a correction to R_0 , has an influence on the first-order change in the bound-state energy, ΔE_1 , which was seen to depend on R_0^2 .

It should be pointed out, in conclusion, that the approximations used here in solving the second Born

approximation are based on the fact that in the region of electron energy considered, the nucleons remain nonrelativistic. For incoming electrons with energies of the order of the nucleon rest mass energy, or more, this method will fail.

PHYSICAL REVIEW

VOLUME 115, NUMBER 2

JULY 15, 1959

Anticommutator for a Nonlinear Field Theory

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(Received February 26, 1959)

The anticommutator for the Thirring model is computed by ordering the operator $\psi(x)\psi^*(x')$ and evaluating its renormalized vacuum expectation value. The infrared divergence is defined by introducing an *ad hoc* cutoff. The final expression does not agree with the approximations obtained by using perturbation theory or by using expansion over intermediate states (with the same cutoff). It is also found that Heisenberg's procedures cannot be applied to this two-dimensional problem.

I

IN recent years, there has been much discussion concerning the form of the anticommutator for bare-particle spinor operators which satisfy nonlinear equations of motion such as¹

$$i\gamma^\mu \partial_\mu \psi + 2g(\bar{\psi}\psi)\psi = 0. \quad (1)$$

It has been suggested² that

$$S_{\alpha\beta}'(x_1, x_2) = i\langle 0 | \{ \psi_\alpha(x_1), \bar{\psi}_\beta(x_2) \} | 0 \rangle$$

resembles a classical solution of Eq. (1) modified by the addition of a mass term near the light cone, and that (in four dimensions) it is "effectively" more regular in this region than $S_{\alpha\beta}(x_1, x_2)$, the corresponding free-field function. The soluble two-dimensional Thirring model³ allows one to check on the first of these speculations for a nonlinear theory, but since g is dimensionless in Thirring's case and $[g] = L^{-2}$ in Heisenberg's, the actual forms of $S'(x)$ in the two problems cannot be directly compared.

At first sight it would seem that the calculation is trivial for the two-dimensional example. Since no lengths are present, it has been pointed out⁴ that the most general form for Lehmann's spectral function⁵ is $\rho(m) = a(g)\delta(m) + b(g)|m|^{-1}$, giving $S'(x) = S(x)[a(g) + b(g)I]$, $I = P \int_0^\infty dx/x(1-x)$. Perturbation theory and expansion over intermediate states also yield this expression,^{3,6} which does not resemble a c -number solution of Eq. (1). However, the infrared

divergence in I leads to an ill-defined space-time dependence for S' . If one tries to specify the divergent term precisely by introducing a cutoff, $k_{\min} = K$, the dimensional argument fails and singular contributions to $\rho(m)$ such as $m^{-1} \times \sin(K/m)$, etc., cannot be excluded, even as $K \rightarrow 0$.

In order to resolve any ambiguity, it is desirable to write the operator $\psi_\tau(x)\psi_{\tau'}^*(x')$ as an ordered functional of the free-field operators ϕ_τ and $\phi_{\tau'}^*$, as in Glaser's treatment⁷ for ψ_τ . This is done in Secs. II, III. In Sec. IV, the renormalized vacuum expectation value is computed and a "covariant" infrared cutoff is introduced. The result is

$$S_{12}'(x_1, t_1; x_2, t_2) = S_{12}(v_1 - v_2) \exp\{[gg'/(2\pi)^2] \ln(L/|u_1 - u_2|)\} \\ u_1, u_2 \ll L, \quad (2)$$

where $v = x - t$, $u = x + t$, $g' = g + 2\pi n$, so that $|g'/2\pi| < 1$, and L is a constant (with dimension of length) which transforms as $L' = \gamma L(1 - \beta)$ under a Lorentz transformation.

The functional dependence of Eq. (2) does not agree with the predictions of perturbation theory (there is, in fact, an essential singularity at $u/L = 0$) or of the intermediate state expansion. Furthermore, it is shown in the last section that Heisenberg's techniques cannot be applied to this two-dimensional example.

