Fixed Angle Dispersion Relations for Nucleon-Nucleon Scattering

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Dispersion relations in which the scattering angle θ is kept constant are proposed for nucleon-nucleon scattering. These relations are a direct consequence of a general representation proposed by Mandelstam (see reference 4) and can be checked directly up to the fourth-order perturbation theory. An extrapolation procedure based on these dispersion relations is proposed with the purpose of obtaining an independent determination of the pion-nucleon coupling constant and the possibility of discriminating between the different available sets of phase shifts.

A first determination of the pion-nucleon coupling constant from these fixed-angle dispersion relations is attempted by means of a more direct extrapolation procedure, closely analogous to the effective range approximation, and a result in good agreement with the currently accepted value is obtained.

INTRODUCTION

THE theoretical study of nucleon-nucleon scattering is usually made on the basis of potential models. A potential in more or less direct relation with meson theory is derived and used for the computation of physical quantities. It is not always clear whether the good agreement between theory and experiment depends on the meson theoretical or on the phenomenological features of such a potential.

However, from the knowledge of the long-range part of the potential Iwadare *et al.*¹ have obtained a value of the pion-nucleon coupling constant in good agreement with the accepted value 0.08.

A more direct step to check the relation between pion physics and nucleon-nucleon scattering has recently been made by Chew.² He has proposed an extrapolation of the angular distribution based on dispersion relations at fixed energy^{3,4} and has obtained a reasonable determination of the pion-nucleon coupling constant f^2 .

On the other hand, dispersion relations for the scattering amplitude as a function of energy for fixed momentum transfer have been proposed by Goldberger, Oehme, and Nambu.⁵ The appearance of large unphysical regions, even for forward scattering, has prevented until now attempts of numerical evaluation.⁶

In this paper we shall discuss, starting from a different point of view, the problem of obtaining a direct relation

between the energy dependence of nucleon-nucleon scattering and meson theory.

In Sec. 1 we propose dispersion relations in which the scattering angle ϑ is kept constant. These relations have the advantage that they do not involve any unphysical region of nucleon-nucleon scattering. The whole unphysical contribution comes from nucleon-antinucleon scattering. Our relations generalize therefore the forward ones without introducing any new essential difficulty.

In Secs. 2 and 3 we discuss an extrapolation method based on the fixed angle dispersion relations which leads to a determination of the pion-nucleon coupling constant and to consistency checks on the different sets of nucleon-nucleon phaseshifts.

In Sec. 4 we shall introduce a more direct extrapolation method which is in a close analogy with the effective range approximation. This effective range extrapolation is applied to low-energy proton-proton scattering and gives a value of f^2 in good agreement with other determinations.

Finally different possible applications are discussed.

(1) The Fixed Angle Dispersion Relations

We fix our attention on proton-neutron scattering. We denote by p_1, p_2 the initial and final proton four-momenta, by n_1, n_2 the neutron ones. We define:

$$P = \frac{1}{2}(p_1 + p_2), \quad N = \frac{1}{2}(n_1 + n_2), \quad (1)$$

$$s = -(p_1 + n_1)^2; \quad \bar{s} = -(p_1 - n_2)^2; \quad t = -(p_1 - p_2)^2.$$

These three variables are connected by the relation

$$s + \bar{s} + t = 4M^2, \quad (2)$$

where M is the nucleon mass.

In the center-of-mass system we have

$$s = 4(M^2 + \omega), \quad \bar{s} = -2\omega(1 + c), \quad t = -2\omega(1 - c), \quad (3)$$

$$\omega = q^2, \quad c = \cos\vartheta,$$

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¹ Iwadare, Otsuki, Tamagaki, and Watari, Suppl. Progr. Theoret. Phys. Kyoto No. 3, Part III, 32 (1956).

² G. F. Chew, 1958 *Annual International Conference on High-Energy Physics at CERN*, edited by B. Ferretti (CERN, Geneva, 1958), p. 97; University of California Radiation Laboratory Report UCRL-8283, 1958 (unpublished).

³ H. Lehmann, Nuovo cimento **10**, 579 (1958).

⁴ S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

⁵ Goldberger, Nambu, and Oehme, Ann. Phys. **2**, 226 (1957), referred to in the following as GNO.

⁶ See, however, L. Lapidus, Joint Institute for Nuclear Research, Dubna Rept. P. 224.

FIG. 1. Singularities of the scattering amplitude in the plane s, y ($y = t - s$). Shaded regions correspond to the domains where the spectral functions $\Phi_{12}, \Phi_{23}, \Phi_{13}$, are nonvanishing. Solid lines correspond to poles. The region ($\bar{s} \leq 0, t \leq 0$) corresponds to physical $p\bar{n}$ scattering; the two regions ($s \leq 0, t \geq 0$) and ($s \leq 0, \bar{t} \leq 0$) correspond to physical nucleon antinucleon scattering.

where q is the nucleon momentum and ϑ the scattering angle.

