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A Generally Relativistic Field Theory.

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CERN - Geneva

(ricevuto il 10 Giugno 1959)

Summary. — In an effort to make contact between field theory and general relativity it is suggested to study space-times embeddable in flat spaces of higher dimensions, as intermediate steps between special and general relativity. The Lorentz invariant action principle formulation of single particle field theory is found to possess a very natural and simple extension to space-times embeddable in a flat six-space. Viewing space-time as a 4-surface in six-space, the metric properties of the former are determined by requiring the action to be stationary with respect to variations of the 4-surface. A more definite theory is obtained by imposing some rather severe restrictions on the form of the Lagrangian. A simple example is studied in detail and is found to describe a pair of Dirac fields interacting through the gravitational field. The main virtue of the theory is that it takes into account the mutual interaction between matter and metric in a consistent manner. It seems plausible that Mach's principle is contained. Second quantization is not carried out, but may be fairly simple by Schwinger's method.

1. — Introduction.

In trying to combine in some way quantum theory with the general theory of relativity it is natural, as with any difficult problem, to do so via several intermediary steps. The stage which may be said to have been reached at present, is the yielding of quantum theory to the requirements of special relativity. It is a very pressing and non-trivial problem to formulate the next stage in such a way that it may be treated more simply than the complete generalization.

A general theorem ⁽¹⁾ states that every 4-dimensional Riemannian space V_4 may be immersed in a 10-dimensional pseudo-Euclidean space S_{10} . On the other hand the special theory of relativity may be said to be the result of specializing to such 4-dimensional spaces as may be immersed in a 4-dimensional flat space. This suggests the successive studies of V_4 's in S_4 , S_5 , S_6 , S_7 , S_8 , S_9 and S_{10} , of which the first has been completed and the last is the ultimate aim.

There is a good reason not to expect the study of V_4 's imbeddable in an S_5 to be fruitful. It may be proved that if such a V_4 satisfies Einstein's equations, then it is a flat space ⁽²⁾. In contrast, at least three facts indicate that the case of V_4 's immersed in an S_6 is sufficiently general to be interesting. First, the Schwarzschild line element, on which the three « crucial » tests of general relativity are based, is a solution of Einstein's equations which is embeddable in an S_6 ⁽³⁾. The cosmological solution of Friedman is also embeddable in S_6 ⁽⁴⁾. Second, a partial generalization of special relativity which has been much discussed is the extension of the class of equivalent observers to include those with relative uniformly accelerated motion ⁽⁵⁾. This requires

⁽¹⁾ It was conjectured by SCHLÄFLI: *Ann. Math.*, (2) 5, 170 (1871), and proved by CARTAN: *Ann. Soc. Polon. Math.*, 6, 1 (1927), that a V_n is embeddable locally in an S_m , where $m = \frac{1}{2}n(n+1)$; see e.g. L. P. EISENHART: *Riemannian Geometry* (Princeton, 1926), p. 188.

When we say that a space is embeddable in a higher dimensional space we mean that it may be embedded locally. Singular points of the embedding may occur, as well as singular lines, etc. The only restriction is that the manifold of singular points must be of dimension less than that of the embedded space. This is the terminology of C. B. ALLENDOERFER: *Duke Math. Journ.*, 3, 317 (1937).

⁽²⁾ E. KASNER: *Am. Journ. Math.*, 43, 126 (1921).

⁽³⁾ E. KASNER: *Am. Journ. Math.*, 43, 130 (1921).

⁽⁴⁾ The common cosmological solutions are all embeddable in an S_6 , except for the Goedel solution. (Private communication from R. J. FINKELSTEIN).

Having been unable to find a proof in the literature, we give a simple proof that the line-element

$$ds^2 = (dx_1^2 + dx_2^2 + dx_3^2)\psi^2(r, t) - dt^2\varphi^2(r, t),$$

is embeddable in an S_6 . In fact, take $Z_i = \psi x_i$, to obtain

$$ds^2 = dZ_1^2 + dZ_2^2 + dZ_3^2 + \left[\left(\frac{\partial r \psi}{\partial r} \right)^2 - \psi^2 \right] dr^2 + 2 \frac{\partial r \psi}{\partial r} \frac{\partial r \psi}{\partial t} dr dt + \left[\left(\frac{\partial r \psi}{\partial t} \right)^2 - \varphi^2 \right] dt^2.$$

Thus the problem has been reduced to that of embedding a V_2 in an S_3 , which is always solvable (ref. ⁽¹⁾).

⁽⁵⁾ See for example E. L. HILL: *Phys. Rev.*, 72, 143 (1947); and J. A. SCHOUTEN: *Rev. Mod. Phys.*, 21, 421 (1949).

physical laws to be invariant under the group C_{4+} of conformal transformations. This group is isomorphic to the group R_6 of rotations in an S_6 ⁽⁶⁾. Thirdly, every V_4 which is conformal to a flat space is embeddable in an S_6 ⁽⁷⁾. It must be stressed that the examples just quoted are only very special cases of V_4 's embeddable in and S_6 ⁽⁸⁾.

The group R_6 of rotations in an S_6 has been found to make contact with theoretical physics in several ways. Perhaps the most important application is due to the close connection with C_{4+} , as was mentioned above. Another connection is found in the circumstance that the simplest two-valued representation of R_6 is on 8-spinors. It has been conjectured that these 8-spinors may correspond to the nucleon wave functions ⁽⁹⁾. Recently it was pointed out that the group of general co-ordinate transformations can be mapped on the group of special linear homogeneous transformations of the «normal» co-ordinates ⁽¹⁰⁾. This group is also isomorphic to R_6 .

