{x}_1 \dots d\vec{x}_r \frac{e^{-\mu x_1}}{x_1} \frac{1}{|\vec{x}_1 - \vec{x}_2|} \frac{e^{-\mu x_2}}{x_2} \dots \frac{1}{|\vec{x}_{n-1} - \vec{x}_r|} \frac{e^{-\mu x_r}}{x_r} \quad (3.20)$$

$$r = n+1 - 2m \quad (3.21)$$

Provided $r \geq 1$ which is satisfied for $n \geq 2m$ and also provided $2b-m\mu = m\mu \geq 0$ which is always satisfied.

$$\int I_2^{(n)} \leq \left(\frac{4\pi}{\lambda}\right)^{2m} \left(\frac{1}{m!}\right)^2 \left(\frac{1}{4\pi}\right)^n (4\pi)^r N M^{r-1} \quad (3.22)$$

$$\int I_2^{(n)} \leq 4\pi N M^n \quad (3.23)$$

Therefore

$$|f_n^c| \leq \int I_1^{(n)} + \int I_2^{(n)} \leq 8\pi N M^n \quad (3.24)$$

$$\sum_{2m} \lambda^n f_n^c \leq \sum_{2m} |\lambda^n| |f_n^c| \leq 8\pi N \sum_0^\infty |\lambda|^n M^n = \frac{8\pi N}{1-|\lambda|M} \quad (3.25)$$

This converges for

$$\begin{aligned} |\lambda|M &< 1 \\ |\lambda| &< \mu \end{aligned} \quad (3.26)$$

Thus each f_n^c in $\sum_{2m} \lambda^n f_n^c$ is analytic and the sum is uniformly convergent for $0 \leq \text{im } K \leq b = m\mu$ which is what we set out to prove.

4. Exchange Forces.

We next show how the previous analysis can be extended to include exchange forces. We shall suppose for simplicity that the exchange force has the same range as the direct force but a different strength. We write

$$(\nabla^2 + K^2)\psi = \int V(\vec{x}, \vec{y}) \psi(\vec{y}) d\vec{y} \quad (4.1)$$

$$V(\vec{x}, \vec{y}) = \frac{e^{-\mu x}}{x} [\lambda_D \delta(\vec{x} - \vec{y}) + \lambda_E \delta(\vec{x} + \vec{y})] \quad (4.2)$$

Going to the integral equation for scattering one has

$$\psi(\vec{x}) = e^{i\vec{k} \cdot \vec{x}} - \int C_1(\vec{x}, \vec{y}) V(\vec{y}, \vec{z}) \psi(\vec{z}) d\vec{y} d\vec{z} \quad (4.3)$$

$$= e^{i\vec{k} \cdot \vec{x}} + \int (\lambda_D K_D(\vec{x}, \vec{y}) + \lambda_E K_E(\vec{x}, \vec{y})) \psi(\vec{y}) d\vec{y} \quad (4.4)$$

$$K_D(\vec{x}, \vec{y}) = -C_1(\vec{x}, \vec{y}) \frac{e^{-\mu y}}{y} \quad (4.5)$$

$$K_E(\vec{x}, \vec{y}) = K_D(\vec{x}, -\vec{y}) \quad (4.6)$$

and the scattering amplitude is again

$$f(K, \theta) = -\frac{1}{4\pi} \int e^{-i\vec{k}' \cdot \vec{y}} V(\vec{y}, \vec{z}) \psi(\vec{z}) d\vec{y} d\vec{z} \quad (4.7)$$

Iterating the equation for $\psi(x)$ and substituting in the equation for f one finds

$$f(K, \theta) = -\frac{1}{4\pi} \left[\tilde{V} + \sum_{n=1}^{\infty} f_n^c \right] \quad (4.8)$$

$$f_n^c(K, \theta) = \left(-\frac{1}{4\pi}\right)^n \sum_{i=0}^{n-1} \lambda_D^i \lambda_E^{n-i} \sum_p \int d\vec{x} d\vec{y} d\vec{z} d\vec{x}_1 \dots d\vec{x}_{n-1} e^{-i\vec{k}' \cdot \vec{x}} V(\vec{x}, \vec{y}) \underbrace{K_D(\vec{y}, \vec{x}_1) \dots K_D(\vec{x}_{i-1}, \vec{x}_i)}_{i \text{ terms}} \times \underbrace{K_E(\vec{x}_i, \vec{x}_{i+1}) \dots K_E(\vec{x}_{n-1}, \vec{x}_n)}_{(n-i) \text{ terms}} e^{i\vec{k} \cdot \vec{z}} \quad (4.9)$$

where $\sum_p$ denotes a sum over all permutations of the indices D and E on the K 's.

