Abstract. Using the observation that configurations of $N$ polymers with hard core interactions on a closed random surface correspond to random surfaces with $N$ boundary components we calculate the free energy of a gas of polymers interacting with fully quantized two-dimensional gravity. We derive the equation of state for the polymer gas and find that all the virial coefficients beyond the second one vanish identically.
It is well known that some statistical mechanical models are easier to solve on a random triangulated surface than on a fixed lattice. Well known examples are the Ising model \cite{1} and self avoiding random walks \cite{2}. In the latter reference a closed form for the partition function of $n$ mutually self avoiding walks with common endpoints (watermelon diagrams) was obtained and the critical exponents determined; see \cite{3} for a review.

The reason for the simplicity of self avoiding random walks on a random surface is that we can keep the walks fixed and sum over the surfaces in a relatively straightforward fashion. One can for example regard the links which make up the walks as boundary components where the links have been identified pairwise. In fact one can also identify the boundary links in such a way that one obtains a branched polymer on the surface. The statistical sum over surfaces and walks therefore becomes a sum over surfaces with fluctuating boundaries.

In \cite{4} closed expressions were derived for all the multiloop functions for random surfaces made up of even sided polygons. The scaling limit of the loop functions in these models was also calculated and it was shown that the limit is the same when one drops the assumption that the polygons are even sided. The authors of \cite{4} used matrix model techniques. Identical results can be obtained working directly with triangulations \cite{5}. Here we use the explicit results of \cite{4} to study the thermodynamical properties of a gas of polymers interacting with 2d quantum gravity. This might be regarded as a two-dimensional toy model for primordial cosmic strings in thermal equilibrium with the gravitational field.

Let us consider an ensemble of triangulated surfaces with the topology of $S^2$ with $b$ boundary components. By a triangulation we here mean a collection of polygons whose sides are identified pairwise and degenerate boundaries may appear (unrestricted triangulations in the terminology of \cite{5}). We assume that each boundary component consists of an even number of links and carries one marked link. Let us arrange the links cyclically along each boundary component: $i_1, i_2, \ldots i_{2l}$, so that $i_1$ is the marked link. If we identify the link $i_1$ with $i_{2l}$, $i_2$ with $i_{2l-1}$ etc. we obtain a self-avoiding walk on a surface with one less boundary component. One of the endpoints of the walk is marked corresponding to the marking of the original boundary component. Doing this for all the boundary components we end up with a collection of $b$ mutually self-avoiding walks. We shall call the self-avoiding walks linear polymers or just polymers from now on. Given an orientation of the surface we see that there are precisely two distinct markings of a boundary component that
give rise to the same unmarked linear polymer. Similarly one can obtain a branched polymer on the surface by identifying the boundary links in a more general fashion. This can be done in precisely
\[ C_l = \frac{(2l)!}{l!(l+1)!} \]  
(1)
different ways, see e.g. [5]. Here we shall focus on the case of linear polymers. The general case can be treated by the same methods and we give the result below.

It follows from the above considerations that we can write the partition function of a gas consisting of \( b \) polymers of lengths \( l_1, l_2, \ldots, l_b \) with a hard core interaction on a random surface as
\[ Z_b(l_1, \ldots, l_b) = 2^{-b} w(2l_1, \ldots, 2l_b), \]  
(2)
where \( w \) is the \( b \)-loop function of quantum gravity (discretized Hartle–Hawking wave functional). The loop function is given by
\[ w(l_1, l_2, \ldots, l_b) = \sum_{T \in \mathcal{T}(l_1, \ldots, l_b)} e^{-S_T} \]  
(3)
where \( \mathcal{T}(l_1, \ldots, l_b) \) is the collection of all triangulations in the class under study which have \( b \) boundary components of lengths \( l_1, \ldots, l_b \) and \( S_T \) is the action of the triangulation \( T \).

We briefly recall the derivation of the formula for the loop function. For simplicity we consider triangulations which are made up of even sided polygons and take the action to be of the form
\[ S_T = -\sum_{j=1}^{\infty} k_j \log g_{2j}, \]  
(4)
where \( k_j \) is the number of \( 2j \)-gons in \( T \). Only a finite number of the coupling constants \( g_{2j} \) are nonzero and they are assumed to be nonnegative since we are interested in the pure gravity case but in principle one can study the multicritical models by the same method. Denoting the coupling constants collectively by \( \underline{g} \) we can write the Laplace–transform of the \( b \)-loop function as
\[ \hat{w}(\underline{g}; z_1, \ldots, z_b) = \prod_{i=2}^{b} \frac{d}{dV(z_i)} \hat{w}(\underline{g}; z_1), \]  
(5)
where
\[ \hat{w}(\underline{g}; z) = \sum_{l=1}^{\infty} z^{-2l-1} w(\underline{g}; 2l) \]  
(6)
is the Laplace–transform of the one-loop function and the “loop insertion operator” is defined by
\[
\frac{d}{dV(z)} = \sum_{j=1}^{\infty} \frac{d}{dg_{2j}},
\]
see [4, 5]. One should regard the variables \(z_i\) as the cosmological constants for boundaries. After some calculation one obtains for \(b \geq 3\) the formula
\[
\hat{w}(g; z_1, \ldots, z_b) = \left( \frac{2}{2M_1(c^2)} \frac{d}{dc^2} \right)^{b-3} \frac{1}{2c^2 M_1(c^2)} \prod_{k=1}^{b} \frac{c^2}{(z_k^2 - c^2)^{3/2}},
\]
where \(c = c(g)\) is the critical value of the \(z_i\)-variables,
\[
\tilde{M}_1(c^2) = \oint \frac{d\omega}{2\pi i} \frac{\omega V'(\omega)}{(\omega^2 - c^2)^{3/2}}
\]
and
\[
V(\omega) = \sum_{i=1}^{\infty} g_{2j} \omega^{2j}.
\]
Writing \(g_{2j} = gt_{2j}\) where \(g > 0\) and the \(t_{2j}\) are kept fixed we have a critical value \(g_0\) of \(g\) where the loop functions become singular.