Whatever the justification for and merits of theories based on invariance under R_6 , their success has been checked by the very great difficulty of interpretation. Most attempts have been based on the idea that the space-time world must be a surface of such a kind that the transformations induced on it by R_6 be the group C_{4+} . The transformations which correspond to accelerations are then tentatively associated with some intrinsic property of elementary particles ⁽¹¹⁾, e.g. with isotopic spin ⁽⁹⁾. There are several objections to such identifications even within the philosophy on which they are con-

⁽⁶⁾ See for example Y. MURAI: *Progr. Theor. Phys.*, 9, 147 (1953). The precise statement is that C_{4+} is the covering group of R_6 , provided the respective signatures are $+++--$ and $++++--$.

⁽⁷⁾ L. P. EISENHART: loc. cit., p. 214.

⁽⁸⁾ Suppose that a theory can be constructed which will allow us to understand the next stage (that which limits the V_4 's to embeddability in an S_6) as well as we actually understand the physics of special relativity. How would one demonstrate the need for a more complete theory? One would have to devise an experiment to explore a phenomenon which is inconsistent with the metric being embeddable in an S_6 . Such an experiment would have to be quite difficult. At present it is probably not even possible to think of such an experiment. This is because we are unable to formulate a physical problem in a space in which a cosmic time does not exist, which is the reason why Friedman's cosmological solution is embeddable in an S_6 . We might therefore conjecture that space-time is not a general V_4 , but a V_4 which may be thought of as a continuous sequence of 3-spaces. This has the consequence that space-time is embeddable in an S_6 , and excludes e.g. the Goedel solution. This conjecture not only is in very little danger of being contradicted, but has the very attractive feature of restricting the general theory of relativity just enough that we may understand it. This conjecture, however, is not the premise of the present paper.

⁽⁹⁾ See for example R. L. INGRAHAM: *Phys. Rev.*, 106, 595 (1957).

⁽¹⁰⁾ R. J. FINKELSTEIN: *Phys. Rev.*, 109, 1842 (1958).

⁽¹¹⁾ See for example Y. MURAI: *Progr. Theor. Phys.*, 11, 441 (1954).

ceived⁽¹²⁾, although these might conceivably be overcome by a particularly ingenious interpretation. Here, however, we only note that any *fixed* correspondence between quantities in S_6 and observables in V_4 is inconsistent with our present aims. The most important feature of general relativity which we hope to incorporate in a future theory is that the metric is calculable and depends on the matter distribution⁽¹³⁾. Hence the form of the 4-dimensional surfaces in S_6 is to be calculated within the theory, and not axiomatically fixed *a priori*.

2. - A generalized action principle.

Conventional field theory is based on the following considerations. Let a partial specification of a physical system on a 3-space σ be represented by the symbol $\Psi(\sigma)$. The objects under study are the correlations that exist between, say, $\Psi(\sigma_1)$ and $\Psi(\sigma_2)$. Such correlations are represented by the symbol $U(\sigma_2, \sigma_1)$. It is conjectured that to these symbols correspond well-defined mathematical quantities, so that

$$\Psi(\sigma_2) = U(\sigma_2, \sigma_1) \Psi(\sigma_1).$$

The transformation function $U(\sigma_2, \sigma_1)$ may be generated from an infinitesimal transformation

$$\delta \Psi(\sigma) = \delta U(\sigma, \sigma_0) \Psi(\sigma_0),$$

where σ_0 serves as a reference.

The σ 's are restricted to be embeddable in an S_4 , and may be labelled by a single parameter t . Every space-like surface may be considered as a surface in S_4 determined by $t = t(x, y, z)$, where t, x, y, z are the co-ordinates in S_4 . As different surfaces are considered in succession, one may speak of a development of σ in t . It is easy to see that $\delta U(\sigma_2, \sigma_1)$ must have the form of a volume integral

$$(2.1) \quad \delta U(\sigma_2, \sigma_1) = \delta \int_{\sigma_1}^{\sigma_2} L d^4x,$$

in S_4 . Since the co-ordinates in S_4 are determined by the embedded σ only

⁽¹²⁾ See for example R. OMNÈS: *Compt. Rend.*, **245**, 1382 (1957).

⁽¹³⁾ At least to a certain extent, although one would hope to be able to incorporate Mach's principle.

up to a rigid displacement in S_4 ⁽¹⁴⁾, this expression makes sense only if Ld^4x is invariant. If d^4x is the invariant volume element, L must be an invariant.

In the introduction the appropriateness of the S_6 -approximation to general relativity was argued. It is expected that a field theory in such spaces will retain many of the mathematical features of ordinary field theory. One should therefore select a point in the development of the conventional theory where the flatness of 4-space is explicitly assumed, and attempt a generalization. The success of such an attempt may depend critically on the point selected; it is probably important that the generalization, be as direct and as obvious as possible. The scope of the present work is merely to investigate one such point, namely that at which the space-like σ -spaces are assumed to be embeddable in an S_4 . This point would seem particularly appropriate to the S_6 -approximation, since an S_6 is just what is needed to embed an arbitrary 3-dimensional Riemannian space.

Having thus extended the class of allowed space-like σ -spaces, we have the new situation that not all σ -spaces are surfaces in space time. Instead space time is one of many possible «path»'s in the manifold of σ -spaces, and a new dynamical principle is needed to select the actual «path». Fortunately, the action principle discussed above supplies this automatically. To see this note that the integral (2.1) can be viewed geometrically as a surface integral over a four-dimensional surface (*i.e.* space-time) in 6-space. In keeping with the general method of considering all possible variations in (2.1), one must carry out, in addition to the conventional variations, the variation of the «path» of integration. Thus the form of the 4-dimensional surface, *i.e.* the metric of space-time, is determined by making the action principle stationary with respect to the new variation.

Up to now we have only a very general framework for a theory. A definite physical theory is obtained only after restricting in some way the form of the Lagrangian density. The remainder of this paper is devoted to the study of a particular theory, obtained by a set of restrictions imposed on the form of L .

