On scale transformations
Wess, J

Published in Nuovo Cimento 14 (1959) 527-531

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From: Angela Oleandri <oleandri@sif.it>
Date: October 12, 2007 4:07:47 PM GMT+02:00
To: Salvatore Mele <Salvatore.Mele@cern.ch>
Subject: Re: Request for permission to reproduce published material.

Dear Dr Mele,

further to your request, sent in July to Carmen Vasini (herewith attached)
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On Scale Transformations.

J. WESS

CERN - Geneva

(ricevuto il 7 Luglio 1959)

Summary. — Invariance against scale transformation was used by W. HEISENBERG ⁽¹⁾ to define a lepton number. Some properties of this transformation are more easily investigated with scalar fields, the mass dependence of which is known. For the scalar fields and the harmonic-oscillator, some properties of the infinitesimal transformation are listed here.

The field equation

$$(1) \quad \partial^\mu \partial_\mu \varphi(x, m) - m^2 \varphi(x, m) = 0$$

and the commutation relation

$$(2) \quad [\varphi(x, m), \varphi(y, m)] = i \Delta(x - y, m)$$

are invariant against the transformation

$$(3) \quad \left\{ \begin{array}{l} x \rightarrow x' = \frac{1}{\lambda} x, \\ m \rightarrow m' = \lambda m, \\ \varphi(x, m) \rightarrow \varphi'(x', m') = \lambda \varphi(xm), \\ \text{or} \\ \varphi'(xm) = \lambda \varphi\left(\lambda x, \frac{m}{\lambda}\right), \end{array} \right.$$

in the sense that

$$(1a) \quad \partial^\mu \partial_\mu \varphi'(x, m) - m^2 \varphi'(xm) = 0$$

and

$$(2a) \quad [\varphi'(xm), \varphi'(y, m)] = i \Delta(x - y, m)$$

so that we might expect that there exists a unitary transformation matrix U

$$(4) \quad U\varphi(x, m)U^{-1} = \varphi'(x, m).$$

If we develop $\varphi(x, m)$ in a Fourier integral:

$$(5) \quad \varphi(x, m) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3k}{\sqrt{2k_0}} [a(\mathbf{k}) \exp[-ikx] + a^*(\mathbf{k}) \exp[-ikx]]$$

with

$$k_0 = \sqrt{\mathbf{k}^2 + m^2}.$$

the $a(\mathbf{k})$ should only depend on $\mathbf{k}$, and not on m , which means, the $a(\mathbf{k})$ act in the same Hilbert space for all m . The mass dependence of $\varphi(x, m)$ is then simply given through k_0 .

From (5), (4) and (3) follows

$$(4a) \quad Ua(\mathbf{k})U^{-1} = \frac{1}{\lambda^{\frac{3}{2}}} a\left(\frac{\mathbf{k}}{\lambda}\right).$$

We see that the transformation is only a permutation of the k variables in the same Hilbert space.

For the infinitesimal transformation $\lambda = 1 + \varepsilon$ we have

$$(3a) \quad \begin{cases} x'' = (1 - \varepsilon)x, \\ m' = (1 + \varepsilon)m, \\ \varphi'(xm) - \varphi(xm) = \varepsilon \left(\varphi + x_\nu \frac{\partial \varphi}{\partial x_\nu} - m \frac{\partial \varphi}{\partial m} \right) = \delta_0 \varphi, \end{cases}$$

and with

$$\begin{aligned} U &= 1 + i\varepsilon F, \\ [a(\mathbf{k}), F] &= -i\left(\frac{3}{2}a(\mathbf{k}) + \mathbf{k} \operatorname{grad}_{\mathbf{k}} a(\mathbf{k})\right), \\ [a^*(\mathbf{k}), F] &= -i\left(\frac{3}{2}a^*(\mathbf{k}) + \mathbf{k} \operatorname{grad}_{\mathbf{k}} a^*(\mathbf{k})\right). \end{aligned}$$

We wish now to compute F in terms of the field operators with the help of the action principle,

$$(6) \quad \delta W = F(\sigma) - F(\sigma_0),$$

where

$$(7) \quad W = \int_{\Omega} \mathcal{L}(x, m) d^4x$$

and

$$(6a) \quad \delta W = \int_{\Omega'} \mathcal{L}'(x', m') d^4x' - \int_{\Omega} \mathcal{L}(x, m) d^4x.$$

If we use

$$(8) \quad \mathcal{L}'(x' m') = \mathcal{L}'(xm) + \delta x \frac{d\mathcal{L}}{dx} + \delta m \frac{d\mathcal{L}}{dm}$$

we find

$$(6b) \quad \delta W = \left(\int_{\sigma} - \int_{\sigma_0} \right) d\sigma_{\mu} \left(\frac{\partial \mathcal{L}}{\partial \varphi_{\mu}} \delta_0 \varphi + \mathcal{L} \delta x^{\mu} \right) + \delta m \frac{d}{dm} \int_{\Omega} \mathcal{L} d^4x.$$

$\delta_0 \varphi$ is given by (3a) and the φ satisfy the eq. (1).