In GNO the $p\bar{n}$ scattering amplitude is written in the form

$$\mathfrak{M}(p_1, p_2, n_1, n_2) = \sum_i G_i(s, t) I_i, \quad (4)$$

$$I_1 = \bar{u}(p_2) u(p_1) \bar{u}(n_2) u(n_1),$$

$$I_2 = \bar{u}(p_2) u(p_1) \bar{u}(n_2) i\gamma \cdot Pu(n_1) + \bar{u}(p_2) i\gamma \cdot Nu(p_1) \bar{u}(n_2) u(n_1), \quad (5)$$

$$I_3 = \bar{u}(p_2) i\gamma \cdot Nu(p_1) \bar{u}(n_2) i\gamma \cdot Pu(n_1),$$

$$I_4 = \bar{u}(p_2) \gamma_5 i\gamma \cdot Nu(p_1) \bar{u}(n_2) \gamma_5 i\gamma \cdot Pu(n_1),$$

$$I_5 = \bar{u}(p_2) \gamma_5 u(p_1) \bar{u}(n_2) \gamma_5 u(n_1).$$

TABLE I. Residue of the poles of the functions $G_i(s, t)$ due to one pion exchange.

	B	C
G_1	0	$\frac{g^2 (M^2 - \mu^2) [4M^2(1+c)^2 - 2\mu^2(1+c)] + \mu^2 M^2(1+c)^2}{4M^2\mu^2(1-c^2) - 2\mu^4(1-c)}$
G_2	0	$\frac{g^2 M (4M^2 - \mu^2)(1+c)^2 - 2\mu^2(1+c)}{4M^2\mu^2(1-c^2) - 2\mu^4(1-c)}$
G_3	0	$\frac{g^2 (4M^2(1+c)^2 - \mu^2(1-c^2))}{4M^2\mu^2(1-c^2) - 2\mu^4(1-c)}$
G_4	0	$\frac{g^2 (1+c)}{4M^2(1+c) - 2\mu^2}$
G_5	g^2	$-\frac{g^2 (1+c)}{1-c}$

The G_i are scalar functions of s and t . We shall write a representation for these functions which is the generalization of that given by Mandelstam in the case of pion-nucleon scattering. This representation, whose general validity has not yet been proved, has the big advantage of giving explicitly the analytic properties in the variables $s, \bar{s}$, and t . For example, the fixed- t dispersion relations of GNO and the fixed-energy dispersion relations used for the extrapolation of Chew can be obtained as consequences of such representation. We write

$$G(s, t) = \frac{A}{s_d - s} + \frac{B}{\mu^2 - t} + \frac{C}{\mu^2 - \bar{s}} + \frac{1}{\pi^2} \int_2^\infty ds' \int_{4\mu^2}^\infty dt' \frac{\phi_{13}(s', t')}{(s' - s)(t' - t)} + \frac{1}{\pi^2} \int_{4\mu^2}^\infty d\bar{s}' \int_{4\mu^2}^\infty dt' \frac{\phi_{23}(\bar{s}', t')}{(\bar{s}' - \bar{s})(t' - t)} + \frac{1}{\pi^2} \int_2^\infty ds' \int_{4\mu^2}^\infty d\bar{s}' \frac{\phi_{12}(s', \bar{s}')}{(s' - s)(\bar{s}' - \bar{s})}, \quad (6)$$

where the ϕ are three real weighting functions. The pole in the variable s corresponds to the deuteron state (of mass $s_d = 2M - B$); the two poles in t and $\bar{s}$ correspond to the exchange of one meson between nucleons: the first is due to the reaction ($p \rightarrow p + \pi^0$) the second to ($p \rightarrow n + \pi^+$). In Table I the residue of the poles due to one pion exchange are listed; the deuteron effects will be discussed in the next section.

The meaning of the representation (6) is illustrated in Fig. 1 where $y = t - s = 4q^2 \cos \vartheta$ is plotted against s . The shaded regions correspond to the domains where ϕ_{12}, ϕ_{23} , and ϕ_{13} can contribute to the integrals. The solid lines correspond to the poles $s = s_d, t = \mu^2, \bar{s} = \mu^2$. The region ($\bar{s} \leq 0, t \leq 0$) corresponds to physical $p\bar{n}$ scattering. The two regions ($s \leq 0, t \geq 0$) and ($s \leq 0, \bar{t} \leq 0$) correspond to physical nucleon-antinucleon scattering. The GNO dispersion relations are taken along lines parallel to $t = 0$, the Chew extrapolation exploits analytic properties along vertical lines.