In general L is a function of a certain number of field variables, the components of the metric tensor, and the space-time co-ordinates. As the first restriction we write

$$(I) \quad L = L(\varphi, \varphi_i; g_{ij}, x^i).$$

Here x_i and g_{ik} are the space-time co-ordinates and metric tensor; φ stands for the whole set of field variables, and φ_i for $\partial\varphi/\partial x^i$. The meaning of (I) is

⁽¹⁴⁾ It is even less determined if the metric tensor on σ is degenerate.

that L may involve only φ , the first derivatives of φ , plus g_{ik} and x_i in an arbitrary way. This restriction is always made in conventional theory, and is due mainly to the fact that a more general variational principle is not manageable or well understood.

Instead of (2.1) one may always write

$$(2.2) \quad \delta U(\sigma_2, \sigma_1) = \delta \int_{\sigma_1}^{\sigma_2} \frac{1}{2} L^{\alpha\beta} d\sigma_{\alpha\beta},$$

where $d\sigma_{\alpha\beta}$ is the 4-dimensional surface element, and $L^{\alpha\beta}$ is a skew 6-dimensional tensor. (Latin indices run from 1 to 4, Greek indices from 1 to 6). We now impose the crucial second restriction:

(II) The Lagrangian tensor in (2.2) is independent of the path.

This does not mean that the Lagrangian *density* is independent of the properties of space-time, but it does specify this dependence completely, once $L^{\alpha\beta}$ is known. The meaning of (II) is that no inherent properties, such as a « surface tension », are ascribed to the space-time surface. The inclusion of a free gravitational part is also excluded. It is, of course, nevertheless to be hoped that the conditions on the space-time metric, obtained by varying the path of integration in (2.2), be similar to those of general relativity. The main virtue of the present theory, we feel, is that the mutual interaction of matter and metric is treated in a self consistent manner.

Writing z^ν for the 6-space co-ordinates, let space-time be the surface

$$(2.3) \quad z^\nu = Z^\nu(x).$$

where $Z^\nu(x)$ are six reasonably continuous functions of the four parameters x^i . With $Z^\nu_{,i} \equiv \partial Z^\nu / \partial x^i$, the 4-dimensional surface element is

$$(2.4) \quad \begin{cases} d\sigma_{\alpha\beta} = \varepsilon_{\alpha\beta} d^4x, \\ \varepsilon^{\alpha\beta} = \varepsilon_{\alpha\beta;\delta\epsilon\eta} Z^\gamma_1 Z^\delta_2 Z^\epsilon_3 Z^\eta_4. \end{cases}$$

Similarly for the 3-dimensional surface element

$$(2.5) \quad \begin{cases} d\sigma_{\alpha\beta\gamma} = \varepsilon_{\alpha\beta\gamma}^i d\sigma_i, \\ \varepsilon_{\alpha\beta\gamma}^i = \frac{1}{3!} \varepsilon_{\alpha\beta\gamma\delta\epsilon\eta} \varepsilon^{ijkl} Z^\delta_j Z^\epsilon_k Z^\eta_l. \end{cases}$$

Here d^4x is the invariant volume element of space-time, and $d\sigma_i$ is the 4-vector surface element of a 3-dimensional surface in space-time. The following formulae are useful

$$(2.6) \quad \begin{cases} g_{ij} &= g_{\mu\nu} Z^\mu_i Z^\nu_j, \\ g &= \frac{1}{2} \varepsilon_{\alpha\beta} \varepsilon^{\alpha\beta}, \\ g^{ij} &= \left(\frac{1}{3!g} \right) \varepsilon_{\alpha\beta\gamma}^i \varepsilon^{\alpha\beta\gamma j}, \\ [ij, k] &= g_{\mu\nu} Z^\mu_{ij} Z^\nu_k, \\ Z^\nu_{i,j} &= {}^2\delta_\mu^{\nu} Z^\mu_{ij}. \end{cases}$$

In the last formula appears the normal projection of the Kronecker symbol. The latter may be split into a normal and a tangential part as follows

$$(2.7) \quad \delta_\mu^{\nu} = {}^1\delta_\mu^{\nu} + {}^2\delta_\mu^{\nu} = Z^\nu_i Z^\mu_i + \frac{1}{g} \varepsilon_{\mu\alpha} \varepsilon^{\nu\alpha}.$$

Covariant differentiation with respect to x^i is denoted by a subscript following a comma. The product of $\text{Det } g_{\mu\nu}$ and $\text{Det } g_{ij}$ is denoted g .

Varying both the field quantities φ and the surface functions (2.3), calculation of the r.h.s. of (2.2) gives the result (according to I and II, $L^{\alpha\beta}$ depends on $\varphi(z)$ and on $\varphi_\nu(z) \equiv \partial\varphi(z)/\partial z^\nu$ only):

$$(2.8) \quad \delta U(\sigma_2, \sigma_1) = \int_{\sigma_1}^{\sigma_2} \left\{ \left(\frac{\partial L}{\partial \varphi} - \Pi^i_{,i} \right) \delta\varphi + (\Pi^i \delta\varphi)_{,i} + \frac{\partial L}{\partial \varphi_\nu} {}^2\delta_\nu^\mu \frac{\partial \delta\varphi}{\partial z^\mu} + \varepsilon_{\alpha\gamma} \frac{\partial L^{\alpha\beta}}{\partial z^\beta} \delta Z^\gamma + \frac{1}{2} (L^{\alpha\beta} \varepsilon_{\alpha\beta\gamma}^i \delta Z^\gamma)_{,i} \right\} d^4x.$$

Here L stands for $\frac{1}{2} \varepsilon_{\alpha\beta} L^{\alpha\beta}$. The momenta are defined by

$$(2.9) \quad \Pi^i \equiv \frac{\partial L}{\partial \varphi_\nu} Z^\nu_i.$$

Because L is a scalar density, the covariant divergences occurring in the integral actually reduce to ordinary divergences, so that surface terms may be identified. The meaning of ${}^2\delta_\nu^\mu \partial\delta\varphi/\partial z^\mu$ is the gradient of $\delta\varphi$ normal to the 4-surface, which is evidently independent of $\delta\varphi$ on that surface. Hence, taking variations which vanish on σ_1 and on σ_2 we get a field equation, a subsidiary condition and a path equation:

Field Equation:

$$(2.10) \quad \frac{\partial L}{\partial \varphi} - \Pi^i_{,i} = 0,$$

Subsidiary Condition:

$$(2.11) \quad {}^2\delta_\nu{}^\mu \frac{\partial L}{\partial \varphi_\nu} = 0,$$

Path Equation:

$$(2.12) \quad \varepsilon_{\alpha\gamma} \frac{\partial L^{\alpha\beta}}{\partial z^\beta} = 0.$$

If (2.12) is ignored, and a flat space is assumed in (2.10) and (2.11), the usual flat space theory is formally obtained. Of course, there is no assurance that there exists an $L^{\alpha\beta}$ corresponding to every L in the conventional theory. It is, in fact, very probable that only a very restricted class of L 's may be derived from $L^{\alpha\beta}$'s. This is because of the very strong restriction II, which may eventually turn out to be too severe. Until that has been clearly shown, however, we think that it may deserve further study. In the next section detailed study of the main equations (2.10), (2.11) and (2.12) shall lead us to a third, and last, and much less severe restriction on the structure of $L^{\alpha\beta}$.

3. - The structure of $L^{\alpha\beta}$.

It is well known that the Lagrangian density in ordinary field theory can always, by an appropriate choice of field variables, be written in the form

$$(3.1) \quad L = i\tilde{\varphi} A^\mu p_\mu \varphi - \mathcal{H}(\varphi, \tilde{\varphi}),$$

where $\mathcal{H}$ contains φ and $\tilde{\varphi}$ but not their derivatives. Similarly the Lagrangian tensor $L^{\alpha\beta}$ can always be written

$$(3.2) \quad L^{\alpha\beta} = i\tilde{\varphi} A^{\alpha\beta\mu} p_\mu \varphi - \mathcal{H}^{\alpha\beta}(\varphi, \tilde{\varphi}),$$

where

$$(3.3) \quad \tilde{\varphi} \equiv \varphi^\dagger \eta,$$

and η is a numerical matrix to be determined in such a way that L be hermitian. The numerical matrix $A^{\alpha\beta\mu}$ and the «Hamiltonian» $\mathcal{H}^{\alpha\beta}$ are both skew in α, β . The Lagrangian density

$$L \equiv \frac{1}{2} \varepsilon_{\alpha\beta} L^{\alpha\beta}$$

is given by (3.1), if we define

$$(3.4) \quad A^\mu \equiv \frac{1}{2} \varepsilon_{\alpha\beta} A^{\alpha\beta\mu}, \quad \mathcal{H} \equiv \frac{1}{2} \varepsilon_{\alpha\beta} \mathcal{H}^{\alpha\beta}.$$

The subsidiary condition

$$\varepsilon_{\alpha\gamma}(\partial L/\partial\varphi_\gamma) = 0$$

becomes

$$(3.5) \quad \varepsilon_{\mu\gamma}\varepsilon_{\alpha\beta}A^{\alpha\beta\mu}\varphi = 0.$$

Although it may not be absolutely necessary, we shall limit ourselves to Lagrangians such that (2.5) is identically satisfied:

$$\varepsilon_{\mu\gamma}\varepsilon_{\alpha\beta}A^{\alpha\beta\mu} = 0,$$

Since $A^{\alpha\beta\mu}$ is independent of co-ordinates the factor $\varepsilon_{\mu\gamma}\varepsilon_{\alpha\beta}$ may be deleted, provided due regard is given to the symmetries of this factor. We obtain

Restriction III:

$$(III) \quad A^{\alpha\beta\mu} = A^{[\alpha\beta\mu]},$$

where the right hand side is, by definition, completely antisymmetric. In the following paragraphs this restriction to completely antisymmetric $A^{\alpha\beta\mu}$ will be seen to produce a significant simplification. Indeed it is probable that (III) is a necessary condition for the path equation to be meaningful.

The field equation corresponding to (3.2) is

$$(3.6) \quad \frac{1}{2}\varepsilon_{\alpha\beta}A^{\alpha\beta\mu}\varphi_\mu - \frac{1}{4}A^{\alpha\beta\mu}(\varepsilon_{\alpha\beta}Z_\mu^i)_{,i}\varphi - \frac{\partial\mathcal{H}}{\partial\tilde{\varphi}} = 0.$$

The second term vanishes by virtue of the antisymmetric of $A^{\alpha\beta\mu}$. The first term involves tangential derivatives only. Writing

$$A^i = Z_\alpha^i A^\alpha,$$

the field equation (6) reduces to

$$(3.7a) \quad A^i\varphi_i = \frac{\partial\mathcal{H}}{\partial\tilde{\varphi}},$$

$$(3.7b) \quad \tilde{\varphi}_i A^i = -\frac{\partial\mathcal{H}}{\partial\varphi}.$$

Turning next to the path equation (2.7), we separate the terms which formally involve normal derivations, namely

$$\varepsilon_{\alpha\gamma}{}^2\delta_\beta{}^\delta\frac{\partial L^{\alpha\beta}}{\partial z_\delta} + \varepsilon_{\alpha\gamma}\frac{\partial}{\partial z^\beta}(i\tilde{\varphi}A^{\alpha\beta\mu}p_\mu\varphi),$$

because of the antisymmetry of $A^{\alpha\beta\mu}$ this expression contains no second order derivatives, and reduces to

$$\frac{1}{2} \varepsilon_{\alpha\gamma} {}^2\delta_{\beta}{}^{\delta} (\tilde{\varphi}_{\delta} A^{\alpha\beta\mu} \varphi_{\mu} - \tilde{\varphi}_{\mu} A^{\alpha\beta\mu} \varphi_{\delta}) - \varepsilon_{\alpha\gamma} {}^2\delta_{\beta}{}^{\delta} \left(\frac{\partial \mathcal{H}^{\alpha\beta}}{\partial \varphi} \varphi_{\delta} + \tilde{\varphi}_{\delta} \frac{\partial \mathcal{H}^{\alpha\beta}}{\partial \tilde{\varphi}} \right) + \\ + \frac{1}{2} \varepsilon_{\alpha\gamma} {}^2\delta_{\beta}{}^{\delta} (\tilde{\varphi}_{\beta} A^{\alpha\beta\mu} \varphi_{\delta} - \tilde{\varphi}_{\delta} A^{\alpha\beta\mu} \varphi_{\beta}).$$