Now notice that one can change all the E's to D's in each term of the above sum (including the E in $V(\vec{x}, \vec{y})$) by starting with the first exchange term on the left and changing the integration variables to their negatives working out to the left. That is we use

$$K_E(\vec{x}, \vec{y}) = K_D(\vec{x}, -\vec{y}) = K_D(-\vec{x}, \vec{y}) . \quad (4.10)$$

Now if the exchange force acts an even number of times the net result is to change the sign of $\vec{K}' \cdot \vec{x}$ an even number of times and thus we can put all the E's $\rightarrow$ D's and $\vec{K}' \rightarrow \vec{K}'$. This is just the expression we have studied before. If, however, the exchange force acts an odd number of times, we can change all the E's $\rightarrow$ D's but must simultaneously take $\vec{K}' \rightarrow -\vec{K}'$ which means taking $\cos \theta \rightarrow -\cos \theta$ or $\sin^2 \frac{\theta}{2} \rightarrow \cos^2 \frac{\theta}{2}$.

As for proving the uniform convergence of $\sum_{2m}^{\infty} f_n^C$, it will be noticed that the bounds we put on the integrals were independent of the direction of $\vec{K}'$ and hence still hold. The only difference now is that we have more terms and the condition for the convergence of the bounding series is changed slightly. Thus

$$\begin{aligned} |f_n^C| &\leq \sum_{i=0}^n |\lambda_D|^i |\lambda_E|^{n-i} \left[\frac{|\lambda_D|^{n-i} (n-i)!}{i! (n-i)!} + \frac{|\lambda_E|^i i!}{i! (n-i)!} \right] 8\pi N M^n \\ &= 8\pi N (|\lambda_D| + |\lambda_E|) M^n (|\lambda_D| + |\lambda_E|)^n \end{aligned} \quad (4.11)$$

$$\sum_{2m}^{\infty} f_n^C \leq 8\pi N (|\lambda_D| + |\lambda_E|) \frac{1}{1 - M(|\lambda_D| + |\lambda_E|)} \quad (4.12)$$

The factors $\frac{n!}{i! (n-i)!}$ appear because this is the number of terms in $\sum_p$. The condition for convergence is now

$$\begin{aligned} (|\lambda_D| + |\lambda_E|) M &< 1 \\ |\lambda_D| + |\lambda_E| &< \mu \end{aligned} \quad (4.13)$$

The Fourier transform of the potential now gives

$$\tilde{V} = 4\pi \left[\frac{1}{\mu^2 + 4K^2 \sin^2 \frac{\theta}{2}} + \frac{1}{\mu^2 + 4K^2 \cos^2 \frac{\theta}{2}} \right] \quad (4.14)$$

and thus $f(K^2, \theta)$ is an analytic function in the whole K^2 plane except for a cut from $K^2 = 0$ to ∞ , poles at $K^2 = -\frac{\mu^2}{4} \left(\frac{1}{\sin^2 \frac{\theta}{2}} \right)$ and $K^2 = -\frac{\mu^2}{4} \left(\frac{1}{\cos^2 \frac{\theta}{2}} \right)$ and a cut from $K^2 = -\mu^2 \min \left(\frac{1}{\sin^2 \frac{\theta}{2}}, \frac{1}{\cos^2 \frac{\theta}{2}} \right)$ to $-\infty$.

5. Extension to Spectral Representations.

In this section we further extend our results to include potentials which can be written as integrals over Yukawa potentials.

A. Central Force

We write

$$V(r) = \int_{\mu}^{\infty} \rho(\sigma) \frac{e^{-\sigma r}}{r} d\sigma \quad (5.1)$$

and can again write a Born series classifying terms by the number of times the potential acts. Thus

$$f(K, \theta) = -\frac{1}{4\pi} \left[\tilde{V} + \sum_{n=1}^{\infty} f_n \right] \quad (5.2)$$

$$f_n(K, \theta) = \left(-\frac{1}{4\pi} \right)^n \int_{\mu}^{\infty} d\sigma_1 \dots d\sigma_{n+1} \rho(\sigma_1) \dots \rho(\sigma_{n+1}) \int d\vec{y} d\vec{z} d\vec{x}_1 \dots d\vec{x}_{n-1} e^{-i\vec{K} \cdot \vec{y}} \frac{e^{-\sigma_1 y}}{y} \frac{e^{i\vec{K}_1 \cdot \vec{y}}}{|\vec{y} - \vec{x}_1|} \frac{e^{-\sigma_2 x_1}}{x_1} \dots \frac{e^{-\sigma_{n+1} z}}{z} e^{i\vec{K} \cdot \vec{z}} \quad (5.3)$$