We shall use the representation (6) in order to obtain dispersion relations for a fixed value of $\cos \vartheta$, i.e., taken along lines passing through the point $(4M^2, 0)$. We sub-

TABLE II. Residue of the poles of the functions $\alpha, \beta, \gamma, \delta, \epsilon$ due to one pion exchange

	B	C
α	0	$f^2 M$
β	0	$-f^2 M$
γ	0	0
δ	$f^2 M$	$-f^2 M$
ϵ	0	$f^2 M$

stitute Eqs. (3) into Eq. (6). We obtain (see Appendix I)

$$G(s, c) = \frac{A}{s_d - s} + \frac{B}{\mu^2 + (s - 4M^2)(1 - c)/2} + \frac{C}{\mu^2 + (s - 4M^2)(1 + c)/2} + \frac{1}{\pi} \int_{4M^2}^{\infty} ds' \frac{\Phi_A(s', c)}{s' - s} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} dt' \frac{\Phi_B(t', c)}{t' + (s - 4M^2)(1 - c)/2} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} d\bar{s}' \frac{\Phi_C(\bar{s}', c)}{\bar{s}' + (s - 4M^2)(1 + c)/2}, \quad (8)$$

with Φ_A , Φ_B , and Φ_C real functions.

One important advantage of the representation (8) is that for $-1 \leq c \leq +1$ the spectral function Φ_A can be always expressed in terms of physical nucleon-nucleon amplitudes (see Appendix I),

$$\Phi_A(s, c) = \text{Im}G(s, c) \quad \text{for } s \geq 4M^2, \quad -1 \leq c \leq +1. \quad (9)$$

On the other hand very little is known about Φ_B , and Φ_C . One can in principle relate them to unphysical nucleon-antinucleon processes. This lies outside the scopes of the present paper.

We collect in a unique integral all the unknown contribution and we express everything in terms of the variable $\omega = q^2$. We have

$$G(\omega, c) = -\frac{A}{\omega_d + \omega} + \frac{B}{\mu^2 + 2\omega(1 - c)} + \frac{C}{\mu^2 + 2\omega(1 + c)} + \frac{1}{\pi} \int_0^{\infty} \frac{\text{Im}G(\omega', c)}{\omega' - \omega} d\omega' + \frac{1}{\pi} \int_{-\infty}^{-\omega_{\min}} \frac{\Omega(\omega', c)}{\omega' - \omega} d\omega', \quad (10)$$

where

$$\omega_{\min} = 2\mu^2/(1 + |c|).$$

The representation (10) shows explicitly the analytic properties of $G(\omega, c)$. All singularities are on the real axis: poles at $\omega = -\omega_d$ and $\omega = -\mu^2[2(1 \pm c)]^{-1}$ and two cuts between 0 and $+\infty$ and between $-\infty$ and $-\omega_{\min}$.

As we have already said, we shall not attempt to evaluate Ω . However, the very fact that the lower limit of integration on Ω is at least four times larger than the ω corresponding to at least one of the poles in f^2 will enable us to use Eq. (10) as a tool in order to correlate nucleon-nucleon experimental data. Equation (10) has been derived as a consequence of the Mandelstam representation. Its validity has been proved up to fourth order in field theory, and exactly for scattering of a particle by an arbitrary superposition of Yukawa potentials.⁷ In any case, the applications we shall discuss

will depend only on the location and size of the poles and on the order of magnitude of $\omega_{\min}$. They are thus independent of the most questionable features of the Mandelstam representation.

(2) Relations for the Physical Amplitudes

The T matrix for proton-neutron scattering is usually written in the form

$$T = \frac{M}{(M^2 + q^2)^{\frac{1}{2}}} \{ \alpha + \beta(\sigma^{(p)} \cdot \mathbf{n})(\sigma^{(n)} \cdot \mathbf{n}) + \gamma(\sigma^{(p)} + \sigma^{(n)}) \cdot \mathbf{n} + \delta(\sigma^{(p)} \cdot \mathbf{m})(\sigma^{(n)} \cdot \mathbf{m}) + \epsilon(\sigma^{(p)} \cdot \mathbf{l})(\sigma^{(n)} \cdot \mathbf{l}) \}, \quad (11)$$

where $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$, are unit vectors in the directions $\mathbf{q} + \mathbf{q}'$, $\mathbf{q} - \mathbf{q}'$, $\mathbf{q} \times \mathbf{q}'$, respectively.