The first and the last terms are equal by the antisymmetry of $A^{\alpha\beta\mu}$, and we obtain

$$\varepsilon_{\alpha\gamma} {}^2\delta_{\beta}{}^{\delta} \left\{ \tilde{\varphi}_{\delta} \left(A^{\alpha\beta\mu} \varphi_{\mu} - \frac{\partial \mathcal{H}^{\alpha\beta}}{\partial \tilde{\varphi}} \right) - \left(\tilde{\varphi}_{\mu} A^{\alpha\beta\mu} + \frac{\partial \mathcal{H}^{\alpha\beta}}{\partial \varphi} \right) \varphi_{\delta} \right\}.$$

Use of the identity

$$(3.8) \quad \varepsilon_{\alpha\gamma} {}^2\delta_{\beta}{}^{\delta} - \varepsilon_{\beta\gamma} {}^2\delta_{\alpha}{}^{\delta} + \varepsilon_{\beta\alpha} {}^2\delta_{\gamma}{}^{\delta} = 0,$$

enables us to permute the indices γ, β in the first factor, upon which the whole expression is seen to vanish as a consequence of (2.7). The remainder of the path equation, which does not involve normal derivatives, reads

$$(3.9) \quad \varepsilon_{\alpha\gamma} \left\{ \tilde{\varphi}_i A^{\alpha ij} \varphi_j - Z_{\beta}{}^i \frac{\partial \mathcal{H}^{\alpha\beta}}{\partial x^i} \right\} = 0.$$

Summarizing, the effect of (III) is a), to reduce the subsidiary condition to an identity and b), to ensure the absence of normal derivatives from the Lagrangian density (and hence from the field equation) as well as from the path equation.

A special class of Lagrangians may be obtained in the following way. Every first order « wave-equation » in 6-space:

$$(3.10) \quad (I^{\mu} p_{\mu} + im)\varphi = 0$$

defines a set of six form invariant numerical matrices. From these may be formed

$$(3.11) \quad A^{\alpha\beta\mu} = I^{[\alpha} I^{\beta} I^{\mu]} \equiv \frac{1}{3!} \sum I^{\alpha} I^{\beta} I^{\mu},$$

where the sum is the alternating sum over all permutations of α, β, μ .

The remainder of this paper is concerned with a particular example; except for the digression of the next section, the most useful result of which is a general definition of the covariant derivative.

4. - Digression on motions.

Although the main purpose of this section is to give a general definition of the covariant derivative of field quantities, for use in subsequent sections, this question is discussed in the slightly more general context of arbitrary local motions. Consider a 4-dimensional space-time embedded in an S_6 , and a 3-dimensional space-like surface σ in space-time. The set of all points with co-ordinates

$$(4.1) \quad x'^i = x^i + \delta x^i(x),$$

where x^i is a point on σ and $\delta x^i(x)$ is an infinitesimal-valued function, defines a new surface σ' . Alternatively (4.1) may be viewed as a co-ordinate transformation. Adopting an intermediary interpretation of (4.1) we fix our attention on a particular point $x^i = {}^0x^i$ on σ , and say that (4.1) « moves » the point ${}^0x^i$ to the point ${}^0x'^i = {}^0x^i + \delta x^i({}^0x)$, and in addition defines a co-ordinate transformation in an infinitesimal neighbourhood of ${}^0x^i$. This co-ordinate transformation is given by the derivatives of the function $\delta x^i(x)$, evaluated at ${}^0x^i$. It is convenient to choose the co-ordinate transformation in such a way that certain mathematical quantities have the same values at ${}^0x'$ in the new coordinate system as they had at 0x in the old. Thus it is a well known result that

$$(4.2) \quad \delta g_{ij}(x)|_{x={}^0x} \equiv g'_{ij}({}^0x') - g_{ij}({}^0x) = 0$$

if and only if Killing's equations are satisfied at 0x :

$$(4.3) \quad \delta x_{i,j} + \delta x_{j,i}|_{x={}^0x} = 0.$$

A simple calculation shows that

$$(4.4) \quad \delta \Gamma_{ij}^k(x)|_{x={}^0x} \equiv \Gamma'_{ij}^k({}^0x') - \Gamma_{ij}^k({}^0x) = 0$$

if and only if

$$(4.5) \quad \delta x_{i,jk} + \delta x_{k,ij} - \delta x_{k,ji}|_{x={}^0x} = 0.$$

Invariance, in the sense of (4.2) and (4.4), may be extended to mixed variables, *i.e.* to variables with both 4-vector and 6-vector indices. The 4-surface is determined by the equations

$$(4.6) \quad z^\alpha = Z^\alpha(x).$$

If a particular point 0x on σ is characterized by the 6-co-ordinates ${}^0z^\alpha$, the new point ${}^0x'$ on σ' will have the different co-ordinates

$$(4.7) \quad {}^0z'^\alpha = {}^0z^\alpha + Z^\alpha_{i}({}^0x)\delta x^i({}^0x).$$

Introduce a rigid transformation of the 4-surface to compensate for this difference:

$$(4.8) \quad z^\alpha = y^\alpha + b^\alpha + b^\alpha_{\beta}(y_x - {}^0y^\alpha),$$

where b^α and b^α_{β} are infinitesimal and constant, and

$$(4.9) \quad b^{\alpha\beta} = -b^{\beta\alpha}.$$

We obtain

$${}^0y'^\alpha = {}^0z^\alpha,$$

by taking

$$(4.10) \quad b^\alpha = Z^\alpha_{i}\delta x^i|_{x={}^0x}.$$

while the b^α_{β} remain arbitrary. The equation for the 4-surface is

$$(4.11) \quad y^\alpha = Z^\alpha(x) - b^\alpha - b^\alpha_{\beta}[Z^\beta(x) - Z^\beta({}^0x)] \equiv Y^\alpha(x).$$

