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Theory of low energy nucleon-nucleon scatteringIII. Partial wave integral equations for low orbital momentum amplitudesD. Amati, E. Leader^(*) and B. Vitale

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SUMMARY

The theory developed in our previous papers^{1),2)} is applied to the low orbital momentum scattering amplitudes. Integral equations for these amplitudes are constructed, and the entire two-meson contribution to the left hand cut is explicitly given in terms of the effect of the Born term and ρ, ρ resonance in pion-nucleon scattering and a possible $\pi - \pi$ interaction. The contributions to the cut from higher mass intermediate states are assumed to be given by a low order polynomial in the energy, whose coefficients will appear as phenomenological constants in the theory.

The integral equations are exactly of the form treated recently by Noyes³⁾ and the present work may be regarded as a specification of the left hand cut discontinuity to be used in those equations, whose solutions should give the energy variation of the low partial waves.

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1. Introduction

In a previous paper¹⁾ (henceforth referred to as I) we have developed a theory of low energy nucleon-nucleon scattering, based on a two-dimensional representation of the nucleon-nucleon scattering amplitudes. It was shown that a convenient set of scalar amplitudes $c_j^T(w\bar{t}\bar{t})$ could be found which had simple crossing properties and which satisfied a Mandelstam representation at least up to 4th order perturbation theory. The spectral functions of this representation were calculated in terms of functions describing pion-nucleon and pion-pion scattering. In particular the role of a possible $\pi - \pi$ resonance in an s or p state was analysed.

In a second paper²⁾ (henceforth referred to as II) the simplification arising in the case of "high" angular momentum states was explicitly given.

In the present paper we show how in the case of "low" angular momentum states one can use the representation of the $c_j^T(w\bar{t}\bar{t})$ in order to write down integral equations for the partial wave amplitudes of nucleon-nucleon scattering.

The type of integral equations which result, in which the jump on the left-hand cut is known (at least in terms of some parameters of the theory) and in which the jump of the right-hand cut is expressible in terms of a real phase, have been discussed recently by Noyes³⁾, who has proposed a method for their solution. As Noyes has remarked, the specification of a particular meson theory approximation or of a phenomenological model will just be equivalent to the specification of the jump on the left-hand cut in the integral equations for the partial wave amplitudes.

The aim of this paper is, therefore, to give explicitly the form of this jump in the two-meson approximation, exhibiting separately the contributions which arise from the fourth order perturbation terms, the pion-nucleon rescattering corrections, and the effect of a pion-pion interaction.

In section II we discuss the analytic properties of the singlet and triplet partial wave amplitudes as derived from the properties of the $c_j^T(w\bar{t}\bar{t})$ amplitudes, and present the results for the jump on the left-hand cut.

Section III considers the extension of these properties to those of the scattering amplitudes and discusses the validity of the approximations of the theory.

We shall throughout the paper refer to the explicit formulae contained in I and II.

2. Analytic Properties of the Partial Wave Amplitudes

1) Properties of the $c_j^T(w\bar{t}\bar{t})$ functions

It can be easily seen from (I-3.4), (I-3.7) and (I-3.43) that the functions $c_j(w\bar{t}\bar{t})$ are analytic functions in the cut w plane, with cuts only on the real axis. The discontinuity across the left cut can be expressed in terms of the functions ρ defined in (I-3.43), in which all the dependence on the pion-nucleon and pion-pion interaction is made explicit, and the discontinuity across the right cut is connected with the imaginary part of the function c_j for positive values of k^2 .

As was mentioned at the end of I the right hand discontinuities of the $c_j(w\bar{t}\bar{t})$ are (apart from the 4th order perturbation theory contributions) only weakly dependent on $t-\bar{t}$ (or equivalently on $\cos \theta$, where θ is the nucleon-nucleon C.M. scattering angle), so that in the projection into partial wave amplitudes it was possible, for the "high" waves, to retain only the 4th order contribution to the right-hand discontinuity.

In the case of the "low" waves it is of course essential to retain the entire right-hand jump, and to determine it by using the unitarity condition in the nucleon-nucleon channel^(*). This will relate the right-hand discontinuity to the imaginary part of the amplitude, which in turn can be related to the phase-shifts and coupling parameters. There will thus result, instead of the explicit formulae of the high wave case, a set of integral equations for the low wave amplitudes.