Investigating first the uniform convergence of the series $\sum_{n=0}^{\infty} f_n$ we find

$$|f_n| \leq \left(\frac{1}{4\pi}\right)^n \int_{\mu}^{\infty} |p(\sigma_1)| \dots |p(\sigma_{n+1})| d\sigma_1 \dots d\sigma_{n+1} \int d\vec{y} d\vec{z} d\vec{x}_1 \dots d\vec{x}_{n-1} e^{by} e^{-\mu y} \frac{e^{-b|\vec{y}-\vec{x}_1|}}{|\vec{y}-\vec{x}_1|} \frac{e^{-\mu x_1}}{x_1} \dots \frac{e^{-\mu z}}{z} e^{bz} \quad (5.4)$$

where use has been made of $e^{-\sigma x} \leq e^{-\mu x}$. This uncouples the spectral functions and allows one to make the same analysis as previously except we must replace λ by $\int_{\mu}^{\infty} |p(\sigma)| d\sigma$. Therefore, the condition for convergence of the series is

$$\int_{\mu}^{\infty} |p(\sigma)| d\sigma < \mu \quad (5.5)$$

One can carry out the parametrization as before and finds

$$f_n^P(K, \theta) = A_n \int_{\mu}^{\infty} d\sigma_1 \dots d\sigma_{n+1} p(\sigma_1) \dots p(\sigma_{n+1}) \int_0^1 dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{2n-1}} dx_{2n} \frac{1}{(a'_1 \dots a'_n)^{3/2}} \frac{1}{\tilde{D}_n^{n/2+1}} \quad (5.6)$$

$$\tilde{D}_n = -K^2 F_{\theta}(x) + C_1(\sigma, x) \quad (5.7)$$

where (see Appendix I)

$$C_1(\sigma, x) = \sigma_1^2 x_{2n} + \sigma_2^2 (x_{2n-1} - x_{2n}) + \dots + \sigma_{n+1}^2 (x_n - x_{n+1}) \geq \mu^2 x_n \quad (5.8)$$

Therefore, $G(\sigma, x)$ can only vanish for $x_n \rightarrow 0$ in which case $F_{\theta}(x) \neq 0$. Also, to locate the position of the branch cut on the negative axis we must find the minimum of

$$\frac{C_1(\sigma, x)}{2|x|\sin^2 \frac{\theta}{2}} \geq \frac{\mu^2 x_n}{2|x|\sin^2 \frac{\theta}{2}} \quad (5.9)$$

with the equality holding for all $\sigma = \mu$. Therefore, the condition

on the start of the branch cut for each term of the Born series is the same as before. What is different is

$$\tilde{V} = 4\pi \int_{\mu}^{\infty} \frac{\rho(\sigma) d\sigma}{\sigma^2 + (\vec{K}' - \vec{K})^2} \quad (5.10)$$

If $\rho(\sigma)$ is continuous from μ we have a branch cut (no longer a pole) starting from $K^2 = \frac{-\mu^2}{4\sin^2 \frac{\Theta}{2}}$. If, however, $\rho(\sigma)$ has the form

$$\rho(\sigma) = 2\lambda \delta(\sigma - \mu) + g(\sigma) \quad (5.11)$$

where $g(\sigma) = 0$ for $\sigma < 2\mu$ as one might expect from meson theory, we have

$$\tilde{V} = \frac{4\pi\lambda}{\mu^2 + 4K^2 \sin^2 \frac{\Theta}{2}} + \int_{2\mu}^{\infty} \frac{g(\sigma) d\sigma}{\sigma^2 + 4K^2 \sin^2 \frac{\Theta}{2}} \quad (5.12)$$

Again there is an isolated pole and a branch cut starting from $K^2 = \frac{-\mu^2}{4\sin^2 \frac{\Theta}{2}}$ which coincides with the branch cut coming from the first iteration of the Yukawa potential.