Hence

$$(4.12) \quad \delta Z^\alpha(x)|_{x={}^0x} \equiv Y^\alpha({}^0x') - Z^\alpha({}^0x) = 0$$

which expresses the invariance of $Z^\alpha(x)$. If Killing's equations (4.3) are satisfied, it may be arranged that

$$(4.13) \quad \delta Z^\alpha_{i}(x)|_{x={}^0x} = 0,$$

by taking for b^α_{β} the following expression:

$$(4.14) \quad b^\alpha_{\beta} = \{Z^\alpha_{j}Z^\beta_{i}\delta x^j_{,i} + (Z^\alpha_{i,j}Z^\beta_{i} - Z^\alpha_{i}Z^\beta_{i,j})\delta x^j\}|_{x={}^0x} + c^\alpha_{\beta}.$$

Here the first term represents a rotation in a plane tangential to the «path» at 0x . The second and third terms represent rotations in planes with one tangential and one normal direction, respectively two normal directions. The magnitude of the latter remains arbitrary. Finally we investigate the variation of $Z^\alpha_{ij}(x)$. The invariance of the tangential projection:

$$(15) \quad \delta({}^1\delta^\alpha_{\beta}Z^\beta_{ij})|_{x={}^0x} = 0$$

is a consequence of (4) and (13). The variation of the normal projection

$${}^2\delta_\beta{}^\alpha Z^\beta_{ij} = Z^\beta_{i,j}$$

does not vanish, however. Taking $c^\alpha_\beta = 0$, we find

$$(4.16) \quad \delta Z^\alpha_{i,j}(x)|_{x=^0x} = \{ {}^2\delta_\beta{}^\alpha Z^\beta_{i,jk} \delta x^k + Z^\alpha_{i,k} \delta x^k_{,j} + Z^\alpha_{j,k} \delta x^k_{,i} \} |_{x=^0x}.$$

Hence $Z^\alpha_{i,j}(x)$ is a genuine dynamical variable, as is made plausible by noting the following expression for the curvature tensor

$$(4.17) \quad R_{ijk\ell} = g_{\alpha\beta} (Z^\alpha_{i,k} Z^\beta_{j,\ell} - Z^\alpha_{i,\ell} Z^\beta_{j,k}).$$

By a local motion near the point $x = ^0x$ we shall mean the transformation of a space-like surface σ passing through 0x into the space-like surface σ' defined by (1), where $\delta x^i(x)$ differs from zero only infinitesimally close to 0x and satisfies (3) and (5); followed by a rigid displacement of the entire 4-surface given by (8), (10) and (14). A local translation is characterized by

$$(4.18) \quad \delta x_{i,j}|_{x=^0x} = 0, \quad (\text{for translations}),$$

and a local rotation by

$$(4.19) \quad \delta x^i|_{x=^0x} = 0, \quad (\text{for rotations}).$$

Consider an arbitrary geometrical object $\varphi(z)$ in six-space, and let $\varphi(x)$ denote its values on the space-time four-surface. If s^β_α stands for the 6-spin matrices associated with $\varphi(z)$, and s^i_j for their space-time projections, the total change of $\varphi(x)$ under a local motion is

$$(4.20) \quad \delta\varphi(x)|_{x=^0x} = \{ -\delta x^i \nabla_i \varphi(x) + \frac{1}{2} \delta x^j_{,i} \nabla^i_j \varphi(x) \} |_{x=^0x},$$

where

$$(4.21) \quad \nabla_i \varphi(x) = \frac{\partial}{\partial x^i} \varphi(x) - \frac{1}{i} Z^\alpha_{i,\ell} Z^\ell_{\beta} s^\beta_\alpha \varphi(x),$$

$$(4.22) \quad \nabla_{ij} \varphi(x) = [(x_i - ^0x_i) \nabla_j - (x_j - ^0x_j) \nabla_i] \varphi(x) - \frac{1}{i} s_{ij} \varphi(x).$$

The operator ∇_i is the operator of completely covariant differentiation. It contains the conventional definition of the covariant derivative of tensors in the following way, *e.g.*

$$(4.23) \quad \nabla_i \varphi^i \equiv \nabla_i (Z^\alpha_{i} \varphi^\alpha) = Z^\alpha_{i} \nabla_i \varphi^\alpha = \varphi^i_{,i}.$$

For a geometrical object with indices referring to 4-space, to 6-space or to both we have finally

$$(4.24) \quad \nabla_i \varphi(x) = \varphi(x)_{,i} - \frac{1}{i} Z^\alpha_{i,l} Z^\beta_{\beta} s^\beta_{\alpha} \varphi(x).$$

The completely covariant derivatives of g_{ij} , $[ij, k]$, $Z^\alpha(x)$ and Z^α_i are all zero, while

$$(4.25) \quad \nabla_i Z^\alpha_{i,j} = 2\delta^\alpha_{\beta} Z^\beta_{i,jl}.$$

It may be verified that

$$(4.26) \quad \nabla_i s_{ij} = 0.$$

The set of local motions referred to a point 0x almost form a group in the sense that the commutators, *e.g.*

$$[\nabla_i, \nabla_j] \varphi(x),$$

evaluated at $x = {}^0x$, be linearly expressible in terms of ∇_i and ∇_{ij} , evaluated at $x = {}^0x$. A group is obtained by adjunction of a single operator

$$(4.27) \quad \xi = i(-g)^{\frac{1}{2}} \varepsilon_{\alpha\beta} s^{\alpha\beta}.$$