The scaling limit is calculated by introducing a lattice length \(a\) which tends to zero and letting \(g \to g_0\) and \(c \to c_0 \equiv c(g_0)\) such that
\[
\log g - \log g_0 = a^2 \Lambda \quad (11)
\]
\[
\log z_i - \log c_0 = aZ_i, \quad (12)
\]
where \(\Lambda\) and \(Z_i\) are renormalized dimensionful cosmological constants which are kept fixed in the scaling limit. The continuum loop functions are given by
\[
\hat{W}(\Lambda; Z_1, \ldots, Z_b) = \lim_{a \to 0} a^{-5+7b/2} \hat{w}(g; z_1, \ldots, z_b)
\]
\[
= C \left( \frac{d}{d\Lambda} \right)^{b-3} \frac{1}{\sqrt{\Lambda}} \prod_{i=1}^{b} \frac{1}{(Z_i + \sqrt{\Lambda})^{3/2}},
\]
for \(b \geq 3\) and \(C\) is a numerical constant. There are also explicit formulas for \(b = 1\) and 2 but they are not of interest to us here.

It is now straightforward to take the inverse Laplace–transform in the variables \(Z_i\) and \(\Lambda\) and this yields the continuum \(b\)-loop function at fixed volume \(V\) and loop lengths \(L_1, \ldots, L_b\):
\[
W(V; L_1, \ldots, L_b) = CV^{b-7/2} \prod_{i=1}^{b} L_i \exp \left( -\frac{1}{4V} \left( \sum_{i=1}^{b} L_i \right)^2 \right).
\]
Let now all the polymers have the same continuum length $L$. Then the partition function becomes
\[ Z_b(V, L) = C V^{b-7/2} b^{b/2} L^{b^2 L^2 / 4 V}, \] where we have inserted the statistics factor $(b!)^{-1}$ in order to ensure “correct Boltzmann counting” of polymers. Putting
\[ \rho = \frac{b}{V} \] and taking the limit $b \to \infty$ and $V \to \infty$ with the polymer density $\rho$ fixed we obtain the free energy per unit volume
\[ f(\rho, L) = \rho \log \rho - \frac{\rho^2}{2} \log L + \frac{\rho^2 L^2}{4}. \] The first term on the right hand side of (17) corresponds to an ideal gas.

The pressure is given by
\[ p = \frac{\partial}{\partial \rho^{-1}} \left( \rho^{-1} f(\rho, L) \right) \]
\[ = \rho + \frac{\rho^2 L^2}{4}. \] The equation (18) is the exact equation of state for the polymer gas and we see in particular that the hard core repulsion of polymers is reflected in a single quadratic term in the virial expansion of the pressure. This is much simpler than the corresponding formula for the pressure for polymers in flat space where in general all the virial coefficients are non-vanishing [6, 7]. It is clear that this gas has no phase transition as expected for a gas with purely repulsive interaction.

In the case of general branched polymers we obtain an extra entropy factor
\[ \prod_{i=1}^{b} C_l \] on the right hand side of (2). Using
\[ C_l \sim 4^l l^{-3/2} \] for large $l$ we find that the continuum partition function at fixed volume is in this case given by
\[ Z'_b(V, L) = C V^{b-7/2} \frac{b!}{b} L^{-b} e^{-b^2 L^2 / 4 V}, \]
which yields the free energy

\[ f'(\rho, L) = \rho \log \rho + \rho \log L + \frac{\rho^2 L^2}{4} \]  

(22)

and the equation of state is unchanged.

Using the semi-explicit formulas for the higher genus loop functions derived in [5] one can in principle generalize the above calculation to obtain the free energy of a polymer gas on a surface of arbitrary fixed genus. It is not straightforward to obtain an explicit expression in this case since there is no known analogue of (14) for higher genus surfaces. The problem of letting the number of handles go to infinity in the thermodynamic limit so that one has a finite density of “wormholes” can presumably also be studied by this method.

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References