B. General Exchange Mixture

We write

$$V(\vec{x}, \vec{y}) = \int_{\mu}^{\infty} \rho_D(\sigma) \frac{e^{-\sigma r}}{r} d\sigma \delta(\vec{x} - \vec{y}) + \int_{\mu}^{\infty} \rho_E(\sigma) \frac{e^{-\sigma r}}{r} d\sigma \delta(\vec{x} + \vec{y}) \quad (5.13)$$

Exactly the same analysis holds as in the section on exchange forces. We again use $e^{-\sigma r} \leq e^{-\mu r}$ which decouples the spectral functions and must replace

$$|\lambda_{D,E}| \rightarrow \int_{\mu}^{\infty} |\rho_{D,E}(\sigma)| d\sigma \quad (5.14)$$

The arguments on the parametrization are the same as above, where again, if the exchange force acts an odd number of times we must take $\sin^2 \frac{\theta}{2} \rightarrow \cos^2 \frac{\theta}{2}$. The condition for the convergence of the series is

$$\int_{\mu}^{\infty} (|\rho_D(\sigma)| + |\rho_E(\sigma)|) d\sigma < \mu \quad (5.15)$$

and if, in particular

$$\rho_{D,E}(\sigma) = 2\lambda_{D,E} \delta(\sigma - \mu) + g_{D,E}(\sigma) \quad (5.16)$$

with $g_{D,E}(\sigma) = 0$ for $\sigma < 2\mu$ we find, as the analytic properties of $f(K^2, \theta)$, a branch cut from 0 to ∞ poles at $K^2 = \frac{-\mu^2}{4} \frac{1}{\sin^2 \frac{\theta}{2}}$ and $\frac{-\mu^2}{4} \frac{1}{\cos^2 \frac{\theta}{2}}$ and a branch cut from $K^2 = -\mu^2 \min(\frac{1}{\sin^2 \frac{\theta}{2}}, \frac{1}{\cos^2 \frac{\theta}{2}})$ to $-\infty$.

6. General Considerations.

It should be noted that it is now an easy task to study the analytic properties of each phase shift. One multiplies $f(K^2, \theta)$ by a Legendre polynomial and integrates over angles. The partial wave amplitude that one obtains will again be analytic in the whole complex K^2 plane with the exception of the cut on the positive K^2 axis and poles and cuts on the negative K^2 axis which depend on the nature of the potential. These extra singularities appear in addition to any real bound states in the problem and have long been a subject of discussion⁹⁾. An alternative way of investigating the analytic properties of the scattering amplitude might be to demonstrate analytic properties of each partial wave separately and then consider the behaviour of their sum.

It is interesting to compare our results with the fixed angle dispersion relations recently proposed within the framework of field theory ⁸⁾. These dispersion relations follow if one assumes the Mandelstam relations ⁴⁾ to have a general validity. The analytic properties obtained and exploited in ⁸⁾ are just the ones demonstrated here for a potential. We note, for example, that the residue of the pole coming from potential (5.11) is just $\frac{-\lambda}{4\sin^2\frac{\Theta}{2}}$ and if the Yukawa potential arises from a meson exchange, then $\frac{\lambda}{2}$ is just the square of the coupling constant. We have not been able to prove the complete dispersion relations at fixed angle which the above authors suggest since we are not able to make a definite statement about the behaviour of the scattering amplitude at infinity. All we are able to say is that each term in the Born series goes to zero there and so one is perhaps justified in hoping that the scattering amplitude will also have this property ^{*}).

One may make one other interesting comparison with the proposed field theory results ⁸⁾. In the relativistic theory, the branch points on the negative energy axis correspond to the thresholds for processes involving antinucleons, and in particular for the processes

$$N + \bar{N} \rightarrow (n+1)\pi$$

they occur at ⁸⁾

$$(\vec{K} - \vec{K}')^2 = -[(n+1)\mu]^2 \quad (\text{Direct forces})$$

$$\mu = m_\pi$$

or

$$K^2 = -\frac{1}{4\sin^2\frac{\Theta}{2}}[(n+1)\mu]^2$$

Exactly the same result occurs in our potential theory where n is now the order of iteration of the potential and μ is its range.

^{*}) It is easy to show from our analysis that in the first and fourth quadrants of the K^2 plane, the scattering amplitude must go to zero at infinity.