The relations between the physical amplitudes and the G_i are given in GNO Eqs. (5-5 to 9). Using these relations it is possible to obtain in the nonrelativistic approximation uncoupled equations for all physical amplitudes.⁸ These relations are of the form

$$\alpha(\omega, c) = -\frac{A_\alpha(c)}{\omega + \omega_d} + \frac{B_\alpha(c)}{2\omega(1 - c) + \mu^2} + \frac{C_\alpha(c)}{2\omega(1 + c) + \mu^2} + \frac{1}{\pi} \int_0^{\infty} \frac{\text{Im}\alpha(\omega', c)}{\omega' - \omega} d\omega' + \frac{1}{\pi} \int_{-\infty}^{-\omega_{\min}} \frac{\Omega_\alpha(\omega', c)}{\omega' - \omega} d\omega'. \quad (12)$$

It is convenient to express the physical amplitudes in terms of the standard T_{ij} matrices which are functions of the scattering phase shifts [GNO Eqs. (B6) to (B11)]. The dispersion relations for the T_{ij} are of the same form of Eq. (12). The residue of the poles of all amplitudes are collected in Tables II and III. Analogous relations can be obtained for proton-proton scattering by recalling that

$$T_{ij}^{(pp)} = T_{ij}^{(pn)}(\omega, c) - (-1)^{i+j} T_{ij}^{(pn)}(\omega, -c), \quad (13)$$

$$T_{SS}^{(pp)} = T_{SS}^{(pn)}(\omega, c) + T_{SS}^{(pn)}(\omega, -c).$$

Let us now discuss how one can use our relations at the present stage of knowledge. Even the incomplete information on the scattering amplitude obtained by these relations is sufficient to define a procedure which

TABLE III. Residue of the poles of the functions T_{ij} .

	B	C	$A \times \frac{2}{\kappa} (1 - \kappa c) (1 + \rho^2)^{\frac{1}{2}}$
T_{11}	$f^2 M (1 - c)/2$	$f^2 M (1 + c)$	$4 + \sqrt{2} (1 + 3c^2) \rho - \rho^2 (1 + 3c^2)$
T_{00}	$f^2 M c$	$-2f^2 M c$	$4 - 2\sqrt{2} (1 + 3c^2) \rho - 4\rho^2 (1 - 3c^2)$
T_{1-1}	$f^2 M (1 + c)/2$	$f^2 M (1 - c)$	$\sqrt{2} (3 - 3c^2) \rho + 3(1 - c^2) \rho^2$
$T_{10} + T_{01}$	0	0	$12c(1 - c^2)^{\frac{1}{2}} \rho - 6\sqrt{2} \rho^2 c(1 - c^2)^{\frac{1}{2}}$
$T_{10} - T_{01}$	0	0	$-9\sqrt{2} \rho^2 c(1 - c^2)^{\frac{1}{2}}$
T_{SS}	$-f^2 M$	$2f^2 M$	0

^a Notations of GNO.

⁸ In GNO some complications arise when $s - 4M^2 \rightarrow 0$ with t finite, due to artificial singularities. In our case no such complications exist.

⁷ J. Bowcock and G. D. Walecka, Nuclear Phys. (to be published).

enables (a) to determine the pion-nucleon coupling constant, (b) to correlate experimental phase shifts.

Let us consider the function

$$F_\alpha(\omega, c) = \left[\alpha(\omega, c) - \frac{1}{\pi} \int_0^{\omega_{\max}} \frac{\text{Im} \alpha(\omega', c)}{\omega' - \omega} d\omega' + \frac{A_\alpha(c)}{\omega + \omega_d} \right] \times [2\omega(1-c) + \mu^2][2\omega(1+c) + \mu^2]. \quad (14)$$

The function $F_\alpha(\omega)$ has no singularities on the real axis between $-\omega_{\min}$ and $\omega_{\max}$. Therefore in this interval it can be safely approximated by a polynomial. Such a function can be determined for positive ω in terms of the phase shifts of nucleon-nucleon scattering. The polynomial fit of $F_\alpha(\omega, c)$ gives a convenient check of the consistency of the different sets of phase shifts.⁹ The polynomial thus obtained can be used in order to get the value of $F_\alpha(\omega, c)$ at the location of the poles (those included between $-\omega_{\min}$ and $\omega_{\max}$). This leads to many possible determinations of f^2 .

