STRUCTURE AND STABILITY OF COLD SCALAR-TENSOR BLACK HOLES

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Abstract

We study the structure and stability of the recently discussed spherically symmetric Brans-Dicke black-hole type solutions with an infinite horizon area and zero Hawking temperature, existing for negative values of the coupling constant $\omega$. These solutions split into two classes: B1, whose horizon is reached by an infalling particle in a finite proper time, and B2, for which this proper time is infinite. Class B1 metrics are shown to be extendable beyond the horizon only for discrete values of mass and scalar charge, depending on two integers $m$ and $n$. In the case of even $m - n$ the space-time is globally regular; for odd $m$ the metric changes its signature at the horizon. All spherically symmetric solutions of the Brans-Dicke theory with $\omega < -3/2$ are shown to be linearly stable against spherically symmetric perturbations. This result extends to the generic case of the Bergmann-Wagoner class of scalar-tensor theories of gravity with the coupling function $\omega(\phi) < -3/2$.

1. Introduction

In the recent years there has been a renewed interest in scalar-tensor theories (STT) of gravity as viable alternatives to general relativity (GR), mostly in connection with their possible role in the early Universe. Another aspect of interest in STT is the possible existence of black holes (BHs) different from those well-known in GR. Thus, Campanelli and Lousto [1] pointed out among the static, spherically symmetric solutions of the Brans-Dicke (BD) theory a subfamily possessing all BH properties, but (i) existing only for negative values of the coupling constant $\omega$ and (ii) with horizons of infinite area (the so-called Type B BHs [2]). These authors argued that large negative $\omega$ are compatible with modern observations and that such BHs may be of astrophysical relevance.

In Ref. [2] it was shown, in the framework of a general (Bergmann-Wagoner) class of STT, that nontrivial BH solutions can exist for the coupling function $\omega(\phi) + 3/2 < 0$, and that only in exceptional cases these BHs have a finite horizon area. In the BD theory ($\omega = \text{const}$) such BHs were indicated explicitly; they have infinite horizon areas and zero Hawking temperature (“cold BHs”), thus confirming the conclusions of [1]. These BHs in turn split into two subclasses: B1, where horizons are attained by infalling particles in a finite proper time, and B2, for which this proper time is infinite.

The static region of a Type B2 BH is geodesically complete since its horizon is infinitely remote and forms a second spatial asymptotic in a nonstatic frame of reference. For Type B1 BHs the global picture is more complex and is studied here in some detail. It turns out that the

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horizon is generically singular due to violation of analyticity, despite the vanishing curvature invariants. Only a discrete set of B1-solutions, parametrized by two integers \(m\) and \(n\), admits a Kruskal-like extension, and, depending on their parity, four different global structures are distinguished. Two of them, where \(m - n\) is even, are globally regular, in two others the region beyond the horizon contains a spacelike or timelike singularity.

We also discuss the stability of STT solutions under spherically symmetric perturbations. Under reasonable boundary conditions, it turns out that the BD solutions with \(\omega < -3/2\) are linearly stable, and this result extends to similar solutions of the general STT provided the scalar field does not create new singularities in the static domain. For the case \(\omega > -3/2\) the stability conclusion depends on the boundary condition at a naked singularity, which is hard to formulate unambiguously.

Thus some vacuum STT solutions describe regular stable BH configurations with peculiar global structures.

2. Black holes in scalar-tensor theories

The Lagrangian of the general (Bergmann-Wagoner) class of STT of gravity in four dimensions is

\[
L = \sqrt{-g} \left[ \phi R + \frac{\omega(\phi)}{\phi} \phi_{,\rho} \phi^{,\rho} + L_m \right]
\]

(1)

where \(\omega(\phi)\) is an arbitrary function of the scalar field \(\phi\) and \(L_m\) is the Lagrangian of non-gravitational matter. This formulation (the so-called Jordan conformal frame) is commonly considered to be fundamental since just in this frame the matter energy-momentum tensor \(T_{\mu\nu}\) obeys the conventional conservation law \(\nabla_\alpha T^\alpha_{\mu} = 0\), giving the usual equations of motion (the so-called atomic system of measurements). We consider only scalar-vacuum configurations and put \(L_m = 0\).