A P P E N D I X I

The Parametric Integrals

A. Parametrization.

We want to parametrize f_n^K

$$\frac{f_n^K}{4\pi} = \left(-\frac{1}{2\pi^2}\right)^n \int d\vec{K}_1 \dots d\vec{K}_n \frac{1}{\mu^2 + (\vec{K}' - \vec{K}_1)^2} \frac{1}{K_1^2 - K^2 - i\varepsilon} \dots \frac{1}{K_n^2 - K^2 - i\varepsilon} \frac{1}{\mu^2 + (\vec{K}_n - \vec{K})^2}$$

and carry out the momentum integrations. To do this we use ¹⁰⁾

$$\frac{1}{a_1 a_2 \dots a_N} = (N-1)! \int_0^1 dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} \dots \int_0^{x_{N-2}} dx_{N-1} \left[a_1 x_{N-1} + a_2 (x_{N-2} - x_{N-1}) + \dots + a_N (1 - x_1) \right]^{-N}$$

$$1 > x_1 > x_2 > \dots > x_{N-1} > 0$$

We let $N = 2n+1$, the number of terms in f_n^K , and take all the potentials first and the Green's functions last. We find, if J^{-N} is the integrand above

$$J = \mu^2 x_n - (1 - x_n)(K^2 + i\varepsilon) + K'^2 x_{2n} + K^2 (x_n - x_{n+1}) - 2\vec{K}' \cdot \vec{K}_1 x_{2n} - 2\vec{K} \cdot \vec{K}_n (x_n - x_{n+1}) + \sum_{i,j=1}^{n-1} a_{ij} \vec{K}_i \cdot \vec{K}_j$$

where

$$a_{ii} = x_{2n-i} - x_{2n+2-i} + x_{n-i} - x_{n+1-i} > 0$$

$$a_{ij} = -2_1 (x_{2n-i} - x_{2n+1-i}) \delta_{i,i+1} < 0$$

We now complete the square and do the $\vec{K}_1$ integration. We use

$$\begin{aligned} \int_0^\infty \frac{p^2 dp}{(p^2 + C)^{n+1}} &= \frac{\pi}{2} \frac{(2n-3)!!}{n! 2^n} \frac{1}{C^{n-1/2}} && n \text{ integral} \\ &= 2^{n-3/2} \frac{(n-3/2)!}{(2n)!!} \frac{1}{C^{n-1/2}} && n \text{ half-integral.} \end{aligned}$$

These formulae are easily established by taking appropriate derivatives of

$$\int_0^\infty \frac{dp}{p^2 + C} = \frac{\pi}{2\sqrt{C}}$$

$$\int_0^\infty \frac{dp}{(p^2 + C)^{3/2}} = \frac{1}{C}$$

These integrals are well defined as long as C is never purely negative and since the coefficient of $i\varepsilon$, $-(1-x_n)$ does not vanish in the region of integration, this can never happen. Repeating the process n times one finds

$$f_n^K = A_n I_n(K^2, \cos \theta)$$

$$A_n = \left(-\frac{1}{2\sqrt{\pi}}\right)^n \left(\frac{n}{2}\right)! \quad n \text{ even}$$

$$= \frac{\sqrt{\pi}}{2} \left(-\frac{1}{2\sqrt{2\pi}}\right)^n n!! \quad n \text{ odd}$$

$$I_n = \int_0^1 dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{2n-1}} dx_{2n} \frac{1}{(a'_1 \dots a'_n)^{3/2}} \left(\frac{1}{D_n}\right)^{n/2+1}$$

$$D_n = \alpha(x_1 \dots x_{2n}) K^2 + \beta(x_1 \dots x_{2n}) K'^2 + \gamma(x_1 \dots x_{2n}) \vec{K} \cdot \vec{K}' + \mu^2 x_n - i\varepsilon(1-x_n)$$

$$\alpha = -(1-x_n) - \frac{a_{n,n+1}}{2} \left(1 + \frac{a_{n,n+1}}{2a'_n}\right)$$

$$\beta = -\frac{a_0}{2} - \sum_{i=1}^n \left(\frac{1}{2}\right)^{2i} \frac{1}{a'_i} \prod_{j=1}^{i-1} \frac{a_{0i}^2 a_{j,i+1}^2}{(a'_j)^2}$$

$$\gamma = \left(-\frac{1}{2}\right)^n \prod_{i=1}^n \frac{a_{0i} a_{i,i+1}}{a'_i}$$

$$a_{0i} = -2x_{2i}$$

$$a_{n,n+1} = -2(x_n - x_{n+1})$$

$$a'_i = a_{ji} - \frac{a_{i-1,i}^2}{4a'_{i-1}}$$

$$a'_1 = a_{11}$$

Note that all the a_{ij} and a'_i have a dependence on n which is suppressed in the notation. The a'_i appear because when one completes the square for the $\vec{K}_i$ integration, the coefficient of $\vec{K}_{i+1}^2$ is altered.