II

In Thirring's two-component representation ($\gamma^{(1)} = i\sigma_1$, $\gamma^{(2)} = \beta = \sigma_2$), Eq. (1) becomes

$$\partial\psi_1/\partial u = ig\psi_2^*\psi_2\psi_1, \quad \partial\psi_2/\partial v = -ig\psi_1^*\psi_1\psi_2, \quad (3)$$

⁷ V. Glaser, Nuovo cimento **9**, 990 (1958).

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¹ $\hbar = c = 1$.

² W. Heisenberg, Revs. Modern Phys. **29**, 269 (1957).

³ W. E. Thirring, Ann. Phys. **9**, 91 (1958).

⁴ F. L. Scarf, Nuclear Phys. **11**, 475 (1959).

⁵ H. Lehmann, Nuovo cimento **11**, 342 (1954).

⁶ W. E. Thirring, Nuovo cimento **9**, 1007 (1958).

and the general c -number solutions are

$$\psi_1 = \phi_1(v) U_1(u, -\infty) = \phi_1(v) \exp\left(ig \int_{-\infty}^u \rho_2(u') du'\right), \quad (4)$$

$$\psi_2 = \phi_2(u) U_2(v, \infty) = \phi_2(u) \exp\left(-ig \int_{-\infty}^v \rho_1(v') dv'\right), \quad (5)$$

where $\rho_1(v) = \psi_1^* \psi_1 = \phi_1^* \phi_1$, $\rho_2(u) = \psi_2^* \psi_2 = \phi_2^* \phi_2$. Glaser has quantized the problem by treating ϕ as an incoming field with free-field commutation relations

$$\phi_{1,2} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} dp C_{1,2}(p) \exp[ip(v, u)], \quad (6)$$

$$\{C_\tau(p), C_{\tau'}^*(p')\} = \delta_{\tau\tau'} \delta(p - p'), \quad \{C_\tau(p), C_{\tau'}(p')\} = 0.$$

The physical vacuum is defined by

$$C_1^{(+)}|0\rangle = C_2^{(-)}|0\rangle = 0, \quad C_1^{*(-)}|0\rangle = C_2^{*(+)}|0\rangle = 0.$$

Then, since $[\rho_\tau(x), \rho_{\tau'}(x')] = 0$ for all x, x' , it is possible to treat the ρ_τ as c -numbers so that Eqs. (4), (5) remain valid as operator equations relating ψ to the incoming fields (it is convenient, however, to replace ρ by $:\rho$: in these equations; this eliminates a physically meaningless infinite phase factor).

Because of the Hermitian nature of ρ_τ , it follows that $\psi_1(x_1)\psi_1^*(x_2) = \phi_1(x_1)\phi_1^*(x_2)U_1(u_1, u_2)$ and

$$U_1(u_1, u_2) = U_1 = \exp\left(ig \int_{u_1}^{u_2} :\phi_2^*(u')\phi_2(u'): du'\right). \quad (7)$$

In order to evaluate matrix elements of this operator, we would like to have U in the form suggested by Glaser's work:

$$U_1(u_1, u_2) = \langle U_1 \rangle_0 \left\{ \exp\left[\int dx \int dy H(x, y) \phi_2^*(x) \phi_2(y)\right] \right\}, \quad (8)$$

where $\langle U_1 \rangle_0$ and H are c -numbers. Equation (7) leads to

$$\frac{1}{i} \frac{\partial U_1}{\partial g} = \int_{u_1}^{u_2} du' :\phi_2^*(u')\phi_2(u'): U, \quad (9)$$

and if Eq. (8) is inserted and the vacuum expectation value is taken, this becomes