(3) The Physical Meaning of the Extrapolation Procedure

We now want to discuss in some detail the physical implications of our procedure and to compare them with the extrapolation proposed by Chew² for p - n scattering in order to examine what kind of physical information is exploited in both cases. It has been stressed by Chew that the physical content of his method is simply that the interaction between two nucleons corresponding to the exchange of one meson has a range at least twice as large as those corresponding to the exchange of many mesons, and, more important, leads to a contribution to the scattering amplitude which is known exactly. Therefore the scattering amplitude for waves of sufficiently high angular momentum will be almost completely determined by the one-pion exchange amplitude. This fact follows easily from the Mandelstam spectral representation.⁴ Let us consider, for the purpose of discussion, the amplitude ϵ of Eq. (11) for which a representation of the form (6) holds with

$$B=0, \quad C=f^2M, \quad (15)$$

The dispersion relations with fixed s and variable t used by Chew are readily obtained by performing the integrations on $\bar{s}'$ and s' ,

$$\epsilon(s, t) = \frac{f^2M}{s+t-4M^2+\mu^2} - \frac{A_\epsilon}{s_d-s} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{dt'}{t'-t} \alpha(s, t') + \frac{1}{\pi} \int_{-\infty}^{4M^2-4\mu^2-s} \frac{dt'}{t'-t} \beta(s, t'), \quad (16)$$

⁹ Also in the work by W. C. Davidon and M. L. Goldberger [Phys. Rev. **104**, 1119 (1956)] where use is made of the dispersion relations for the spin flip part of the pion-nucleon scattering amplitude in order to discriminate between the Fermi and Yang sets of phase shifts, the unknown contribution of the high-energy region in the integral is approximated with a constant, whose value is determined by the low-energy experimental data.

where α β are unknown functions related to the analytic continuation of the scattering amplitudes in the unphysical region. In the c.m. system we can write Eq. (16) as

$$\epsilon(\omega, c) = \frac{f^2M}{2\omega(1+c) + \mu^2} + \sum_{l < L} C_l(\omega) P_l(c), \quad (17)$$

where the coefficients $C_l(\omega)$ are not known (they arise from the expansion of the integrals on the cuts in partial waves and contain in addition the known contribution from the deuteron term), but they should be negligible for l greater than a given L sufficiently large. The determination of f^2 is clearly based in Chew's method, at least in principle, on the fact that, for $l > L$, one has

$$\int_{-1}^{+1} \epsilon(\omega, c) P_l(c) dc = \int_{-1}^{+1} p(\omega, c) P_l(c) dc, \quad (18)$$

where we have in the left-hand side for a given l an expression which can be determined from the knowledge of the experimental phase shifts, and on the right-hand side the known coefficient of the expansion in partial waves of the term

$$p(\omega, c) = \frac{f^2M}{2\omega(1+c) + \mu^2}. \quad (19)$$

In practice, one multiplies the differential cross section by $[2\omega(1+c) + \mu^2]^2$ and fits the experimental data with a polynomial in $c = \cos \vartheta$ in order to extrapolate to the unphysical value of c corresponding to the pole $c = -1 - \mu^2/2\omega$: but in doing this the contributions to the phase shifts for $l < L$ arising from $p(\omega, c)$ are lumped together with the contributions from the cuts, and are therefore not exploited for the determination of f^2 .

In our method one can write, from Eq. (12),

$$\epsilon(\omega, c) = -\frac{A_\epsilon(c)}{\omega + \omega_d} + \frac{f^2M}{2\omega(1+c) + \mu^2} + \frac{1}{\pi} \int_0^\infty d\omega' \frac{\text{Im} \epsilon(\omega', c)}{\omega' - \omega} + \frac{1}{\pi} \int_{-\infty}^{-\omega_{\min}} d\omega' \frac{\Omega_\epsilon(\omega', c)}{\omega' - \omega}. \quad (20)$$

This can be written as

$$E(\omega, c) \equiv \epsilon(\omega, c) - \frac{1}{\pi} \int_0^\infty d\omega' \frac{\text{Im} \epsilon(\omega', c)}{\omega' - \omega} + \frac{A_\epsilon(c)}{\omega + \omega_d} = p(\omega, c) + \sum_{i < N} C_i(c) \omega^i. \quad (21)$$

Now the unknown coefficients $C_i(c)$ derive from the expansion in powers of ω of the integral on the cut from $-\infty$ to $-\omega_{\min}$ and are negligible for $i > N$ if N is sufficiently large.

The determination of f^2 is now based, in principle, on the comparison between the coefficients of the expansion of $E(\omega, c)$ in powers of ω for $i > N$, and the known coefficients of the expansion in powers of ω of the term $p(\omega, c)$ with a pole at $\omega = -\mu^2/2(1+c)$.