B. Properties of the Parametrization.

$$\underline{1) \ a'_i > 0.}$$

To study the properties of the parametric integrals it is more convenient to introduce

$$-2b_{ij} = a_{ij} \quad i \neq j$$

$$b_i = a_{ij} \quad i = j$$

in which case

$$b_{ij} > 0$$

$$b_i > 0.$$

One easily verifies from the definitions that

$$b_i = b_{i-1,i} + b_{i,i+1} + c_i$$

$$c_i = x_{n-i} - x_{n+1-i} > 0.$$

All three terms in the expression for b_i are thus greater than zero.

One next notices that by substituting in the recurrence relation defining a'_i that the general form of a'_i is

$$a'_i = \frac{1}{a'_1 \dots a'_{i-1}} \left[\prod_{j=1}^i b_j + \sum \text{over all contractions of pairs of neighbouring indices in } \prod_{j=i}^i b_j \right]$$

where by a contraction of a pair of neighbouring indices we mean

$$b_i b_j \rightarrow -b_{ij}^2$$

For example :

$$a'_4 = \frac{1}{a'_1 a'_2 a'_3} \left[b_1 b_2 b_3 b_4 - b_{12}^2 b_3 b_4 - b_1 b_{23}^2 b_4 - b_1 b_2 b_{34}^2 + b_{12}^2 b_{34}^2 \right]$$

If one now substitutes the above expression for b_i it is seen that the $\sum$ over contractions is just the process which removes all the terms in b_{ij}^2 from $\prod_{j=1}^n b_j$ and what is left is a series of positive terms. One can thus write

$$\begin{aligned} a'_1 \dots a'_j &= c_1 c_2 \dots c_j + \sum^1 c_2 \dots c_j + \sum^2 c_1 c_3 \dots c_j + \dots \\ &+ \sum^{1,2} c_3 \dots c_j + \sum^{1,3} c_2 c_4 \dots c_j + \dots \\ &+ \sum^{1,2,3} c_4 \dots c_j + \sum^{1,2,4} c_3 c_5 \dots c_j + \dots \\ &+ \sum^{1,1,3} \dots \end{aligned}$$

where $\sum^{i,j}$ is a sum over all possible products of the b_{ij} 's associated with the missing c_i, c_j through $b_i = b_{i-1,i} + b_{i,i+1} + c_i$ and not containing any b_{ij}^2 .

For example :

$$\begin{aligned} a'_3 a'_2 a'_1 &= c_1 c_2 c_3 + (b_{01} + b_{12}) c_2 c_3 + c_1 (b_{12} + b_{23}) c_3 + c_1 c_2 (b_{23} + b_{34}) \\ &+ (b_{01} b_{12} + b_{01} b_{23} + b_{12} b_{23}) c_3 + (b_{01} b_{23} + b_{12} b_{23} + b_{01} b_{34} + b_{12} b_{34}) c_2 + \\ &+ (b_{12} b_{23} + b_{12} b_{34} + b_{23} b_{34}) c_3 + (b_{01} b_{12} b_{23} + b_{01} b_{12} b_{34} + b_{01} b_{23} b_{34} + b_{12} b_{23} b_{34}). \end{aligned}$$

Since this is a series of positive terms and holds for all $i \leq n$ one sees that all the $a'_i > 0$ *).

*) Note that the $a'_1 \dots a'_n$ may vanish at the boundary of the integration region but that the integrals over the vanishing denominators are convergent. This has been shown for $n=1,2,3$ and probably holds in general.

2) Show $-(\alpha + \beta + \gamma \cos \theta) = F \neq 0$ when $X_n = 0$.

From their definition

$$-(\alpha + \beta + \gamma) = 1 + g(x) - 2x_n - x_{2n}$$

$$g(x) \geq 0$$

also
$$-(\alpha + \beta + \gamma \cos \theta) = -(\alpha + \beta + \gamma) + 2\gamma \sin^2 \frac{\theta}{2}$$

$$\equiv A - 2|\gamma| \sin^2 \frac{\theta}{2}$$

(Notice that $\gamma < 0$). We shall show in the next section that $\frac{x_n}{|\gamma|}$ has a finite minimum and therefore $\gamma \rightarrow 0$ as $X_n \rightarrow 0$ *).