The commutation relations are

$$(28a) \quad [\nabla_i, \nabla_j] \varphi(x)|_{x={}^0x} = \{\frac{1}{2} R_{ijkl} \nabla_{kl} + \kappa_{ij} \xi\} \varphi(x)|_{x={}^0x},$$

$$(28b) \quad [\nabla_i, \nabla_{jk}] \varphi(x)|_{x={}^0x} = \{g_{ij} \nabla_k - g_{ik} \nabla_j\} \varphi(x)|_{x={}^0x},$$

$$(4.28c) \quad [\nabla_{ij}, \nabla_{kl}] \varphi(x)|_{x={}^0x} = \{g_{jk} \nabla_{il} + g_{il} \nabla_{jk} - g_{ik} \nabla_{jl} - g_{jl} \nabla_{ik}\} \varphi(x)|_{x={}^0x},$$

$$(4.28d) \quad [\nabla_i, \xi] \varphi(x) = [\nabla_i, \xi] \varphi(x) = 0,$$

where κ_{ij} are the functions

$$(4.29) \quad \kappa_{ij} = \frac{1}{2} (-g)^{-\frac{1}{2}} \varepsilon_{\alpha\beta} Z^\alpha_{k,j} Z^\beta_{l,i} g^{kl}.$$

5. - Free 8-spinor field.

As a first example let (3.10) describe a spinor field. Then the Γ^μ are 8-by-8 anticommuting hermitian matrices. Taking

$$(5.1) \quad \mathcal{H}^{\alpha\beta} = \bar{\psi} \frac{1}{2} m (\Gamma^\alpha \Gamma^\beta - \Gamma^\beta \Gamma^\alpha) \psi,$$

or

$$(5.2) \quad \mathcal{H} = \bar{\psi} m \sqrt{-g} \xi \psi, \quad \xi \equiv \frac{1}{2\sqrt{-g}} \varepsilon_{\alpha\beta} \Gamma^\alpha \Gamma^\beta,$$

we obtain, using (3.11), the Lagrangian density

$$(5.3) \quad L = i\bar{\psi} \sqrt{-g} \xi (\Gamma^i p_i + im) \psi.$$

A very useful formula is

$$(5.4) \quad \varepsilon_{\alpha\beta} \varepsilon_{\gamma\delta} = g({}^2g_{\alpha\gamma} {}^2g_{\beta\delta} - {}^2g_{\alpha\delta} {}^2g_{\beta\gamma}).$$

By means of (4) it becomes trivial to prove that $\xi^2 = 1$. Furthermore Γ^i commute with ξ . The field equation is

$$(5.5) \quad \left\{ \Gamma^i \frac{\partial}{\partial x^i} + \frac{1}{2} \xi (\xi \Gamma^i)_{,i} - m \right\} \psi = 0.$$

The vanishing of the second term was pointed out already in the general theory. It is nevertheless convenient to study separately the two terms obtained by differentiating ξ and Γ^i . Thus we define

$$(5.6) \quad \Sigma_i \equiv -\frac{1}{2} \xi \xi_{,i}$$

and verify, noting that ξ is a scalar, that

$$(5.7) \quad \Sigma_i = \frac{1}{2} \Gamma^j_{,i} \Gamma_j.$$

Since $\Gamma^j_{,i}$ anticommutes with Γ_k , there follows that

$$(5.8) \quad [\Sigma_j, \Gamma_i] = \Gamma_{i,j}.$$

This property of Σ_i allows us to define spinor-covariant differentiation in such a way that the spinor covariant derivative of Γ_i vanishes. Retaining the notation « $_{,j}$ » for tensor-covariant differentiation, we introduce an operator ∇_i in the following way. If $\psi^{\dots}$ stands for a spinor-tensor with any number of tensor indices:

$$(5.9) \quad \nabla_i \psi^{\dots} \equiv (\psi^{\dots})_{,i} - \Sigma_i \psi^{\dots}.$$

With this notation (8) becomes

$$(5.10) \quad [\nabla_i, \Gamma^j] = 0,$$

and the field equation may be written in two equivalent forms:

$$(5.11) \quad \left(\Gamma^i \frac{\partial}{\partial x^i} - m \right) \psi = 0,$$

or

$$(5.12) \quad (\Gamma^i \nabla_i - m) \psi = \frac{1}{2} \Gamma^i{}_{,i} \psi.$$

We may verify that the present definition of the operator ∇_i agrees with the general definition (4.21). Inserting the expression

$$(5.13) \quad s^\beta{}_\alpha = \frac{1}{4i} (\Gamma^\beta \Gamma_\alpha - \Gamma_\alpha \Gamma^\beta),$$

for the spin operator into (4.21) gives

$$(5.14) \quad \nabla_i \psi = \frac{\partial}{\partial x^i} \psi - \frac{1}{2} \Gamma^j{}_{,i} \Gamma_j \psi,$$

which does indeed coincide with the present definition. The matrices Σ_i correspond to what is conventionally denoted Γ_i .

In order to interpret the theory, we must exhibit the reduction of the wave-function according to the Lorentz-group. Because Ψ furnishes an irreducible representation of the six-dimensional « Lorentz » group this is to diagonalize the projection of (13) on the normal plane. This projection is equal to $\frac{1}{2}\xi$. Hence the irreducible parts of ψ are obtained by the projections:

$$(5.15) \quad \psi_\pm \equiv \frac{1}{2}(1 \pm \xi)\psi.$$

Verifying that

$$(5.16) \quad \xi \Gamma^i \nabla_i = \Gamma^i \nabla_i \xi, \quad \xi \Gamma^i{}_{,i} = -\Gamma^i{}_{,i} \xi,$$

we may write the field equation (12) in four-component form

$$(5.17) \quad \begin{cases} (\Gamma^i \nabla_i - m) \psi_+ = \frac{1}{2} \Gamma^i{}_{,i} \psi_-, \\ (\Gamma^i \nabla_i - m) \psi_- = \frac{1}{2} \Gamma^i{}_{,i} \psi_+. \end{cases}$$

Hence the present example describes a pair of Dirac four-spinor fields interacting through the « gravitational field ».