$$\frac{1}{i} \frac{\partial \langle U_1 \rangle_0}{\partial g} = -\frac{\langle U_1 \rangle_0}{(2\pi)^2} \int_{u_1}^{u_2} du' \int dx \int dy \frac{H(x, y)}{(u' - x - i\epsilon)(u' - y - i\epsilon)}, \quad (10)$$

so that $\langle U_1 \rangle_0$ can be calculated in terms of H . To find

H , assume that

$$[U_1, \phi_2(x)] = G; \quad \phi_2(x) U_1 = \int dy \times \mathcal{G}(x, y) [\phi_2^{(+)}(y) U_1 + U_1 \phi_2^{(-)}(y)] \quad (11)$$

has been evaluated. Then, by considering $\langle 0 | \{C_2^*(p), [C_2(p'), U_1]\} | 0 \rangle$ and inserting in turn Eq. (8) for U_1 and Eq. (11) for the commutator, it can be verified that $H(x, y) = -\mathcal{G}(x, y)$. Thus, the calculation of U_1 in the ordered exponential form is reduced to evaluation of $\mathcal{G}(x, y)$.⁸ We shall order $\psi_1 \psi_1^*$ below; the corresponding result for $\psi_2 \psi_2^*$ can be obtained from this by letting $x \rightarrow -x$.

III

Glaser has shown that

$$\phi_2(x) U_1(u, -\infty) = U_1(u, -\infty) \phi_2(x) \exp[ig\theta(u-x)].$$

This equation and the relationship

$$U_1(u_1, u_2) = U_1^{-1}(u_2, -\infty) U_1(u_1, -\infty)$$

lead to

$$U\phi = \phi U - (e^{-ig} - 1) e^{ig\theta(u_2 - u_1)} \times [\theta(u_2 - x) - \theta(u_1 - x)] \phi U, \quad (12)$$

and

$$\phi U = U\phi - (e^{ig} - 1) e^{-ig\theta(u_2 - u_1)} \times [\theta(u_2 - x) - \theta(u_1 - x)] U\phi, \quad (13)$$

where the indices on ϕ, U are suppressed. The positive- and negative-frequency parts of ϕ are defined by $\phi^{(+)} + \phi^{(-)} = \phi$,

$$\phi^{(\pm)}(x) = P^{(\pm)} \phi(x) = \pm \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dy \phi(y)}{y - x \mp i\epsilon}. \quad (14)$$

When $P^{(-)}$ is applied to (12) and $P^{(+)}$ to (13), they become

$$[U, \phi^{(-)}] = K^{(-)} \phi U = \int_{u_1}^{u_2} dy \times [\phi^{(+)}(y) U + \phi^{(-)}(y) U] \mathcal{K}^{(-)}(x, y), \quad (15)$$

$$[\phi^{(+)}, U] = K^{(+)} U \phi = \int_{u_1}^{u_2} dy \times [U \phi^{(+)}(y) + U \phi^{(-)}(y)] \mathcal{K}^{(+)}(x, y), \quad (16)$$

⁸ In the original calculation, U_1 was ordered by transforming Eq. (11) to momentum space. This led to an integral equation,

$$f(p) + \lambda \int_0^\infty dk f(k) \sin(p-k)/(p-k) = g(p),$$

which was solved by letting $\sin x/x \rightarrow e^{-\alpha|x|} \sin x/x$ and using the Wiener-Hopf technique for the modified equation. The method of solution is nonrigorous and the solution and consequent integrations have an extremely complicated form which could not be simplified appreciably. I am greatly indebted to Dr. V. Glaser for suggesting the present rigorous and relatively simple configuration-space approach and especially for pointing out that the latter method naturally leads to a reduction of the number of integrations in Eq. (10) [see Eq. (23), below]. Subsequent comparison shows that the two calculations give equivalent expressions for $H(x, y)$ but different values for $\langle U_1 \rangle_0$ (unrenormalized).

with

$$\mathcal{K}^{(\pm)}(x, y) = \mp \frac{(1 - e^{\mp i\vartheta})}{2\pi i} \frac{1}{y - x \mp i\epsilon}, \quad u_2 > u_1. \quad (17)$$