Thus

$$F_\theta \xrightarrow{X_n \rightarrow 0} 1 + g(x) \geq 1$$

3) Location of the branch cuts.

We first make the observation that for forward scattering the scattering amplitude is analytic in the whole complex K^2 plane with a cut from 0 to ∞ . The Fourier transform of the potential reduces to $\frac{4\pi}{\mu^2}$ and the series $\sum \gamma^n f_n$ is easily shown to be uniformly convergent under the condition $\lambda < \mu$ by using the inequality

$$\left| \exp \left[i \left\{ K (|\vec{y} - \vec{x}_1| + |\vec{x}_1 - \vec{x}_2| + \dots + |\vec{x}_n - \vec{z}|) - \vec{k} \cdot (\vec{y} - \vec{z}) \right\} \right] \right| \leq \exp \left[-\gamma_m K |\vec{y} - \vec{z}| (1 - \hat{n}_{\vec{k}} \cdot \hat{n}_{\vec{y} - \vec{z}}) \right]$$

$$\leq 1$$

*) This is easily seen from the definition of γ . All the a_{ij} are differences of X_i with $i \geq n$ while $a'_i a'_n$ has a term $c_i \dots c_j$ which contains differences of X_i 's with $i \leq n$. Therefore as $X_n \rightarrow 0$, the numerator must go to zero while the denominator contains a non-vanishing term inside the region of integration.

for K in the upper half plane. Since we have already shown that f_n^C and f_n^P represent the same analytic function we can conclude f_n^P only has a branch cut on the positive K^2 axis for forward scattering. We thus get the condition

$$-(\alpha + \beta + \gamma) \equiv A \geq 0$$

To find the location of the branch cuts in the general case we write

$$D_n = -F_\theta K^2 + \mu^2 X_n$$

$$F_\theta = A - 2|\gamma| \sin^2 \frac{\theta}{2}$$

We must therefore find what values of $K^2 = \frac{\mu^2 X_n}{A - 2|\gamma| \sin^2 \frac{\theta}{2}}$ are permissible on the negative axis. Since both A and X_n are positive we get an upper limit by finding the minimum of

$$-K^2 = \frac{\mu^2 X_n}{2|\gamma| \sin^2 \frac{\theta}{2}}$$

Now $|\gamma| = 2 \frac{b_{01} b_{12} \dots b_{n,n+1}}{a_1' \dots a_n'}$, to get a minimum for the above expression we want $|\gamma|$ as large as possible. Since we can set all the $C_i = 0$ in $a_1' \dots a_n'$ without changing the values of $b_{01} b_{12} \dots b_{n,n+1}$ and X_n , we first do this. This leaves

$$|\gamma| = 2 \frac{b_{01} b_{12} \dots b_{n,n+1}}{\sum_{12 \dots n}} = 2 \left(\frac{1}{\frac{1}{b_{01}} + \frac{1}{b_{12}} + \frac{1}{b_{13}} + \dots + \frac{1}{b_{n,n+1}}} \right)$$

We must therefore minimize

$$\frac{\mu^2}{4 \sin^2 \frac{\theta}{2}} X_n \left(\frac{1}{b_{01}} + \frac{1}{b_{12}} + \dots + \frac{1}{b_{n,n+1}} \right)$$

It is evident that the minimum of this expression occurs when the interval from 0 to X_n is split into $n+1$ equal parts each of length $\frac{X_n}{n+1}$. Therefore *)

$$-K_{\min}^2 = \left(\frac{\mu(n+1)}{2 \sin \frac{\theta}{2}} \right)^2$$

*) Notice that we have just set an upper limit to the start of the cut. We speak in the text of the cut actually starting there since in the lowest order terms it is true.

$$4) \quad \frac{f_n^P \rightarrow 0}{|K^2| \rightarrow \infty}$$

Write $K^2 = \rho e^{i\phi}$ and $dX = \frac{dx_1 \dots dx_{2n}}{a_1' \dots a_n'}$. First split the X integration into two parts, one where

$$|F_\theta| < \varepsilon \mu^2 X_n \quad (\text{region } \delta)$$

and the rest of the region $(\Gamma - \delta)$. Then

$$\frac{f_n^P}{A_n} = \int_\delta dX \frac{1}{(-K^2 F_\theta + \mu^2 X_n)^{\frac{n}{2}+1}} + \int_{\Gamma-\delta} dX \frac{1}{(-K^2 F_\theta + \mu^2 X_n)^{\frac{n}{2}+1}}$$