Therefore we can say that the Chew method exploits the information contained in the parts of the amplitude rapidly varying with the angle at a given energy, while our method exploits the information contained in the part rapidly varying with the energy at a given angle. Furthermore while the high angular momentum waves are important in Chew's extrapolation, the low angular momentum waves contribute the most in our extrapolation. This can be easily seen if we notice that, at a given angle, say 90° , one has from Eq. (19)

$$p(0,0) = f^2 M / \mu^2; \quad p(\mu^2, 0) = 0.33 f^2 M / \mu^2; \\ p(2\mu^2, 0) = 0.20 f^2 M / \mu^2. \quad (22)$$

If one instead considers the s - and d -wave parts of $p(\omega, c)$ (the p wave does not contribute at 90°) one has:

$$p_s(\omega, c) = f^2 \frac{M}{4\omega} \ln \left(\frac{4\omega + \mu^2}{\mu^2} \right), \\ p_d(\omega, c) = f^2 \frac{5M}{16\omega} (3c^2 - 1) \\ \times \left\{ \left(\frac{3\mu^4}{4\omega^2} + \frac{3\mu^2}{\omega} + 2 \right) \ln \left(\frac{4\omega + \mu^2}{\mu^2} \right) - 6 - \frac{3\mu^2}{\omega} \right\}. \quad (23)$$

From these expressions one obtains:

$$p_s(0,0) = f^2 M / \mu^2; \quad p_s(\mu^2, 0) = 0.40 f^2 M / \mu^2; \\ p_s(2\mu^2, 0) = 0.28 f^2 M / \mu^2, \quad (24) \\ p_d(0,0) = 0; \quad p_d(\mu^2, 0) = -0.08 f^2 M / \mu^2, \\ p_d(2\mu^2, 0) = -0.09 f^2 M / \mu^2.$$

Comparison with the corresponding results in Eq. (22) shows that the lowest angular momentum waves account for almost all the energy variation of $p(\omega, c)$, at least in the region of interest.

In practice, as we have seen in Sec. 3, in our method we multiply $E(\omega, c)$ by $2\omega(1+c)$ and fit the experimental data with a polynomial in ω . The extrapolation at $\omega = -\mu^2/2(1+c)$ of this polynomial will give $f^2 M$.

The method described above will be the guiding principle of our applications. However, in the form given by Eq. (14) the appearance of the integral on $\text{Im}T$ creates a practical problem because such an integral requires a very detailed knowledge of the p - n phase shifts as a function of energy. In the next section we shall propose a simplified extrapolation method which is closely related to the well-known effective range theory.

(4) The Effective Range Approximation

We fix our attention on the singlet scattering amplitude at 90° :

$$T_{ss}(\omega) = - \sum_l \frac{1}{q} (2l+1) e^{i\delta_l} \sin \delta_l P_l(0). \quad (25)$$

The dispersion relation is

$$T_{ss}(\omega) = \frac{Mf^2}{2\omega + \mu^2} + \frac{1}{\pi} \int_0^\infty \frac{\text{Im}T_{ss}(\omega')}{\omega' - \omega} d\omega' \\ + \frac{1}{\pi} \int_{-\infty}^{-\omega_{\min}} \frac{\Omega_{ss}(\omega')}{\omega' - \omega} d\omega'. \quad (26)$$

Let us consider the function

$$g(\omega) = \frac{1}{T_{ss}(\omega)}. \quad (27)$$

Equation (26) shows that $g(\omega)$ has the following analytic properties: cuts between 0 and $+\infty$ and between $-\infty$ and $-\omega_{\min}$ and possibly some poles corresponding to the zeros of $T(\omega)$. Finally we have:

$$g(-\mu^2/2) = 0, \quad (28) \\ g'(-\mu^2/2) = 2/Mf^2.$$

One can therefore write for $g(\omega)$ the following representation:

$$g(\omega) = g(0) + R(\omega) + \frac{\omega}{\pi} \int_0^\infty \frac{\text{Im}g(\omega')}{\omega'(\omega' - \omega)} d\omega' \\ + \frac{\omega}{\pi} \int_{-\infty}^{-\omega_{\min}} \frac{\rho(\omega')}{\omega'(\omega' - \omega)} d\omega', \quad (29)$$

where $R(\omega)$ is a rational function of ω .

The advantage of using Eq. (29) is that for small ω , $\text{Im}g(\omega)$ can be evaluated. Indeed when the contribution from the s wave dominates in Eq. (25), one has

$$\text{Reg}(\omega) = q \cot \delta_s, \quad (30) \\ \text{Im}g(\omega) = -q.$$

From the preceding discussion it follows that the function

$$f(\omega) = g(\omega) + \frac{\omega}{\pi} \int_0^\infty \frac{q'd\omega'}{\omega'(\omega' - \omega)}, \quad (31)$$

apart for the possible appearance of poles, is analytic between $-\omega_{\min}$ and $\omega_{\max}$, $\omega_{\max}$ being the value of ω for which Eqs. (30) start being invalid. From the existing phase shift analysis we know that the s wave dominates in Eq. (25) at least up to 50 Mev.