In symbolic notation, Eqs. (15) and (16) are

$$\begin{aligned} [1 + K^{(+)}]U\phi^{(+)} &= \phi^{(+)}U - K^{(+)}U\phi^{(-)}, \\ [1 + K^{(-)}]\phi^{(-)}U &= U\phi^{(-)} - K^{(-)}\phi^{(+)}U. \end{aligned}$$

Inverse operators, defined by

$$[1 + G^{(\pm)}][1 + K^{(\pm)}] = 1,$$

obey

$$[1 + G^{(\pm)}]K^{(\pm)} = -G^{(\pm)},$$

so that

$$[U, \phi^{(+)}] = G^{(+)}; \phi U; \quad \text{and} \quad [\phi^{(-)}, U] = G^{(-)}; \phi U;$$

Thus $[U, \phi] = G; \phi U$; with $G = G^{(+)} - G^{(-)}$. The inverse operators are constructed in the Appendix by solving the integral equations $[1 + K^{(\pm)}]f = g$ in the form $f = [1 + G^{(\pm)}]g$. The results are

$$\begin{aligned} G^{(\pm)}f(x) &= \int_{u_1}^{u_2} dy \mathcal{G}^{(\pm)}(x, y) f(y) = \pm \frac{(1 - e^{\pm i\vartheta})}{2\pi i} \int_{u_1}^{u_2} dy \\ &\times \left(\frac{y_{\pm} - u_1}{x_{\pm} - u_1} \right)^{\vartheta'/2\pi} \left(\frac{u_2 - y_{\pm}}{u_2 - x_{\pm}} \right)^{-\vartheta'/2\pi} \frac{f(y)}{x_{\pm} - y}, \end{aligned} \quad (18)$$

where $x_{\pm} = x \pm i\epsilon$ and $(x_{\pm} - u_1)^{\vartheta'/2\pi} (u_2 - x_{\pm})^{-\vartheta'/2\pi}$ is defined to be real when x is real and $u_1 < x < u_2$.

The subsequent calculations can be simplified with the aid of the identity $[(x - i\epsilon)^{-1} - (x + i\epsilon)^{-1}] = 2\pi i \delta(x)$; then $\mathcal{G}^{(-)}$ may be expressed in terms of $\mathcal{G}^{(+)}$ and

$$\begin{aligned} \mathcal{G}(x, y) &= (1 - e^{-i\vartheta})[\delta(x - y) + \mathcal{G}^{(+)}(x, y)], \\ &= 0, \quad \text{otherwise,} \end{aligned} \quad u_1 \leq (x, y) \leq u_2 \quad (19)$$

so that Eq. (10) can be written as

$$\begin{aligned} \frac{1}{i} \frac{\partial \langle U_1 \rangle_0}{\partial g} &= \frac{(1 - e^{-i\vartheta})}{(2\pi)^2} \int_{u_1}^{u_2} du \int_{u_1}^{u_2} dx \\ &\times \int_{u_1}^{u_2} dy \frac{[\delta(x - y) + \mathcal{G}^{(+)}(x, y)]}{(u' - x - i\epsilon)(u' - y - i\epsilon)} \langle U_1 \rangle_0. \end{aligned} \quad (20)$$

The factor $(u' - y - i\epsilon)^{-1}$ in Eq. (20) is equal to $\mathcal{K}^{(+)}(y, u')$ times a constant. Accordingly, using

$$[1 + G^{(+)}]K^{(+)} = -G^{(+)},$$

the integration over y may be performed to give

$$\frac{1}{i} \frac{\partial \langle U_1 \rangle_0}{\partial g} = \frac{-i}{2\pi} \int_{u_1}^{u_2} du' \int_{u_1}^{u_2} dx \frac{\mathcal{G}^{(+)}(x, u')}{(u' - x - i\epsilon)} \langle U_1 \rangle_0, \quad (21)$$