(*) We stress once more that the division into "low" or "high" waves is somewhat arbitrary a priori, but is determined a posteriori by consistency requirements. We shall assume (as in II) that the "low" waves are those for which $\ell < 2$.

Concerning the left-hand cut in the k^2 plane, it should be realised that our knowledge of the jump is limited to that part of the negative real axis where the one and two pion contributions predominate. This region is, however, just that one in which the energy dependence is strong. What is left of the negative real axis will therefore contribute only a weak dependence on k^2 . This means that we shall not be able to determine completely the low angular momentum amplitudes (as was indeed possible for the high waves), but that we can hope to be able to give their energy dependence, determining therefore from our theory their effective ranges and shape dependent parameters. The physical reason for this is rather clear: low waves are sensitive to the inner region of the interaction, which is just determined by the far lying singularities; because of our lack of knowledge on these singularities, our knowledge of the low wave amplitudes will always be unknown at least within a constant.

2) Properties of the Singlet and Triplet Amplitudes

The scattering amplitudes α_i^T (as defined by Blatt and Biedenharn⁴⁾ or Stapp, Ypsilantis and Metropolis⁵⁾) are simply related to the partial wave singlet and triplet amplitudes $T_{ij}^T(\ell)$ (see Goldberger, Nambu and Oehme⁶⁾) which arise very naturally in the dispersion relation treatment of nucleon-nucleon scattering. They are in fact just linear combinations (with numerical coefficients) of $k T_{ij}^T(\ell)$.

The $T_{ij}^T(\ell)$ were defined⁶⁾ in terms of the total singlet-triplet amplitudes $T_{ij}^T(w, \cos \theta)$ by

$$T_{ij}^T(\ell) = \frac{2\ell+1}{2} \frac{(\ell-m)!}{(\ell+m)!} \int_{-1}^{+1} T_{ij}^T(w, \cos \theta) P_{\ell}^m(\cos \theta) d(\cos \theta) \quad (2.1)$$

with $m = |i-j|$.

The $T_{ij}^T(w, \cos \theta)$ on the other hand are related to the $c_j^T(w\bar{t}\bar{t})$ and therefore to the perturbative invariants $p_j^{\pm}(w\bar{t}\bar{t})$ [see I, equations (2.32), (3.4), (3.5)] by

$$T_{ij}^T(w, \cos \theta) = \sum_{\beta} W_{ij}^{\beta} p_{\beta}^T(wt\bar{t}) \quad (2.2)$$

where

$$W_{ij}^{\beta}(w, \cos \theta) = \sum_{\alpha} V_{ij}^{\alpha} U_{\alpha\beta} \quad (2.3)$$

and the matrices $V_{ij}^{\alpha}, U_{\alpha\beta}$ are given at the end of II, section III^(*).

Let us define

$$\nu = k^2 \quad \text{and} \quad \cos \theta = c$$

so that

$$\begin{aligned} w &= 4m^2 + 4\nu \\ t &= -2\nu(1-c) \\ \bar{t} &= -2\nu(1+c) \end{aligned} \quad (2.4)$$

The matrix W_{ij}^{β} , since it contains terms in $\sqrt{w}$, will introduce an extra cut with branch point at $\nu = -m^2$ when going from the $p^{\pm}(wt\bar{t})$ to the $T_{ij}(w, c)$. However, in so far as our discussion is concerned, this is a very far-lying singularity and will not affect our results, so that for our purpose the $T_{ij}(w, c)$ may be considered to have the same analytic behaviour as the $p^{\pm}(wt\bar{t})$, apart, of course, from the deuteron pole whose contribution to the $T_{ij}^T(w, c)$ has been given in reference⁶⁾, and which occurs only in the triplet, $T = 0$ amplitudes.

We had shown in (I-3.7) that the $p^{\pm}(wt\bar{t})$ could be represented as

$$\begin{aligned} p_j^{\pm}(wt\bar{t}) &= \frac{p_j^{\pm}}{\mu^2 - t} + \frac{1}{\pi} \int_{4\mu^2}^{t'_{\max}} p_j^{\pm}(wt') \frac{1}{t' - t} dt' \\ &+ \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{\chi_j^{\pm}(x, t)}{x - w} dx + (-1)^j (w \rightarrow \bar{t}) \end{aligned} \quad (2.5)$$

(*) One must simply replace y by $\cos \theta$ in these formulae for $U_{\alpha\beta}$ and V_{ij}^{α} .