The transition to the Einstein conformal frame, \(g_{\mu\nu} = \phi^{-1} \bar{g}_{\mu\nu}\), transforms Eq. (1) (up to a total divergence) to the form of GR with a minimally coupled scalar field \(\phi\),

\[
L = \sqrt{-\bar{g}} \left( \bar{R} + \epsilon \bar{g}^{\alpha\beta} \frac{\partial}{\partial \phi} \bar{g}_{\alpha\beta} \right), \quad \epsilon = \text{sign}(\omega + 3/2), \quad \frac{d\varphi}{d\phi} = \left| \frac{\omega + 3/2}{\varphi^2} \right|^{1/2}.
\]

(2)

The field equations are

\[
R_{\mu\nu} = -\epsilon \phi \varphi_{,\mu} \varphi_{,\nu}, \quad \nabla^{\alpha} \nabla_\alpha \varphi = 0
\]

(3)

where we have suppressed the bars marking the Einstein frame. The value \(\epsilon = +1\) corresponds to normal STT, with positive energy density in the Einstein frame; the choice \(\epsilon = -1\) is anomalous. The BD theory corresponds to the special case \(\omega = \text{const}\), so that \(\phi = \exp(\varphi/\sqrt{\omega + 3/2})\).

Let us consider a spherically symmetric field system, with the metric

\[
ds_E^2 = e^{2\gamma} dt^2 - e^{2\alpha} du^2 - e^{2\beta} d\Omega^2, \quad d\Omega^2 = d\theta^2 + \sin^2 \theta d\Phi^2,
\]

(4)

where \(E\) stands for the Einstein frame, \(u\) is the radial coordinate, \(\alpha, \beta, \gamma\) and the field \(\varphi\) are functions of \(u\) and \(t\). Up to the end of Sec. 3 we will be restricted to static configurations.

The general static, spherically symmetric scalar-vacuum solution of the theory (1) is given by [3, 4]

\[
ds^2 = \frac{1}{\phi} ds_E^2 = \frac{1}{\phi} \left[ e^{-2\varphi_0} dt^2 - \frac{e^{2\varphi_0}}{s^2(k, u)} \left( \frac{du^2}{s^2(k, u)} + d\Omega^2 \right) \right], \quad s(k, u) = \begin{cases} k^{-1} \sinh k u, & k > 0, \\ u, & k = 0, \\ k^{-1} \sin k u, & k < 0, \end{cases}
\]

\[
\varphi = Cu + \varphi_0, \quad C, \varphi_0 = \text{const},
\]

(5)
where \( J \) denotes the Jordan frame and \( u \) is the harmonic radial coordinate in the static spacetime, such that \( \alpha(u) = 2\beta(u) + \gamma(u) \). The constants \( b, k \) and \( C \) (the scalar charge) are related by
\[
2k^2 \text{sign } k = 2b^2 + \varepsilon C^2. \tag{7}
\]
The range of \( u \) is \( 0 < u < u_{\text{max}} \), where \( u = 0 \) corresponds to spatial infinity, while \( u_{\text{max}} \) may be finite or infinite depending on \( k \) and the behaviour of \( \phi(\varphi) \). In normal STT (\( \varepsilon = +1 \)), by (7), we have only \( k > 0 \), while in anomalous STT \( k \) can have either sign.

According to the previous studies [3, 4], all these solutions in normal STT have naked singularities, up to rare exceptions when the sphere \( u = \infty \) is regular and admits an extension of the static coordinate chart. An example is a conformal scalar field in GR viewed as a special case of STT, leading to BHs with scalar charge [5, 6]. Even when it is the case, such configurations are unstable due to blowing-up of the effective gravitational coupling [7].