where the integrand is now of the form $\mathcal{G}^{(+)}(x, u')$

$\times \mathcal{K}^{(-)}(u', x)$. As before, $[1 + G^{(-)}]K^{(-)} = -G^{(-)}$ or

$$\begin{aligned} &\int_{u_1}^{u_2} du' \int_{u_1}^{u_2} dx \mathcal{G}^{(-)}(z, u') \mathcal{K}^{(-)}(u', x) f_z(x) \\ &= \int_{u_1}^{u_2} du' [\mathcal{G}^{(-)}(z, u') + \mathcal{K}^{(-)}(z, u')] f_z(u'), \end{aligned} \quad (22)$$

and furthermore,

$$\mathcal{G}^{(+)}(x, u') = [e^{i\vartheta} \mathcal{G}^{(-)}(x, u') + (e^{i\vartheta} - 1) \delta(x - u')].$$

When $\mathcal{G}^{(+)}$ in Eq. (21) is written in terms of $\mathcal{G}^{(-)}$, and $\mathcal{G}^{(-)}\mathcal{K}^{(-)}$ is reduced using Eq. (22) [with $\lim_{f_z(x)} = \delta(x - z)$], an indeterminate form is obtained for the last integrand. Using L'Hôpital's rule, this gives

$$\frac{1}{i} \frac{\partial \langle U_1 \rangle_0}{\partial g} = \frac{ig'}{(2\pi)^2} \int_{u_1}^{u_2} du' \left[\frac{1}{u_+ - u_1} + \frac{1}{u_2 - u_+} \right] \langle U_1 \rangle_0, \quad (23)$$

so that

$$\langle U_1 \rangle_0 = \exp \left\{ \frac{-gg'}{2(2\pi)^2} \int_{u_1}^{u_2} du' \left[\frac{1}{u_+ - u_1} + \frac{1}{u_2 - u_+} \right] \right\}. \quad (24)$$

IV

The final expression [Eq. (24)] has $\langle U_1 \rangle_0 = 1$ for $u_1 = u_2$; $\langle U_1 \rangle_0 = 0$, otherwise. The result is not meaningful because it is a matrix element of unrenormalized operators.⁹ Although Glaser has calculated $Z^{\frac{1}{2}}$ in momentum space, it is necessary to repeat the calculation in configuration space (the two computations give different values for the renormalization constant). In the latter case, $Z^{\frac{1}{2}} = \langle U_1^{-1}(u_2, -\infty) \rangle_0$ may be obtained by setting $u_1 = -\infty$ in Eqs. (18), (20):

$$Z^{\frac{1}{2}} = \exp \left\{ \frac{-gg'}{2(2\pi)^2} \int_{-\infty}^{u_2} \frac{du'}{u_2 - u_+} \right\}. \quad (25)$$

A corresponding substitution gives $\langle U_1(u_1, -\infty) \rangle_0$. Note that $\lim_{u_2 \rightarrow -\infty} \langle U_1(u_1, u_2) \rangle_0$ is *not* equal to $\langle U_1(u_1, -\infty) \rangle_0$. The renormalized matrix element, defined as $\langle U \rangle_0^R = Z^{\frac{1}{2}} Z^{*\frac{1}{2}} \langle U \rangle_0$, has the form

$$\begin{aligned} \langle U_1 \rangle_0^R &= \exp \left\{ \frac{gg'}{2(2\pi)^2} \left[\int_{-\infty}^{u_2} \frac{du'}{u_1 - u_+} \right. \right. \\ &\quad \left. \left. + \int_{-\infty}^{u_1} \frac{du'}{u_2 - u_+} \right] \right\}. \end{aligned} \quad (26)$$

The divergent quantity $\langle U_1 \rangle_0^R$ is independent of u_1, u_2 . In this sense, the results of perturbation theory,

