A Note on Gauge Principle and Spontaneous Symmetry Breaking in Classical Particle Mechanics

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Abstract

The $U(1)$ gauge field is usually induced from the gauge principle, that is, the extension of global $U(1)$ phase transformation for matter field. However the phase itself is realized only for quantum theory. This makes us feel that the gauge fields are induced only in quantum theory by gauge principle. To make clear this point, we start from the non-relativistic classical particle, and we transform its action to the field theoretical form classically. We then have a multiplier field which insures the current conservation, and we get a global symmetry which translate this multiplier field. Using the gauge-principle to extend this global translation to the local one, we can introduce the gauge field, and obtain the minimal gauge coupling with matter. This shows that the gauge principle is alive for the symmetry of translation in classical theory without any phase variable to obtain the gauge field. Using this field theory we show the spontaneous symmetry breaking and Higgs mechanism which is similar to the super-conductivity. The relation of these two phenomenon is discussed.

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1 Introduction

Gauge field is usually explained by gauge-principle for the transformation of phase of matter fields, such as $U(1)$, $SU(N)$ gauge fields, and recently introduced one is the higgs field as the gauge field on $Z^2 \otimes M^4$ space-time in non-commutative geometry [1]. There are, however, different classes of gauge fields which are not related to the quantum mechanical phase. Examples are the gravity, spin-connection, and recently introduced one for classical particle is angular velocity which is related to the local (in the time direction) $SO(3)$ transformation [2]. Another example is the geometric gauge field induced by embedding procedure [3]. In this paper, we would like to pay attention to the $U(1)$ gauge field which are usually introduced by local phase transformation in quantum theory, and we seek its origin in classical mechanics. To realize this program, first order Lagrangian similar to the Polynomial-Formulation [4] is necessary. The local transformation governing gauge-field is the translation of the multiplier field which insure the current conservation. The current conservation is only the common word to the quantum theory for introducing the gauge field but not the phase. We introduce $U(1)$ gauge field by gauge principle in section 2. The field equation is solved and we discuss its solution in section 3, and spontaneous symmetry breaking and Higgs mechanism is discussed in section 4. This field theory is similar to the Schrödinger field, but is essentially different on the formulation and our theory does not come out through the first quantization. Our theory does not have an interference effect as wave, because it does not include the phase variable just having a classical phase-like variable. The translation symmetry through which we obtained gauge field is spontaneously broken from the outset, and we see that the matter field is Nambu-Goldstone boson. Higgs mechanism occurs, and the coupled gauge field gets mass term which comes from the plasma oscillation. This is similar to the super conductivity theory.

2 $U(1)$ gauge field in classical mechanics

Let us start with $N$ non-relativistic free particles in D-dimensional space by the Lagrangian

$$L(z(t), \dot{z}(t)) = \frac{m}{2} \sum_{k=1}^{N} \dot{z}_k^2.$$  

(1)
Then we can construct the conserved current field as
\[ j(x, t) = \sum_{k=1}^{N} \dot{z}_k(t) \delta^D(x - z_k(t)), \quad \rho(x, t) = \sum_{k=1}^{N} \delta^D(x - z_k(t)), \]
(2)
where we take the convention \( z_k \neq z_m \) if \( k \neq m \). Using above current field, we can change the free particle’s action into the form.
\[ L = \int d^Dx \left[ \frac{m}{2\rho} j^2 + \alpha (\nabla \cdot j + \dot{\rho}) \right], \]
(3)
where the first term reduces to the original Lagrangian by putting the form of current and after the space integration. The second term insures the current conservation and \( \alpha \) is the Lagrange multiplier field. For further constraints and hydrodynamical interpretation, we discuss in the last section. This Lagrangian has a strange form carrying no usual kinetic term, but the similar action is well known to treat the non-linear sigma model as Polynomial-Formulation [4]. Since this action has no local gauge invariance, the system is second class and its constraint analysis is done straightforward. The Euler-Lagrange equation leads to three equations
\[ \nabla \cdot j + \dot{\rho} = 0, \quad \nabla \alpha = \frac{m}{\rho} j, \quad \dot{\alpha} = -\frac{m}{2\rho^2} j^2. \]
(4)
By using the second equation of above ones, we can change the Lagrangian into the form
\[ L = \int d^Dx \left[ -\rho \dot{\alpha} - \frac{\rho}{2m} (\nabla \alpha)^2 \right]. \]
(5)
This is equal to the Lagrangian for Schrödinger field
\[ L = \int d^Dx \left[ i\hbar \Psi^* \partial_t \Psi - \Psi^* \hat{H} \Psi \right], \quad \hat{H} = -\frac{\hbar^2}{2m} \nabla^2, \]
(6)
with
\[ \Psi \sim \sqrt{\rho(t)} \exp\left(\frac{i}{\hbar} \alpha(x, t)\right), \]
(7)
where we take the approximation that \( \rho \) changes slowly in space, and we have neglected the total derivative. From this action we find correspondence between Lagrange multiplier field \( \alpha \) and quantum mechanical phase variable.
Now we go back to Lagrangian (3). This Lagrangian has the symmetry of global translation for multiplier field.