In anomalous STT (\( \varepsilon = -1 \)) the following cases without naked singularities can be found:

1. \( k > 0 \). Possible event horizons have an infinite area (Type B black holes), i.e. \( g_{22} \to \infty \) as \( r \to 2k \). In BD theory, after the coordinate transformation \( e^{-2bu} = 1 - 2k/r \equiv P(r) \) the solution takes the form
\[
ds_2^2 = P^{-\xi}ds_\varphi^2 = P^{-\xi}\left(P^a dt^2 - P^{-a} dr^2 - P^{1-a} r^2 d\Omega^2\right), \quad \phi = P^{\xi} \tag{8}
\]
with the constants related by \( (2\omega + 3)\xi^2 = 1 - a^2 \), \( a = b/k \). The allowed range of \( a \) and \( \xi \), providing a nonsingular horizon at \( r = 2k \), is
\[
a > 1, \quad a > \xi \geq 2 - a. \tag{9}
\]
For \( \xi < 1 \) particles can arrive at the horizon in a finite proper time and may eventually (if geodesics can be extended) cross it, entering the BH interior (Type B1 BHs [2]). When \( \xi \geq 1 \), the sphere \( r = 2k \) is infinitely far and it takes an infinite proper time for a particle to reach it. As in the same limit \( g_{22} \to \infty \), this configuration (a Type B2 BH [2]) resembles a wormhole.

2. \( k = 0 \). Just as for \( k > 0 \), in a general STT, only Type B black holes are possible [2], with \( g_{22} \to \infty \) as \( u \to \infty \). In particular, the BD metric is
\[
ds^2 = e^{-su}\left[e^{-2bu} dt^2 - \frac{e^{2bu}}{u^2} \left(\frac{du^2}{u^2} + d\Omega^2\right)\right], \quad s^2(\omega + 3/2) = -2b^2. \tag{10}
\]
The allowed range of the integration constants is \( b > 0, \quad 2b > s > -2b \). This range is again divided into two halves: for \( s > 0 \) we deal with a Type B1 BH, for \( s < 0 \) with that of Type B2 (\( s = 0 \) is excluded since leads to GR).

3. \( k < 0 \). In the general STT the metric (5) typically describes a wormhole, with two flat asymptotics at \( u = 0 \) and \( u = \pi/|k| \), provided \( \phi \) is regular between them. In some STT the sphere \( u_{\text{max}} = \pi/|k| \) may be an event horizon, with \( \phi \sim 1/\Delta u^2, \Delta u \equiv |u - u_{\text{max}}| \). In this case it has a finite area and \( \omega(\phi) + \frac{3}{2} \to -0 \) as \( u \to u_{\text{max}} \). Such metrics behave near the horizon as the extreme Reissner-Nordström metric. In BD theory we have only a wormhole solution.

For all the BH solutions mentioned, the Hawking temperature is zero.
3. Analytic extension and causal structure of Type B1 Brans-Dicke black holes

Let us discuss possible Kruskal-like extensions of Type B1 BH metrics (8) and (10) of the BD theory.

For (8), with \(a > 1 > \xi > 2 - a\), we introduce, as usual, the null coordinates \(v\) and \(w\):

\[
    v = t + x, \quad w = t - x, \quad x \overset{\text{def}}{=} \int P^{-a} dr
\]

where \(x \to \infty\) as \(r \to \infty\) and \(x \to -\infty\) as \(r \to 2k\). The asymptotic behaviour of \(x\) as \(r \to 2k\) \((P \to 0)\) is \(x \propto -P^{1-a}\), and in a finite neighbourhood of the horizon \(P = 0\) one can write

\[
    x \equiv \frac{1}{2}(v - w) = -\frac{1}{2}P^{1-a}f(P),
\]

where \(f(P)\) is an analytic function of \(P\), with \(f(0) = 4k/(a - 1)\). Then, let us define new null coordinates \(V<0\) and \(W>0\) related to \(v\) and \(w\) by

\[
    -v = (-V)^{-n-1}, \quad w = W^{-n-1}, \quad n = \text{const.}
\]

The mixed coordinate patch \((V, w)\) is defined for \(v<0\) \((t< -x)\) and covers the whole past horizon \(v = -\infty\). Similarly, the patch \((v, W)\) is defined for \(w>0\) \((t> x)\) and covers the whole future horizon \(w = +\infty\). So these patches can be used to extend the metric through one or the other horizon.