⁹ The unrenormalized expression for U does, in fact, occur in certain equations, such as $U_R(u_2, -\infty) = U(u_2, u_1)U_R(u_1, -\infty)$. This would appear to be nonsensical for $u_2 \neq u_1$, but the product operator is actually defined by $UU_R = \sum_N U|N\rangle\langle N|U_R$, and the expansion diverges. These considerations are connected with Haag's theorem [R. Haag, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **29**, 12 (1955)].

expansion over intermediate states, and arguments based on Lehmann's representation are reproduced; $S^{(+)'}$ is simply $S^{(+)}$ multiplied by an infinite constant. However, inspection of Eq. (26) shows that the divergence is entirely of the infrared type. There is no singular contribution for $u' \simeq u_1$, u_2 and it should be possible to introduce a covariant cutoff into the definition of $Z^{\frac{1}{2}}$.

Since an infrared divergence is associated with massless virtual particles, one can try to account for a finite mass, μ , in a phenomenological manner. We observe that the integral in Eq. (25) can be interpreted as a sum of contributions from all space-time points $u' = (x' + t') < u_2$ along the past light cone, weighted by the propagator, $(u_2 - u')^{-1}$. For massive particles, it would then be expected that propagation over distances $\Delta x \gg \lambda_{\text{Compton}}$ or $\Delta x \gg \mu^{-1}$ is attenuated. Furthermore, virtual particles would not exist for times $\Delta t \gg E^{-1}$, or $\Delta t \gg \mu^{-1}$. However, $Z^{\frac{1}{2}}$ contains contributions only from points on the light cone and all trajectories have the same proper distance and proper time; an exact calculation with $\mu \neq 0$ at all stages is required to eliminate the infrared divergence in this manner.

Another, more successful, *ad hoc* cutoff is introduced by considering the system placed in a box of length L . Then the maximum contribution to the integral comes from the space-time point $(-L/2, -L/2)$, or $u_{\text{max}} = x_{\text{max}} + t_{\text{max}} = -L$. [Note: u_{max} transforms like $x + t$ so that $u_{\text{max}}' = \gamma u_{\text{max}} (1 - \beta)$ under a Lorentz transformation.] Of course, the exact solution with box normalization should use standing waves, but if the box is sufficiently large, the cutoff can be inserted directly into Eq. (26) with negligible error. The result is

$$\langle U_1(u_1, u_2) \rangle_0^R = \exp\{[gg'/(2\pi)^2] \ln(L/|u_1 - u_2|)\}, \quad u_1, u_2 \ll L. \quad (27)$$

The L dependence of Eq. (27) differs strongly from the logarithmic divergence predicted by perturbation theory with the same cutoff. In fact, $\langle U_1 \rangle_0^R$ can be expanded in a power series in g only if $(gg'/8\pi^2) \times [\ln(L/|u_2 - u_1|)] < 1$ and the radius of convergence tends to zero as $L \rightarrow \infty$.

Furthermore, $\langle U_1(u_2, u_2) \rangle_0^R = \infty$ whether or not an infrared cutoff is present [this follows from $\{\psi^R, \psi^{R*}\} = Z^{\frac{1}{2}} Z^{*\frac{1}{2}} \delta(x - x')$, $t = t'$]. However,

$$\langle U_1(u_2, u_2) \rangle_0^R = \sum_n \sum_p \langle 0 | \psi^R(u_2) | n, p \rangle \langle n, p | \psi^{R*}(u_2) | 0 \rangle$$

(n is the number of particles and p refers to their momenta), and with an infrared cutoff, each term in the series over n is finite. Thus, the expansion over intermediate states *must* diverge.

Therefore, to the extent that this cutoff is meaningful, one can conclude that both expansions yield poor approximations to the exact function, S' . These results also indicate that the spectral density $\rho(m)$ for a system in a box may be considerably different from that pre-

dicted on dimensional grounds when no length is present.