$$\delta \alpha(x, t) = C,$$

where $C$ is constant. The Noether current for this translation is the current appearing in action as dynamical variable. Let us extend this translation to the local one. The local translation,

$$\delta \alpha(x, t) = \theta(x, t),$$

does not change the action only when we extend the derivative for $\alpha$ field as

$$\partial_\mu \alpha(x, t) \rightarrow D_\mu \alpha(x, t) \equiv \partial_\mu \alpha(x, t) + A_\mu(x, t)$$

with transformation law for $A_\mu$ is

$$\delta A_\mu(x, t) = -\partial_\mu \theta(x, t).$$

We used here the relativistic notation. Notice that this is not the usual covariant derivative, and $D_\mu \alpha(x, t)$ is not only covariant but also invariant under the local translation. Using the notation

$$j^\mu = (\rho, j^1, j^2, j^3),$$

our local gauge invariant action takes the form

$$S = \int d^D x dt \left[ \frac{m}{2\rho} j^2 - j^\mu A_\mu + \alpha \partial_\mu j^\mu \right].$$

So we get the usual gauge coupling. Solving the constraint (current conservation), we find the solution as the form of currents (2), and by putting them into the above action we obtain the gauge coupled action for particles.

$$S = \int dt \sum_{k=1}^{N} \left[ \frac{m}{2} \dot{z}_k^2 - \dot{z}_k^i A_i(z_k, t) - A_0(z_k, t) \right].$$

In this way we can introduce $U(1)$ gauge field by gauge principle completely in classical theory. The local symmetry is not the $U(1)$ transformation but 1-dimensional translation. The reason why the multiplier field corresponds to the phase is the following. In quantum theory the translation of phase induced the current conservation, and in our case the same job is done by the multiplier field. So they are corresponding each other in the sense of gauge transformation.
3 Field equation

We consider the free field equation given by the action,

\[ S = \int d^Dx dt \left[ -\rho \dot{\alpha} - \frac{\rho}{2m} (\nabla \alpha)^2 \right]. \] (14)

The Euler-Lagrange equation gives

\[ \dot{\alpha} + \frac{1}{2m} (\nabla \alpha)^2 = 0, \] (15)

\[ \dot{\rho} + \frac{1}{m} \nabla \cdot (\rho \nabla \alpha) = 0. \] (16)

The solution of the first non-linear equation has the form,

\[ \alpha(x,t) = 2mC(|x - x_0| - Ct) + D, \] (17)

where \( x_0, C, D \) are the integration constant. By using the notation \( r = |x - x_0| \), the solution of the second equation is given as,

\[ \rho(x,t) = \int d\omega A(\omega) \frac{1}{r^2} \exp[i\omega(t - \frac{r}{2C})], \ r \neq 0. \] (18)

The green function described by

\[ \dot{\rho}_G + \frac{1}{m} \nabla \cdot (\rho_G \nabla \alpha) = \delta^3(x - x_0)\delta(t), \] (19)

is given as

\[ \rho_G(x, x_0, t) = \frac{(2m)^{-1/2}}{8\pi C r^2} \delta(t - \frac{r}{2C}). \] (20)