Consider the future horizon. As is easily verified, a finite value of the metric coefficient \(g_{vW}\) at \(W = 0\) is achieved if we take \(n+1 = (a - 1)/(1 - \xi)\), which is positive for \(a > 1 > \xi\). The metric (8) can be written in the coordinates \((v, W)\) as follows:

\[
    ds^2 = -(n+1)f^{(n+2)/(n+1)} \cdot (1 - vW^{n+1})^{-(n+2)/(n+1)} dv dW - \frac{4k^2}{(1-P)^2}f^{-m/(n+1)} \cdot (1 - vW^{n+1})^{m/(n+1)} W^{-m} d\Omega^2
\]

where \(m = (a - 1 + \xi)/(1 - \xi)\).

The metric (14) can be extended at fixed \(v\) from \(W > 0\) to \(W < 0\) only if the numbers \(n+1\) and \(m\) are both integers (since otherwise the fractional powers of negative numbers violate the analyticity). This leads to a discrete set of values of the integration constants \(a\) and \(\xi\):

\[
    a = \frac{m+1}{m-n}, \quad \xi = \frac{m-n-1}{m-n}.
\]

where, according to the regularity conditions (9), \(m > n \geq 0\). Excluding the Schwarzschild case \(m = n + 1, \ (\xi = 0)\), we see that regular BD BHs correspond to integers \(m\) and \(n\) such that

\[
    m - 2 \geq n \geq 0.
\]

The extension through the past horizon can be performed in the coordinates \((V, w)\) in a similar way and with the same results.

It follows that, although the curvature scalars vanish on the Killing horizon \(P = 0\), the metric cannot be extended beyond it unless the constants \(a\) and \(\xi\) obey the “quantization condition” (15) and is generically singular. The Killing horizon, which is at a finite affine distance, is part of the boundary of the space-time, i.e. geodesics terminate there. A similar property was found in a (2+1)–dimensional model with exact power–law metric functions \([8]\).
The $k = 0$ solution (10) of the BD theory also has a Killing horizon ($u \to \infty$) at finite geodesic distance if $s > 0$. However, this space-time does not admit a Kruskal–like extension and so is singular. The reason is that in this case the relation giving the tortoise–like coordinate $x$,

$$x = \int \frac{e^{2bu}}{u^2} \, du = \frac{e^{2bu}}{2bu^2} f(u)$$

(with $f(\infty) = 1$) cannot be inverted near $u = \infty$ to obtain $u$ as an analytic function of $x$.

To study the geometry beyond the horizons of the metric (8), or (14), let us define the new radial coordinate $\rho$ by

$$P \equiv e^{-2ku} \equiv 1 - \frac{2k}{r} \equiv \rho^{m-n}.$$

The resulting solution (8), defined in the static region I ($\rho > 0$, $\rho > 0$), is

$$ds^2 = \rho^{n+2} \, dt^2 - \frac{4k^2(m-n)^2}{(1-P)^4} \rho^{-n-2} \, dp^2 - \frac{4k^2}{(1-P)^2} \rho^{-m} \, d\Omega^2, \quad \phi = \rho^{m-n-1}.$$

By (12), $\rho$ is related to the mixed null coordinates ($v,W$) by

$$\rho(v,W) = W \left[ f(P) \right]^{1/(n+1)} \left[ 1 - vW^{n+1} \right]^{-1/(n+1)}.$$

1. $m - n$ is even, i.e. $P(\rho)$ is an even function. The two regions $\rho > 0$ and $\rho < 0$ are isometric ($g_{\mu\nu}(-\rho) = g_{\mu\nu}(\rho)$) for (1a) $n$ even ($m$ even) and anti-isometric ($g_{\mu\nu}(-\rho) = -g_{\mu\nu}(\rho)$) for (1b) $n$ odd ($m$ odd). In this last case the metric tensor in region II ($\rho < 0$) has the signature $(-+++)$ instead of $(+---)$ in region I. Nevertheless, the Lorentzian nature of the space-time is preserved, and one can verify that all geodesics are continued smoothly from one region to the other (the geodesic equation depends only on the Christoffel symbols and is invariant under the anti-isometry $g_{\mu\nu} \to -g_{\mu\nu}$). The maximally extended space-time is regular, its Penrose diagram being the same in both cases 1a and 1b (Fig. 1), an infinite tower of alternating regions I and

Figure 1: Penrose diagram for a BH with even $m - n$
II. However, while in case 1a timelike geodesics periodically cross the horizons from one region to the next, in case 1b these geodesics become tachyonic in region II (just as in the case of a Schwarzschild black hole), and generically extend to spacelike infinity $(\lambda \to \infty)$.