V

Heisenberg considers the quantity

$$\chi_\alpha(x, x') = \{\exp[iR(x')]\} \psi_\alpha(x) \{\exp[-iR(x')]\}, \quad (28)$$

where $R(x') = [a^\nu \psi_\nu^*(x') + a^{*\nu} \psi_\nu(x')]$, ψ is a solution of Eq. (1), and a^ν is constant spinor with

$$\{a^\nu, \psi\} = \{a^{*\nu}, \psi\} = 0.$$

Clearly, χ satisfies Eq. (1) as a function of x , but not as a function of x' . A further property of χ is

$$\lim_{a^\nu \rightarrow 0} \partial \chi_\alpha / \partial a^\nu = i \{\psi_\nu^*(x'), \psi_\alpha(x)\}. \quad (29)$$

In the four-dimensional case, Heisenberg assumes $\chi = \chi^0 + c$ where c is a singular c -number function and χ^0 is an operator regular near the relative light cone. It is then argued that $\langle (\bar{\chi} \chi) \chi \rangle_0 \simeq (\bar{c} c) + \langle \bar{\chi}^0 \chi^0 \rangle_0 c$ near the light cone. Setting $\langle \chi \rangle_0 \simeq c$ in this region, Heisenberg examines classical solutions of

$$i\gamma^\mu \partial_\mu c + 2g(\bar{c} c) c + \kappa c \simeq 0, \quad \kappa \simeq g \langle \bar{\chi}^0 \chi^0 \rangle_0, \quad (30)$$

for $x \simeq x'$ and takes

$$\lim_{a^\nu \rightarrow 0} \partial \chi_\alpha / \partial a^\nu = i \langle 0 | \{\psi_\nu^*(x'), \psi_\alpha(x)\} | 0 \rangle. \quad (31)$$

This procedure is not valid in the two-dimensional case. First, consider Eq. (1): the most general solution for χ consistent with $[\psi_\alpha(x)]^2 = 0$ or $[\chi_\alpha(x, x')]^2 = 0$ is

$$\chi_1(x, x') = \chi_1^{\text{in}}(v, x')$$

$$\times \exp \left[ig \int_{-\infty}^u \chi_2^{\text{in}*}(y, x') \chi_2^{\text{in}}(y, x') dy \right], \quad (32)$$

$$\chi_2(x, x') = \chi_2^{\text{in}}(u, x')$$

$$\times \exp \left[-ig \int_{-\infty}^v \chi_1^{\text{in}*}(y, x') \chi_1^{\text{in}}(y, x') dy \right],$$

where the operator χ^{in} is defined by setting $\psi_\alpha(x) \rightarrow \psi_\alpha^{\text{in}}(x)$ in Eq. (28). These expressions cannot be decomposed into sums of regular operators plus singular c -numbers: one can, of course, set $\chi = c + (\chi - c)$ but then $\chi^0 = (\chi - c)$ is also singular.¹⁰

Secondly, although one component of Eq. (30) is

$$\partial c_1 / \partial u - ig(c_2^* c_1 - c_1^* c_2) c_2 + i\kappa c_2 \simeq 0, \quad (33)$$

the exact function satisfies

$$\frac{\partial S_{12}'}{\partial u} - \frac{gg'}{(2\pi)^2 |u - u'|} S_{12}' = 0. \quad (34)$$

¹⁰ A more significant observation is that $\{\psi_1(x), \psi_1^*(x')\} = \{\phi_1(v), \phi_1^*(v')\} U_1(u, u')$ is a singular operator. Thus, the anti-commutator itself cannot be written as a singular c -number plus a bounded operator.