By a fairly straightforward calculation the path equation (2.12) may be reduced to

$$(5.18) \quad \partial_{\alpha\gamma} [m^2 \bar{\psi} \Gamma^\alpha \psi + g^{ij} \bar{\psi}_i \Gamma^\alpha \psi_j].$$

The simplicity of the set (5.11), (5.18) is remarkable, indicating the possibility of obtaining at least approximate solutions. In a flat space any plane wave solution of (11) is also a solution of (5.18). As soon as one tries to obtain a localized solution by forming a wave packet, however, it becomes impossible to satisfy (5.18). This very reasonable feature of the theory nourishes our hope that Mach's principle may be contained.

It is very simple to construct another theory which may be interpreted as describing a pair of two-component neutrino fields. Equation (3.10) may also be interpreted as a Kemmer equation. The resulting theory has been investigated in detail, but the physical interpretation is not immediately obvious.

6. - Discussion.

The main idea of the theory just presented is that space-times which are locally embeddable in an S_6 are probably sufficiently general that a physical theory in such space-times must exhibit all important features of a generally relativistic theory. If this limited class of space-times can be fully understood, the ultimate generalization is probably a simple matter. What is not the premise of this paper, though an interesting speculation, is the possibility that such final generalization may not be needed. Usually in physics, when the need arises to generalize a theory, it is imprudent to carry the generalization further than absolutely necessary. There is no experimental evidence that space-time is not locally embeddable in an S_6 , and the existence of a cosmic time is very reassuring.

In attempting to build a physical theory in the S_6 approximation, the Lorentz invariant action principle discussed in Section 2 suggests a very direct and obvious generalization. Special relativity is viewed as arising from restricting 3-space to embeddability in an S_4 . Lifting this restriction an action principle in S_6 is obtained, and with it is automatically included the possibility—or indeed the necessity—of varying the «path» of integration. To each «path» corresponds a metric structure of space-time, and that which makes the action stationary is interpreted as that which is actually realized. It is hard to imagine how a lesser departure from conventional theory can be accomplished.

The further development of the theory is not nearly so obvious. We have chosen to reduce the field of possible theories by imposing certain conditions on the Lagrangian. Excluding higher derivatives of field quantities than the first from L (Restriction I) is necessary to make the theory susceptible to a physical interpretation. Replacement of L by the Lagrangian tensor $L^{\alpha\beta}$ is no restriction. Restriction I is then nearly equivalent to the requirement that $L^{\alpha\beta}$ depend on the field quantities and their first derivatives with respect to the six-co-ordinates only. (This near equivalence becomes complete after Restriction III is imposed). Restriction II is probably as controversial as it is restrictive. It is based on an analogy with classical mechanics. The «path» of integration in our action principle is analogous to the path of a particle in classical mechanics. In the latter case the path is not endowed with any intrinsic properties. Restriction II makes exactly this same assumption about the «path» of integration which is space-time.

The example of Section 5 is the simplest one that we have been able to construct. There is no theory corresponding to a single scalar field. Boson theory can be constructed, but involves at least 2 scalar and 2 vector mesons. This is mainly due to Restriction II, but also reflects the extra symmetries introduced by the 6-dimensional formalism. Rotations in the normal plane correspond to an intrinsic symmetry similar to isotopic spin.

The picture of space-time as a surface in 6-space is similar to the physics of a soap-film, except that we have ruled out the possibility of a surface tension. If Restriction II can be shown to be too severe, the re-introduction of surface tension is probably the most plausible generalization. This would ensure the flatness of empty space, and would probably facilitate quantization. The theory as it now stands contains no free gravitational field, and a canonical quantization is not at all obvious. If Schwinger's action principle⁽¹⁵⁾ is really an action principle for quantized fields, the matter field may be quantized in the flat 6-space, but the quantization of the «fields» $Z^a(x)$ remains obscure. A completely different possibility is that the present theory stands in the same relation to a quantized theory as does classical mechanics to quantum theory. The question of quantization is not the most pressing one, however. Of more immediate interest is the comparison with general relativity. This is obstructed by the non-existence of a theory of a single scalar field. The construction of examples must necessarily begin with an $L^{\alpha\beta}$. Hence the simplest way to compare with general relativity is through the example of Section 5. Work on this is in progress. In the meantime it is encouraging that the path equation excludes the formation of wave-packets in flat space, at least in this example.

⁽¹⁵⁾ J. SCHWINGER: *Phys. Rev.*, **82**, 914 (1951).

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RIASSUNTO (*)

Nel tentativo di creare un punto di contatto fra la teoria del campo e la relatività generale si propone di studiare gli spazi-tempi contenibili negli spazi piani a maggior numero di dimensioni come passi intermedi fra la relatività speciale e generale. Si trova che la teoria del campo di una singola particella formulata per mezzo del principio dell'azione invariante di Lorentz è suscettibile di una estensione molto naturale e semplice agli spazi-tempo contenibili in uno spazio piano a sei dimensioni. Considerando lo spazio-tempo come una quadrisuperficie nello spazio a sei dimensioni, si determinano le proprietà metriche del primo richiedendo che l'azione sia stazionaria rispetto alle variazioni della quadrisuperficie. Si ottiene una teoria più esatta imponendo alcune restrizioni piuttosto severe alla forma del lagrangiano. Si studia dettagliatamente un esempio semplice e si trova che esso descrive una coppia di campi di Dirac interagenti attraverso il campo gravitazionale. Il principale pregio della teoria è che essa prende in considerazione in maniera coerente le mutue interazioni fra materia e metrica. Appare plausibile che sia contenuto in essa il principio di Mach. Non si esegue la seconda quantizzazione, tuttavia essa può essere eseguita abbastanza semplicemente ricorrendo al metodo di Schwinger.

(*) Traduzione a cura della Redazione.