Maximizing the integrand in I as a function of F we can write

$$I_\delta \leq \left(\frac{1}{\sin \phi}\right)^{\frac{n}{2}+1} \int_\delta dX \left(\frac{1}{\mu^2 X_n}\right)^{\frac{n}{2}+1}$$

Letting $\rho \rightarrow \infty$, we can write for the second

$$I_{\Gamma-\delta} \leq \left(\frac{1}{\rho \varepsilon}\right)^{\frac{n}{2}+1} \int_{\Gamma-\delta} dX \left(\frac{1}{\mu^2 X_n}\right)^{\frac{n}{2}+1}$$

Now

$$\int dX \left(\frac{1}{\mu^2 X_n}\right)^{\frac{n}{2}+1} = \frac{f_n^P(K^2=0)}{A_n} < \infty.$$

Therefore

$$a) \quad \text{Choose } \varepsilon \text{ so that } I_\delta < \frac{1}{2}$$

$$b) \quad \text{Choose } \rho \text{ so that } I_{\Gamma-\delta} < \frac{1}{2}$$

$$\text{and hence } I < \frac{1}{2} \quad \rho > \rho(\eta)$$

It will be noticed that this proof only holds for $\sin \phi \neq 0$. However, for $\phi=0$, one sees from the coordinate representation that $f_n^C \xrightarrow{\rho \rightarrow \infty} 0$ by the rapid oscillation of the exponentials ¹⁾. Also, noting from the form of f_n^P that we can relate the integrals for complex K^2 to the integrals for complex μ^2 and real K , and that if we make μ^2 complex we can still go from the coordinate to the parametric representations, we conclude again that the integral $\tilde{f}_n^C$ will oscillate to zero as K^2 goes out along the negative axis. Therefore

$$\frac{f_n^P \rightarrow 0}{\rho \rightarrow \infty}.$$

A P P E N D I X II

A. Show $I_1 \equiv \int_0^\infty \frac{d\vec{z}}{|\vec{x}-\vec{z}|} f(z) \leq 4\pi \int_0^\infty z dz |f(z)|$

$$I_1 = 2\pi \left[\int_0^x \frac{z^2 dz \sin\theta d\theta}{|\vec{x}-\vec{z}|} f(z) + \int_x^\infty \frac{z^2 dz \sin\theta d\theta}{|\vec{x}-\vec{z}|} f(z) \right]$$

let

$$u^2 = (\vec{x}-\vec{z})^2 = x^2 + z^2 - 2xz \cos\theta$$

$$u du = xz \sin\theta d\theta$$

The limits on u are $x+z \geq u \geq |x-z|$

$$I_1 = 2\pi \left[\int_0^x \frac{z f(z) dz}{x} \int_{x-z}^{x+z} du + \int_x^\infty \frac{z f(z) dz}{x} \int_{z-x}^{z+x} du \right]$$

$$I_1 = 4\pi \left[\int_0^x \left(\frac{z}{x} \right) z f(z) dz + \int_x^\infty z f(z) dz \right]$$

$$I_1 \leq 4\pi \int_0^\infty z |f(z)| dz$$

B. Show $I_2 \equiv \int_0^\infty \frac{d\vec{z}}{|\vec{x}-\vec{z}|} e^{-b|\vec{x}-\vec{z}|} \frac{e^{-\mu z}}{z} e^{bz} = \frac{4\pi}{\mu} e^{(b-\mu)x} \left[\frac{1 - e^{-(2b-\mu)x}}{(2b-\mu)x} \right]$

Use the same transformation to the variable u as above.

$$\begin{aligned} I_2 &= \frac{2\pi}{x} \left[\int_0^x dz e^{(b-\mu)z} \int_{x-z}^{x+z} e^{-bu} du + \int_x^\infty dz e^{(b-\mu)z} \int_{z-x}^{z+x} e^{-bu} du \right] \\ &= \frac{4\pi}{\mu} e^{(b-\mu)x} \left[\frac{1 - e^{-(2b-\mu)x}}{(2b-\mu)x} \right] \end{aligned}$$

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