This solution says that particle density moves at the constant speed \( 2C \) which is determined by the initial condition. This is essentially classical picture, and much different from the Schrödinger equation, where the equation is diffusion type and propagation belongs to that one. If we work with Schrödinger equation, we should add

\[ -\frac{\hbar^2}{8m\rho} (\nabla \rho)^2 \] (21)

to the action. Then the first equation changes to include \( \rho \) dependent term to be highly non-linear equation, even though the Schrödinger equation for \( \Psi \sim \sqrt{\rho} \exp(i\frac{\hbar}{\sqrt{2m}}\alpha(x,t)) \) is simple linear one.


4 S.S.B. and Higgs mechanism

Let us include the gauge field into the action as in section 2.

$$S = \int d^3x dt \left[ -\rho \dot{\alpha} - \frac{\rho}{2m} (\nabla \alpha - A(x,t))^2 + \rho A^0 \right],$$

where we write $A = A^k = -A_k$ as usual relativistic manner. The Noether current corresponds to the translation invariance is

$$j = \frac{\rho}{m} (\nabla \alpha - A), \quad j^0 = \rho.$$

Therefore the Noether charge $Q$ is given by

$$Q = \int d^3x \rho(x),$$

and induce the infinitesimal translation of $\alpha$. Let us consider the quantization of this field theory. The canonical momentum conjugate to density $\rho$ is the “classical-phase” $\alpha$ and we get

$$[\rho(x), \alpha(y)] = i\hbar \delta^3(x-y).$$

This relation induce the relation

$$< 0 | [Q, \alpha(x)] | 0 > = i\hbar = const.$$

This means the spontaneous symmetry breaking (SSB), and Nambu Goldstone boson is the $\alpha$ field. If we add the kinetic term of gauge field, we can perform the gauge transformation for gauge field freely. The gauge transformation

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \alpha,$$

“gauge out” the $\alpha$ field, and action takes the form

$$S = \int d^3x dt \left[ -\rho A^2(x,t) - \rho A^0 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right].$$

$\rho$ is no more dynamical variable. The equation for gauge field shows that gauge field gets mass term

$$M = (\frac{\rho q^2}{m})^{1/2},$$
where we add the electric charge \( q \) to the gauge coupling. This is the plasma frequency. Even if we do not gauge out the \( \alpha \) field, the electric current has the form

\[
j_q = -\frac{\bar{\rho} q^2}{m} (A - \nabla \alpha), \quad (29)
\]

and taking rotation, we obtain

\[
\nabla \times j_q = -\frac{\bar{\rho} q^2}{m} B. \quad (30)
\]

This is the same as London equation. Here we gave the condition by hand

\[
\rho = \bar{\rho} + \delta \rho \sim \bar{\rho} = \text{const.}
\]

So we have the anti-magnetism just like super-conductivity.

5 **Discussion**

Let us discuss on the super-conductivity like solution obtained above. Our classical field theory should be compared to the charged boson theory rather than BCS theory since for fermion \( \Psi \sim \sqrt{\rho(t)} \exp(\frac{i}{\hbar} \alpha(x, t)) \) can not be applied. In the bosonic quantum theory with the Schrödinger term coupled with gauge field and screened coulomb repulsion term, the repulsive interaction with kinetic term of \( \rho : (21) \) determines the coherent length \( \xi \) to make the density flat. Then we get the London equation in the same way as we have explained. Comparing to our classical theory, we have no \( \nabla \rho \) term (21) and Coulomb repulsion term, the effect of them is given by hand as initial condition: the flatness of density in our case. However we did not pay attention to the stability of London equation under other scattering interaction of particle. Really we do not have energy gap which insures the stability of super-conductivity under other perturbation.