$2. m - n$ is odd, i.e. $P(\rho)$ is an odd function; $P \to -\infty (r \to 0)$ as $\rho \to -\infty$, so that the metric (19) is singular on the line $\rho = -\infty$ which is spacelike for $n$ odd and timelike for $n$ even. The resulting Penrose diagram and geodesic motion are similar to those of Schwarzschild spacetime in subcase 2a ($n$ odd, $m$ even), and of extreme ($e^2 = m^2$) Reissner–Nordström spacetime in subcase 2b ($n$ even, $m$ odd).

4. Stability analysis

Let us now study small (linear) spherically symmetric perturbations of the above static solutions (or static regions of the BHs), i.e. consider, instead of $\varphi(u)$,

$$\varphi(u,t) = \varphi(u) + \delta \varphi(u,t)$$

and similarly for the metric functions $\alpha, \beta, \gamma$, where $\varphi(u), \text{etc.}$, are taken from the static solutions of Sec. 2. We are working in the Einstein conformal frame and use Eqs. (3).

Let us choose the coordinates in the perturbed space-time so that $5\delta \alpha = 2\delta \beta + \delta \gamma$, i.e. apply the same coordinate condition as in the metric $ds_E$ in (5). In this and only in this case the scalar equation for $\delta \varphi$ decouples from other perturbation equations following from (3) and reads

$$e^{4\beta(u)} \delta \ddot{\varphi} - \delta \varphi'' = 0.$$  (22)

Since the scalar field is the only dynamical degree of freedom, this equation can be used as the master one, while others only express the metric variables in terms of $\delta \varphi$, provided the whole set of field equations is consistent. That it is indeed the case, can be verified directly: firstly, among the four different Einstein equations in (3) only three are independent and, secondly, Eq. (22) may be derived from the Einstein equations. Therefore we have three independent equations for the three functions $\delta \varphi, \delta \beta$ and $\delta \gamma$.

The following stability analysis rests on Eq. (22). Separating the variables,

$$\delta \varphi = \psi(u) e^{i\omega t},$$  (23)

we reduce the stability problem to a boundary value problem for $\psi(u)$. Namely, if there is a nontrivial solution to (22) with $\omega^2 < 0$ under physically reasonable boundary conditions, then perturbations can exponentially grow with $t$ (instability). Otherwise it is stable in the linear approximation.

Suppose $-\omega^2 = \Omega^2, \Omega > 0$. We can use two forms of the radial equation (22): the one directly following from (23),

$$\psi'' - \Omega^2 e^{4\beta(u)} \psi = 0,$$  (24)

(as before, primes denote $d/du$) and the normal Liouville (Schrödinger-like) form

$$d^2 y/dx^2 - [\Omega^2 + V(x)] y(x) = 0, \quad V(x) = e^{-4\beta} (\beta'' - \beta'^2).$$  (25)
obtained from (24) by the transformation
\[ \psi(u) = y(x) e^{-\beta}, \quad x = -\int e^{2\beta(u)} du. \] (26)

The boundary condition at spatial infinity \((u = 0, x = +\infty)\) is \(\delta \phi \to 0\), or \(\psi \to 0\). For our metric (5) the effective potential \(V(x)\) has the asymptotic form \(V(x) \approx 2b/x^3\) as \(x \to +\infty\), hence the general solution to (25) and (24) has the asymptotic form
\[ y \sim c_1 e^{\Omega x} + c_2 e^{-\Omega x}, \quad \text{or} \quad \psi \sim u(c_1 e^{\Omega/u} + c_2 e^{-\Omega/u}) \] (27)
with arbitrary constants \(c_1, c_2\). Our boundary condition leads to \(c_1 = 0\).