Let us now use these analytic properties in order to determine f^2 . Equation (31) tells us that on the real axis $f(\omega)$ is given by

$$f(\omega) = g(\omega) + \lambda(\omega), \quad (32)$$

where

$$\lambda(\omega) = 0 \quad \omega > 0, \\ \lambda(\omega) = -(-\omega)^{\frac{1}{2}} \quad \omega < 0.$$

FIG. 2. ---- Blatt-Jackson straight line. — Best fit of the experimental points together with the theoretical point at $\omega = -\mu^2/2$ by means of a function of the form (34).

On the other hand, using Eqs. (30) and (28) we have

$$\begin{aligned} f(\omega) &= q \cot \delta_s, \quad \omega > 0 \\ f(-\mu^2/2) &= -(1/\sqrt{2})\mu, \\ f'(-\mu^2/2) &= 1/\sqrt{2}\mu + 2/Mf^2. \end{aligned} \quad (33)$$

Equation (33) shows clearly that an extrapolation of $f(\omega)$ from $\omega > 0$ to $\omega = -\mu^2/2$ leads to a determination of f^2 . This program has been carried out using the data coming from low-energy proton-proton scattering. In Fig. 2 we have plotted the experimental values of $q \cot \delta_s$ ¹⁰ shifted of a constant amount in order to account for the Coulomb corrections to the scattering length.¹¹ The solid line represents our best fit of $f(\omega)$ giving: $f^2 = 0.11 \pm 15\%$.

The interpolating curve has been taken of the form

$$f(\omega) = a + b\omega + c\omega^2/(1+d\omega), \quad (34)$$

$a = 0.06$ and $b = 0.97$ are fixed by the low-energy points; $c = -0.33$ and $d = 1.40$ are determined by the theoretical relation $f(-0.5) = -0.70$ and by the high-energy experimental points.¹²

The reason for our choice (34) is that in this manner we are able to account rather well for the deviations from the Blatt-Jackson straight line of all the high energy experimental points and in addition of the theoretical point at $\omega = -\mu^2/2$ with only two extra parameters. It is easy to see that it would not be possible to fit all points with a fourth order polynomial: this is

because the deviations from the straight line for positive values of ω are comparatively much smaller than for $\omega = -0.5$.[†]

Our determination of f^2 , which is already of the good order of magnitude, has only to be considered as an illustration of the method since an accurate analysis of the energy dependence of the Coulomb effect is probably necessary. However we feel that these corrections cannot change radically our result since the proton-proton and proton-neutron effective ranges agree rather well.

The generalization of the method to other phase shifts does not offer any new theoretical difficulty: the essential problem is to find a phase shift analysis sufficiently accurate for our purpose. For example one can write a dispersion relation for

$$T_{ss'}(\omega) = \lim_{c \rightarrow 0} \frac{d}{dc} T_{ss'}(\omega, c).$$

At low ω , $T'(\omega)$ is completely dominated by the singlet p -wave phase shift. The very poor experimental data on such phase shift at low energy have been used for an independent determination of f^2 . We obtain the correct sign and order of magnitude of f^2 .

The method outlined here has many formal analogies with the Chew low effective range approximation for pion-nucleon scattering.

In that work the effective range plot gave an extrapolation of the p -wave scattering amplitude to its own pole. In our case the amplitude $T(\omega)$ is a superposition of all waves and its pole is due to the cumulative effect of all angular momenta.

In our extrapolation we are helped by the fact that in a limited range of energy $T(\omega)$ can be well approximated by keeping only the s wave contribution. However the curve of Fig. 2 is not an extrapolation of $q \cot \delta_s$ which would have much more complicated analytic properties.¹³

CONCLUSION

The result of this paper confirms the meson theoretical point of view in the theory of nuclear forces.

[†] Note added in proof.—The experimental singlet wave phase shift used for the extrapolation procedure has no real basis because it is derived by completely neglecting all p and d contributions. We thank Noyes for pointing this out to us.

However, our theoretical prediction (34) can be used to determine the low energy deviation from the shape independent approximation in terms of the well known pion mass and coupling constant.

Indeed the two shape dependent parameters c and d are uniquely determined by the two conditions (33) for $f(-\mu^2/2)$. Therefore the use of information from meson theory predicts that at low energy $k \cot \delta_s$ lies below the effective range line.

Noyes and Wong (to be published) came independently to the same conclusion using similar physical ideas and a different mathematical technique.

It is somewhat satisfactory [MacGregor, Moravcsik, and Noyes, preprint (UCRL 5582-T)] that improved analysis of the data indicate possible agreement with the theoretical prediction.

¹³ It would have a branch point at $\omega = -\mu^2/4$.

¹⁰ B. Cork, Phys. Rev. **80**, 323 (1950).

¹¹ J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), p. 93.

¹² We have taken the pion mass as unity.

Indeed our successful determination of f^2 together with the Chew result follow from the fact that the external part of the nucleon-nucleon potential is very well given by the Yukawa one meson exchange forces.