Next we should note another important point which is usually discussed on super-conductivity compared to the classical mechanics. Hereafter we take \( q = 1 \) and \( A^0 = 0 \). In the usual classical theory the electric current satisfies the equation under the external electric field.

\[
\frac{\partial j}{\partial t} \sim \frac{\bar{\rho}}{m} E,
\]

7
or in another form:  
\[
\frac{\partial}{\partial t} [j + \bar{\rho}mA] \sim 0.
\]
But in London equation (which is the same as our case), total time derivative is disappeared. This is the usual discussion about difference between classical conductivity and super one. Let us consider carefully on this point. Using Euler-Lagrange equation of (13) and current form (2) explicitly, we obtain the field theoretical Lorentz force.

\[
\frac{\partial j}{\partial t} = \frac{1}{m}[\rho E + j \times B] - \frac{1}{\rho}(j \cdot \nabla)j, 
\]
where the relation
\[
\sum_k \dot{z}_k \dot{z}_k \cdot \nabla \delta^D(x - z_k) = \frac{j}{\rho} (\nabla \cdot j) = \frac{j}{\rho^2} (j \cdot \nabla \rho) = \frac{1}{\rho}(j \cdot \nabla)j. 
\]
is utilized. The second equality can be interpreted as
\[
D_t \rho \equiv \dot{\rho} + \mathbf{v} \cdot \nabla \rho = 0, \quad \mathbf{v} \equiv \frac{j}{\rho}, 
\]
\(D_t\) means the Euler-derivative, and this equation shows that the density is preserved along the co-moving frame with this fluid. This is induced from the current conservation law by using non-compressive condition \(\nabla \cdot \mathbf{v} = 0\). The third equality can be interpreted as
\[
\mathbf{v} \cdot \nabla \mathbf{v} = 0.
\]
This relation holds in a small class of fluids approximately.

On the other hand if we start from action (12), we have Euler-Lagrange equation,

\[
\mathbf{j} = \frac{\rho}{m}(\nabla \alpha - \mathbf{A}), \quad \nabla \cdot \mathbf{j} + \dot{\rho} = 0, \quad \dot{\alpha} = -\frac{m}{2\rho^2} j^2. 
\]
To obtain the closed equation for current, we need to take time derivative of current, and we get

\[
\frac{\partial \mathbf{j}}{\partial t} = \frac{1}{m}[\rho \mathbf{E} + \mathbf{j} \times \mathbf{B}] - \mathbf{j} \cdot \nabla (\frac{\dot{\rho}}{\rho}) - \mathbf{j} \frac{\nabla \cdot \mathbf{j}}{\rho}. 
\]
It seems to be different from the another one, but by using the relation (33), we see the equivalence. We see the last two terms in r.h.s. are necessary to form the Euler-derivative together with l.h.s. [5]. If we use the variables $v$ and $\rho$ for above equation, by using the current conservation we obtain

$$mD_t v = E + v \times B.$$  \hspace{1cm} (36)

This is the well known Navier-Stokes (NS) equation under external electro-magnetic field without viscosity and gradient of pressure. The conditions (33) are hold for the current expression defined by (2), but in general it does not hold. Instead starting with the action (12) only, we obtain the quite general NS equation. Therefore we do not contain (33) as constraints into our theory to consider with more general fluids.

Now we come to discuss the main point. Even if we start from the action (12), we have the equation of motion with Lorentz force and only time derivative of current are determined as usual. But by using the multiplier field $\alpha$, we can write down the form of current itself, and obtain the usual (without time derivative) London equation. This is a trick by multiplier field and this kind of paradox may generally occur in field theory. At least for oscillating gauge field, London equation holds, and electromagnetic wave gets mass. This is well known as the reflection of the ones by ionosphere. In this phenomena our formulation makes a sense.

**Acknowledgement.**

After the first appearance of this article on hep-th, the author recieved the comment from Dr.Schakel that he has already done the quite similar work [6]. Though his physical view point is different from this article, starting Lagrangian is almost the same which is originally introduced by C.Eckart [7]. He started from the similar effective Lagrangian with internal energy term and studied the meaning of multiplier field, the relation to non-linear Schroedinger equation, super-fluid and super-conductivity. He introduced the gauge coupling to obtain the hydrodynamical-vortex but it is not the electro-magnetic field. The view point of “gauge principle in classical mechanics” is not appeared there. This is one of the differences to this article.

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References