Recall also that the p^T are given by

$$\begin{aligned} p^4 &= 3p^+ + 2p^- \\ p^0 &= 3p^+ - 6p^- \end{aligned}$$

The limit $t'_{\max}$ as discussed in II should strictly be $\leq 9\mu^2$ for an exact two-pion treatment, but because of the slow threshold increase of the three pion contribution beginning at $9\mu^2$, it is reasonable to take $t'_{\max}$ greater than $9\mu^2$ since the two pion contribution will still be the dominant one.

The weight functions $\rho_j^\pm(w, t')$ of (2.5) have been given explicitly in II (see equations (3.13, 16, 17, 18) and (4.17)) where they were split up into various terms arising from the effect in pion-nucleon scattering of the Born term, $3,3$ resonance, and a possible $\pi - \pi$ interaction.

Using these formulae for the $\rho_j^\pm(w, t')$, and equations (2.5), (2.2) and (2.1), we are now in a position to give in detail the analytic properties of the $T_{ij}^T(\ell)$. One sees that the $T_{ij}^T(\ell)$ are analytic functions of ν in the cut ν plane with the following singularities on the real axis:

a) The one meson exchange term (pole contribution in (2.5)) gives rise to a branch line along the negative axis, with branch point at $-\mu^2/4$ and the discontinuity across this cut is given by

$$-\frac{2+l}{2} \frac{(\ell-m)!}{(\ell+m)!} \pi i \sum_{\beta} \frac{1}{\nu} P^\pm W_{ij}^\beta(\nu, c') P^m(c') \quad (2.6)$$

with

$$c' = \frac{2\nu + \mu^2}{2\nu}$$

and the residues $P^\pm$ are given in (I-3.7).

b) The integral

$$\int \frac{\rho}{t'-t} dt'$$

gives rise to a branch line along the negative axis, with branch point at $-\mu^2$. The two-pion contribution to the discontinuity across this cut is given by:-

$$\begin{aligned}
 & - \frac{2\ell+1}{2} \frac{(\ell-m)!}{(\ell+m)!} i \sum_{\beta} \frac{1}{\nu} \int_{4\mu}^{-4\nu} \rho_{\beta}^{\pm(wt')} W_{ij}^{\beta}(\nu, y) P_{\ell}^m(y) dt' \\
 & = (2\ell+1) \frac{(\ell-m)!}{(\ell+m)!} i \sum_{\beta} \int_{-1}^{1+\frac{2\mu^2}{\nu}} \rho_{\beta}^{\pm(wt)} W_{ij}^{\beta}(\nu, c) P_{\ell}^m(c) dc
 \end{aligned} \tag{2.7}$$

for

$$\frac{-t'_{\max}}{4} \leq \nu \leq -\mu^2$$

where

$$y = \frac{t' + 2\nu}{2\nu}$$

c) The crossed term $w \rightarrow \bar{t}$ in (2.5) gives rise to a branch line along the negative axis, but with branch point at $-\mu^2$. Such far lying singularities have been consistently disregarded in all our calculations as explained in I.

d) Both the second and the third term in (2.5) give rise to a branch line along the positive axis, with branch point at 0. The discontinuity across this cut will just be equal to $2i$ times imaginary part of $T_{ij}(\ell)$ for positive values of ν .

If one now takes the correct combinations of $\pm$ amplitudes so as to form the amplitudes of definite T value (see note at page 4) then formulae (2.6), (2.7) together with the values of the $\rho_{\beta}^{\pm}$ functions given in II, give explicitly the left-hand discontinuities of the $T_{ij}^T(\ell)$ amplitudes in the two-meson approximation.

Finally, it must be remembered that, in order to take account of the crossing $t \xrightarrow{\leftarrow} \bar{t}$ of I-3.4, one should multiply the discontinuities in

the $T_{ij}^T(\ell)$ as given above by a factor

$$(T, m, \ell, s) = 1 - (-1)^{\ell+m+T+s} \quad (2.8)$$

where s is the total spin (i.e. 1 for triplet, 0 for singlet states) of the nucleon-nucleon system.