The anticommutator does indeed obey a differential equation which differs from $\gamma^\mu \partial_\mu S = 0$ but it is a singular, linear equation with no mass term [as classical equations, (1) and (33) contain terms of the form $\phi_2^* \phi_1$ but since $\phi_1^{2\gamma} = 0$ for a spinor operator such a term does not appear in the operator equations]. It may also be noted that Eq. (33) would seem to predict oscillatory behavior for S' , however, as $a' \rightarrow 0$, $c_\alpha \rightarrow 0$ [$\chi_\alpha \rightarrow \psi_\alpha$ which is not singular] and Eqs. (31), (33) may be combined to give

$$\partial S_{12}' / \partial u + i\kappa S_{22}' = 0. \quad (35)$$

Since $S_{22}' = 0$, it can be seen that Heisenberg's method is not applicable to the Thirring model.

ACKNOWLEDGMENTS

I am indebted to Professor V. Glaser for his valuable suggestions and for many discussions. I also wish to acknowledge helpful conversations with Professor W. Thirring and Dr. R. Ascoli, and to thank Professor C. J. Bakker and Professor B. Ferretti for the hospitality extended to me at CERN.†

APPENDIX

The equations $[1 + K^{(\pm)}]f = g$ may be solved using the methods of reference 7. For instance, if one lets $h(x) = f(x) - g(x)$, the $K^{(+)}$ equation is

$$h(x) + \frac{(e^{-ig} - 1)}{2\pi i} \int_{u_1}^{u_2} dy \frac{[h(y) + g(y)]}{y - x} = 0, \quad (A.1)$$

which implies that $h(x)$ is the limiting value of a func-

† *Note added in proof.*—We have recently determined the exact solutions for the Thirring model with $g = g(x, t)$. No infrared divergence appears if $g \rightarrow 0$ as $u, v \rightarrow -\infty$. A variation $g = g_0 \theta(L + u) \theta(L + v)$ gives the same results as those obtained with the *ad hoc* cutoff of Sec. IV, and all of the conclusions contained here are rigorously valid for this modified field theory.

tion $h(x_+)$, analytic in the entire complex plane, except for a branch cut along the real axis between u_1 and u_2 . Then,

$$h(x_+) - h(x_-) = -(e^{-ig} - 1)[h(x_+) + g(x)], \quad u_1 < x < u_2, \quad (A.2)$$

$$= 0, \quad \text{otherwise.}$$

Let $h(x) = (x - u_1)^{-g'/2\pi} (u_2 - x)^{g'/2\pi} S(x)$, where $g' = g + 2\pi n$, and define the factors to be real for $x = |x| + i\epsilon$, $u_1 < |x| < u_2$. If $|g'/2\pi|$ is always chosen to be less than unity, there is a single-valued continuation from x_+ to x_- and

$$e^{-ig} h(x_+) - h(x_-) = e^{-ig} (x_+ - u_1)^{-g'/2\pi} (u_2 - x_+)^{g'/2\pi} [S(x_+) - S(x_-)], \quad u_1 < x < u_2. \quad (A.3)$$

Equations (A.2) and (A.3) yield

$$S(x_+) - S(x_-) = -(1 - e^{+ig})(x_+ - u_1)^{g'/2\pi} \times (u_2 - x_+)^{-g'/2\pi} g(x), \quad (A.4)$$

which finally leads to

$$f(x) = g(x) + \int_{u_1}^{u_2} dy \mathcal{G}^{(+)}(x, y) g(y)$$

$$= g(x) + \frac{(1 - e^{ig})}{2\pi i} \int_{u_1}^{u_2} dy \left(\frac{y_+ - u_1}{x_+ - u_1} \right)^{g'/2\pi} \left(\frac{u_2 - y_+}{u_2 - x_+} \right)^{-g'/2\pi} \frac{g(y)}{x_+ - y}. \quad (A.5)$$

The equation corresponding to (A.2) for the $K^{(-)}$ operator is just the complex conjugate of (A.2) for real valued h, g . Thus, $\mathcal{G}^{(-)}$ is defined by taking the complex conjugate of (A.5), assuming f, g real. The factor $(x_- - u_1)^{-g'/2\pi} (u_2 - x_-)^{g'/2\pi}$ is chosen to be real when x is real and between u_1 and u_2 .