For \(u \to u_{\max}\), where in many cases \(\phi \to \infty\), a boundary condition is not so evident. Refs. [9, 10] and others, dealing with minimally coupled or dilatonic scalar fields, used the minimal requirement providing the validity of the perturbation scheme in the Einstein frame:
\[ |\delta \phi/\phi| < \infty. \] (28)

In STT, where Jordan-frame and Einstein-frame metrics are related by \(g_{\mu\nu}^J = (1/\phi)g_{\mu\nu}^E\), it seems reasonable to require that the perturbed conformal factor \(\phi^{-1}\) behave no worse than the unperturbed one:
\[ |\delta \phi/\phi| < \infty. \] (29)

An explicit form (29) depends on the specific STT and can differ from (28). For example, in BD theory where \(\phi \propto \ln |\phi|\), (29) leads to
\[ |\delta \phi| < \infty. \] (30)

We will refer to (28) and (29) (or (30)) as the “weak” and “strong” boundary conditions respectively. If \(\phi\) and \(\varphi\) are regular at \(u \to u_{\max}\), these conditions both give \(|\delta \varphi| < \infty\).

Let us discuss different cases of the STT solutions, assuming that the field \(\phi\) is regular for \(0 < u < u_{\max}\), so that the factor \(\phi^{-1}\) in (5) does not affect the range of \(u\).

1. \(\varepsilon = +1, k > 0\). This is the singular solution of normal STT. As \(u \to \infty\), \(\beta(u) \sim (b - k)u\) with \(b < k\), so that \(|x| < \infty\) and we can put \(x \to 0\); the potential \(V(x)\) has there a negative double pole, \(V \sim -1/(4x^2)\), and (25) gives
\[ y(x) \approx \sqrt{x}(c_3 + c_4 \ln x). \] (31)

The weak boundary condition (28) leads to the requirement \(|\delta \varphi/\varphi| \approx |y|/ue^\beta \approx |y|/(\sqrt{x} \ln x) < \infty\), which is met by the general asymptotic solution (31) and hence by its special case that joins the allowed solution at spatial infinity ((27) with \(c_1 = 0\)). On the other hand, in the BD theory the strong condition (30) leads to \(|\psi| < \infty\) as \(u \to \infty\), which is incompatible with the boundary condition at spatial infinity \(\psi(u = 0) = 0\), since by (24) \(\psi''/\psi > 0\).

We see that in this singular case the choice of a boundary condition is crucial for the stability conclusion. If we use the weak boundary condition, we conclude that the static configuration is unstable, in agreement with the previous work [9]. This choice of the weak boundary condition certainly appears appropriate in the case of GR with a minimally coupled scalar field, which is thus unstable. On the contrary, in the BD case the strong condition seems more reasonable and leads to the conclusion that the BD singular solution is stable. For any other STT the situation must be considered separately.
2. $\varepsilon = -1, \ k > 0$. This case includes some singular solutions and cold BHs, exemplified by (8), (9) for the BD theory. Now $b > k$, so that as $u \to \infty, \ x \to -\infty$ and $V(x) \approx -1/(4x^2) \to 0$. The general asymptotic solution to Eq. (25) has again the form (27). The weak condition (28) leads, as in the previous case, to the requirement $|y|/(\sqrt{x} \ln |x|) < \infty$ and, applied to (27), to $c_2 = 0$. This means that $\psi$ must tend to zero as both $u \to 0$ and $u \to \infty$, which is impossible due to $\psi''/\psi > 0$. Thus the static system is stable. Even the weak condition is sufficient to conclude to stability, and a stronger one is not necessary.

3. $\varepsilon = -1, \ k = 0$. There are again singular solutions and cold BHs. As $u \to \infty, \ x \to -\infty$ and the potential $V(x) \to 0$. The same reasoning as in item 2 leads to the same conclusion.

4. $\varepsilon = -1, \ k < 0$. This is generically a wormhole, or in exceptional cases (see Sec. 2) a cold BH with a finite horizon area. In all cases one has $x \to \infty$ and $V \sim 1/|x|^3$ as $u \to u_{\text{max}}$; even the weak boundary condition leads to $|\psi| < \infty$, and the stability is concluded just as in items 2 and 3.

Thus, generically, scalar-vacuum spherically symmetric solution of anomalous STT are linearly stable against spherically symmetric perturbations. Excluded are only the cases when the field $\phi$ behaves in a singular way inside the coordinate range $0 < u < u_{\text{max}}$; such cases should be studied individually.

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References