3. Analytic Properties of the Scattering Amplitudes

From the above discussion of the $T_{ij}^T(\ell)$ and their relation with the scattering amplitudes $\alpha_\ell(\nu)$, it follows that the amplitudes

$$a_\ell(\nu) = \sqrt{\frac{m^2 + \nu}{\nu}} \alpha_\ell(\nu) \quad (3.1)$$

have the same singularities as the $T_{ij}^T(\ell)$.

Furthermore, since the $\alpha_\ell(\nu)$ go to zero as $k^{(2\ell+1)}$ it follows that

$$a_\ell(\nu) \rightarrow 0 \quad \text{as} \quad \nu^\ell \quad (3.2)$$

Thus we may divide the $a_\ell(\nu)$ by a factor ν^ℓ without introducing any extra poles.

Finally, then, the set of amplitudes

$$b_\ell(\nu) = \frac{a_\ell(\nu)}{\nu^\ell} = \frac{1}{\nu^\ell} \sqrt{\frac{m^2 + \nu}{\nu}} \alpha_\ell(\nu) \quad (3.3)$$

will have no zeros (which will be useful for the application of the unitarity condition to the right cut, where the imaginary part of the inverse of the amplitude is required) and will have the required convergence properties (for $\ell \neq 0$). As a matter of fact, the unitarity condition implies that the α_ℓ are bounded at infinity, so that the amplitudes b_ℓ defined by (3.3) give rise to convergent integrals at infinity when $\ell \neq 0$. Only the s waves, therefore, will require convergence factors or, what is the same, they will require a subtraction.

It is now possible to write down for the amplitudes b_ℓ a general integral representation of the form:

$$b_\ell(\nu) = \frac{D_i}{\nu_b - \nu} + \frac{1}{\pi} \int_{-\mu^2/4}^{\infty} \frac{\text{Im } b_\ell(\nu')}{\nu' - \nu} d\nu' + \frac{1}{\pi} \int_0^{\infty} \frac{\text{Im } b(\nu')}{\nu' - \nu} d\nu' \quad (3.4)$$

where the discontinuity across the left cut is known, as already discussed, and that across the right cut is connected through unitarity to the phase shifts and coupling parameter in b_ℓ .

Our determination of the discontinuity across the left cut of the partial amplitudes b_ℓ is limited, as already emphasized, to the two-meson approximation, which means that our jumps are exact only down to $-(9/4)\mu^2$, where the three pion cut begins. We have already noted in section 2.2 that the validity of this approximation can be reasonably extended beyond this value, down to a value $-t'_{\text{max}}/4$ (see Eq. (2.7)). The final results will be of course insensitive to the actual choice of t'_{max} , as the contributions due to a variation in the value of t'_{max} will be just lumped into the low power polynomial in ν that will describe phenomenologically what remains of the left cut discontinuity - namely, that going from $-t'_{\text{max}}/4$ to $-\infty$. The actual number of free constants to be so introduced in the theory will clearly depend only on the possibility of a good fit with experiment.

We note also that our assumption that the negative jump for large values of ν can be expressed by means of a low power polynomial is coincident with the hypothesis that the three pion contribution to the spectral function varies slowly with the energy. Such would not be the case, for instance, if a three pion resonance would turn out to be dominant in a region near the three pion threshold.

The discontinuity across the right cut can be expressed, through unitarity, in terms of the phase shifts and coupling parameters that define the elastic nucleon-nucleon scattering. This is readily done for

the uncoupled partial wave amplitudes (singlet and triplet with $j = \ell$), where α is of the form $e^{i\delta} \sin \delta$ but represents a more complex problem in the case of coupled triplet amplitudes, with $j = \ell + 1$ and $j = \ell - 1$.

This procedure is valid only as long as the phase shifts are real, which means below the threshold for inelastic collisions. As the inelastic cross-section is still rather small at energies well above threshold, it could be safely assumed that the phase shifts are almost real also in a region of energy above the inelastic threshold. Thus what remains of the right cut will contribute only a weak dependence on ν in the partial wave amplitudes, and can be lumped into the constant or constants introduced already, in the phenomenological treatment of the left cut for large negative values of ν . The integral equations for the phase shifts and coupling parameters which result from equation (3.4) are exactly of the type discussed by Noyes³⁾ and the reader is referred to that paper for the method of solution.

A determination of the energy dependence of the singlet phase shifts is at present in progress, using the left-cut discontinuities given in the present paper.

References

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