The lagrangian for a scalar field is

$$(9) \quad \mathcal{L} = -\frac{1}{2}(m^2 \varphi^2 - \varphi_{,\lambda} \varphi^{,\lambda})$$

and δW becomes for a volume Ω , which is the whole space between two planes $x_0 = x_0 = \text{const}$ and $x_0 = x'_0 = \text{const}$

$$(10) \quad \delta W = \left(\int_{x_0} - \int_{x'_0} \right) d^3x \left(-\mathcal{L} x_0 + \frac{\partial \varphi}{\partial x_0} \left(\varphi + \frac{\partial \varphi}{\partial x_{\nu}} x_{\nu} - m \frac{\partial \varphi}{\partial m} \right) \right) + m \frac{d}{dm} \int_{\Omega} \mathcal{L} d^4x.$$

In the momentum space, we can carry out the mass differentiation and we get:

$$(11) \quad \frac{d}{dm} \int_{\Omega} \mathcal{L} d^4x = -\frac{mx_0}{2} \int \frac{d^3k}{k_0} [a(\mathbf{k}) a(-\mathbf{k}) \exp[-2ik_0 x_0] + a^*(\mathbf{k}) a^*(-\mathbf{k}) \exp[2ik_0 x_0]] \Big|_{x'_0}^{x_0}$$

and finally

$$(10a) \quad F(x_0) = \frac{i}{2} \int d^3k \left\{ a(\mathbf{k}) a(-\mathbf{k}) \exp[-2ik_0 x_0] \left(\frac{3}{2} - i \frac{x_0}{k_0} k^2 \right) + a^*(\mathbf{k}) a^*(-\mathbf{k}) \exp[2ik_0 x_0] \left(-\frac{3}{2} - i \frac{x_0}{k_0} k^2 \right) + \mathbf{k} [\exp[-2ik_0 x_0] (\text{grad}_{\mathbf{k}} a(\mathbf{k})) a(-\mathbf{k}) - \exp[2ik_0 x_0] (\text{grad}_{\mathbf{k}} a^*(\mathbf{k})) a^*(-\mathbf{k}) - (\text{grad}_{\mathbf{k}} a(\mathbf{k})) a^*(\mathbf{k}) + (\text{grad}_{\mathbf{k}} a^*(\mathbf{k})) a(\mathbf{k})] \right\}.$$

It can easily be shown that (4b) is satisfied. We also see that F is not diagonal in the particle number $N_k = a(\mathbf{k})a^*(\mathbf{k})$, although $\int N_k d^3k$ commutes with F .

Under the condition that the expression (11), integrated over the mass, gives zero, HEISENBERG derived in his paper ⁽¹⁾, a constant of motion, which in our case is $\int (dm/m) F(x_0)$. Generally, this is only constant when we integrate from $-M$ to $+M$, but then in our case also this constant of motion vanishes, since $F(x_0)$ is an even function in m .

For the harmonic oscillator with the Lagrangian

$$(12) \quad \mathcal{L} = \frac{1}{2}(\dot{q}^2 - \omega^2 q^2),$$

$$(13) \quad q = \frac{1}{\sqrt{2\omega}} [a \exp[i\omega t] + a^* \exp[-i\omega t]],$$

where a is independent of ω , we can define the scale transformation:

$$(14) \quad \begin{cases} t' = \lambda t, \\ \omega' = \frac{1}{\lambda} \omega, \\ q'(t', \omega') = \sqrt{\lambda} q(t, \omega), \end{cases}$$

which clearly leaves the commutator relation $[q, \dot{q}] = i$ invariant. Here we have

$$(15) \quad \delta q_0 = q'(t, \omega) - q(t, \omega) = \varepsilon \left(\frac{1}{2} q - t \dot{q} + \omega \frac{\partial q}{\partial \omega} \right) = 0;$$

this follows from (13).

Also from the action principle, we find $F=0$ or $U=1$.

For the harmonic oscillator we can also define the transformation

$$(16) \quad \begin{cases} t' = \lambda t, \\ q'(t') = \sqrt{\lambda} q(t), \end{cases} \quad \delta_0 q = q'(t) - q(t) = \varepsilon \left(\frac{1}{2} q - t \dot{q} \right),$$

which also leaves the commutator invariant, and we can try to find the matrix

$$(17) \quad U q(t) U^{-1} = q'(t),$$

$$(17a) \quad i[F, q] = \frac{1}{2} q - t \dot{q} \quad \text{with } U = 1 + i\varepsilon F.$$

⁽¹⁾ H. P. DÜRR, W. HEISENBERG, H. MÜLLER, S. SCHLIEDER and K. YAMAZOKI: *Zur Theorie der Elementarteilchen*, in *Zeits. f. Naturfor.*

With the help of the action principle we find

$$(18) \quad F = -\frac{1}{2}(t\dot{q}^2 - \dot{q}q + \omega tq^2).$$

The transformation (17) would, for a scalar field, correspond to the transformation

$$(19) \quad U\varphi(x, m)U^{-1} = \varphi'(x, m')$$

but this matrix U does not exist, as was shown by J. M. JAUCH⁽²⁾.

So we see that in the case where we have only a permutation of the variables in the same Hilbert space, we can deduce from the action principle the infinitesimal generator F , eq. (10a). However, this F does not yield a conservation law, nor does it commute with the particle number. In the other case where the transformation would connect fields with different mass, the transformation does not exist at all.

* * *

The author would like to express his thanks to Professor C. J. BAKKER and CERN for the hospitality extended to him. He is grateful to Professor M. FIERZ, Professor W. THIRRING and Professor J. M. JAUCH for useful comments and suggestions.

(²) J. M. JAUCH: *On Scale Transformations*, not published.

RIASSUNTO (*)

Per definire un numero leptonico, W. HEISENBERG si è servito dell'invarianza rispetto alla trasformazione scalare. Alcune proprietà di tale trasformazione sono esaminate in maniera più semplice facendo uso di campi scalari di cui sia nota la dipendenza dalla massa. Si elencano qui, per i campi scalari e per l'oscillatore armonico, alcune proprietà della trasformazione infinitesimale.

(*) Traduzione a cura della Redazione.