The Chew extrapolation exploits the facts that the higher angular momentum waves are essentially given by the external part of the potential. Our effective range extrapolation uses the fact that the deviation of the s wave from the effective range straight line is also due to the external part of the potential. In both cases the effects coming from smaller distances are eliminated through the extrapolation procedure.

The method proposed here can of course be applied to low-energy p waves. In the case of the singlet more accurate values of the phase shifts are needed. For the triplet (where the pp experimental data can be used) we hope that the proposed extrapolation, together with a further check of f^2 could help in selecting between the different sets of phase shifts.

APPENDIX I

We want to obtain dispersion relations along a straight line through the point $(4M^2, 0)$ in the s, y plane. From Eqs. (3) we have

$$\begin{aligned} t &= (4M^2 - s)(1 - c)/2, \\ \bar{s} &= (4M^2 - s)(1 + c)/2. \end{aligned} \quad (\text{A1.1})$$

Accordingly we write the denominators in the integrals of Eq. (6) as follows:

$$\begin{aligned} \frac{1}{(s' - s)(t' - t)} &= \left[\frac{1}{s' - s} + \frac{(1 - c)}{2} \frac{1}{t' - (4M^2 - s)(1 - c)/2} \right] \\ &\quad \times \frac{1}{t' - (4M^2 - s')(1 - c)/2}, \\ \frac{1}{(t' - t)(\bar{s}' - \bar{s})} &= \left[\frac{1 - c}{t' - (4M^2 - s)(1 - c)/2} - \frac{1 + c}{\bar{s}' - (4M^2 - s)(1 + c)/2} \right] \\ &\quad \times \frac{1}{\bar{s}'(1 - c) - t'(1 + c)}, \end{aligned} \quad (\text{A1.2})$$

$$\begin{aligned} \frac{1}{(s' - s)(\bar{s}' - \bar{s})} &= \left[\frac{1}{s' - s} + \frac{(1 + c)}{2} \frac{1}{\bar{s}' - (4M^2 - s)(1 + c)/2} \right] \\ &\quad \times \frac{1}{\bar{s}' - (4M^2 - s')(1 + c)/2}. \end{aligned}$$

Now one can easily see that, defining

$$\begin{aligned} \Phi_A(s', c) &= \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{dt'}{t' - (4M^2 - s')(1 - c)/2} \phi_{13}(s', t') \\ &\quad + \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{d\bar{s}'}{\bar{s}' - (4M^2 - s')(1 + c)/2} \phi_{12}(s', \bar{s}'), \end{aligned} \quad (\text{A1.3a})$$

$$\begin{aligned} \Phi_B(t', c) &= \frac{1}{\pi} \frac{(1 - c)}{2} \int_{4M^2}^{\infty} \frac{ds'}{t' - (4M^2 - s')(1 - c)/2} \phi_{13}(s', t') \\ &\quad + \frac{1}{\pi} \frac{(1 - c)}{2} \int_{4\mu^2}^{\infty} \frac{d\bar{s}'}{\bar{s}'(1 - c) - t'(1 + c)} \phi_{23}(\bar{s}', t'), \end{aligned} \quad (\text{A1.3b})$$

$$\begin{aligned} \Phi_C(\bar{s}', c) &= \frac{1}{\pi} \frac{(1 + c)}{2} \int_{4M^2}^{\infty} \frac{ds'}{\bar{s}' - (4M^2 - s')(1 + c)/2} \phi_{12}(\bar{s}', s') \\ &\quad - \frac{1}{\pi} \frac{(1 + c)}{2} \int_{4\mu^2}^{\infty} \frac{dt'}{\bar{s}'(1 - c) - t'(1 + c)} \phi_{23}(\bar{s}', t'). \end{aligned} \quad (\text{A1.3c})$$

Eq. (8) follows immediately.

The integrals with vanishing denominators in the definition of Φ_B and Φ_C can be safely defined as principal value integrals because

$$\begin{aligned} \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{\text{Im}\Phi_B(t', c) dt'}{t' + (s - 4M^2)(1 - c)/2} \\ + \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{\text{Im}\Phi_C(\bar{s}', c) d\bar{s}'}{\bar{s}' + (s - 4M^2)(1 + c)/2} = 0. \end{aligned} \quad (\text{A1.4})$$

Furthermore

$$\text{Im}\Phi_A(s', c) = 0. \quad (\text{A1.5})$$

Therefore Φ_A , Φ_B , and Φ_C can be taken as real functions and we obtain

$$\Phi_A(s, c) = \text{Im}G(s, c), \quad s \geq 4M^2, \quad -1 \leq c \leq +1 \quad (\text{A1.6})$$