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ANALYTIC DESCRIPTION OF THE FUSION AND FISSION PROCESSES THROUGH COMPACT QUASI-MOLECULAR SHAPES

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Analytic description of the fusion and fission processes through compact quasi-molecular shapes

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Abstract

Recent studies have shown that the characteristics of the entrance and exit channels through compact quasi-molecular shapes are compatible with the experimental data on fusion, fission and cluster radioactivity when the deformation energy is determined within a generalized liquid drop model. Analytic expressions allowing to calculate rapidly the main characteristics of this deformation path through necked shapes with quasi-spherical ends are presented now ; namely formulas for the fusion and fission barrier heights, the fusion barrier radius, the symmetric fission barriers and the proximity energy.

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1. Introduction

Heavy-ion collisions around the Coulomb barrier are the main way to form very heavy and possible superheavy elements [1], rotating super and hyperdeformed states [2], new isotopes along the drip line [3] as well as nuclear molecules in ^{24}Mg [4]. In this entrance channel, the starting configuration is two close quasi-spherical nuclei. Later on, neck formation occurs while the shape keeps almost spherical ends. Finally, the deformed system descends till the quasi-spherical compound nucleus or remains in excited states in higher potential wells. New observed phenomena like cluster radioactivity [5], asymmetric fission of intermediate mass nuclei [6], cold and asymmetric fission of ^{252}Cf [7] and quasi-fission of heavy dinuclear systems [8] suggest that this deformation path through compact quasi-molecular shapes with a deep and narrow neck is also taken as an exit channel.

Within a generalized liquid drop model including an accurate radius, a proximity energy term and the mass asymmetry, it has been shown that this fusion-like deformation valley is compatible with most of the fusion but also the fission data : fusion barrier heights and positions [9-10], symmetric and asymmetric fission barrier heights [11-12], Businaro-Gallone point [13], fragment kinetic energies [12], critical angular momenta for light and medium nuclei [14] and partial half-lives of radioactive nuclei emitting heavy clusters [15]. The rotational hyperdeformed states possibly observed recently [2] might also come up and survive in this second fission valley [16].

The fusion cross sections for low and moderate energies and light and medium systems are essentially governed by the fusion barrier heights and positions. Several potentials and formulas have been proposed [17-20]. The effects of the dissipative forces must be added to describe the fusion of very heavy systems or of medium systems at high energies while the introduction of the deformations allows to explain the subbarrier fusion [21-22, 9]. To

simulate the fission barriers, different phenomenological analytic models have been advanced [23-26].

In this work, are proposed analytic expressions reproducing the fusion barrier heights and positions, the fission barrier heights and profiles and the nuclear proximity energy calculated in the fusion-like deformation valley within our generalized liquid drop model. In sect. 2 the shape parametrisation is recalled while in sect. 3 the potential energy is defined. The formulas allowing to calculate the characteristics of the fusion barriers are provided in sect. 4 and the ones relative to the fission process in sect. 5. Finally, the nuclear proximity energy may be evaluated from the expressions given in sect. 6.

2. Nuclear deformation through compact quasi-molecular shapes

The fusion-like deformation valley has been simulated using a two parameter shape sequence [9] describing the continuous transition from one sphere to two tangent spheres (or vice versa, see fig. 1) along with the progressive formation of a deep and narrow neck. In polar coordinates, the shape is defined by :

$$R(\theta)^2 = \begin{cases} a^2 \sin^2 \theta + c_1^2 \cos^2 \theta & (0 \leq \theta \leq \pi/2) \\ a^2 \sin^2 \theta + c_2^2 \cos^2 \theta & (\pi/2 \leq \theta \leq \pi) \end{cases} \quad (1)$$

a is the transverse semi-axis while c_1 and c_2 are the two elongations along the axis of revolution.

Fig.1. Shape evolution in the deformation path through compact quasi-molecular configurations. The nuclei are spherical when they are separated. S is the ratio between the transverse semi-axis (neck radius when the crevice is formed) and the highest elongation on the deformation axis.

This family of shapes is derived from the elliptic lemniscatoids obtained by inversion of spheroids [11]. Assuming volume conservation, the two parameters $s_1 = a/c_1$ and $s_2 = a/c_2$ completely define the shape. The ratio $\eta = R_2/R_1$ between the radii of the colliding nuclei or future fragments allows to connect s_1 and s_2 :

$$s_2^2 = \frac{s_1^2}{s_1^2 + (1 - s_1^2)\eta^2}. \quad (0 \leq s_1, s_2, \eta \leq 1) \quad (2)$$

Analytic expressions have been obtained for the various shape-dependent functions : volume, surface, distance between mass centres, moment of inertia and quadrupole moment [9].

In the case of symmetric deformation, the equation of the surface is, in Cartesian coordinates :

$$x^4 + y^4 + z^4 + 2x^2y^2 + 2x^2z^2 + 2y^2z^2 - c^2x^2 - s^2c^2y^2 - s^2c^2z^2 = 0. \quad (3)$$

3. Generalized liquid-drop model

For an arbitrary deformed nucleus, the macroscopic total energy [9] is defined as :

$$E = E_{RLDM} + E_N, \quad (4)$$

where E_{RLDM} and E_N are respectively the rotational liquid-drop model energy and the nuclear proximity energy. Constant density and volume conservation are assumed.

$$E_{RLDM} = E_V + E_S + E_C + E_{Rot}. \quad (5)$$

For one-body shapes, the volume E_V , surface E_S and Coulomb E_C energies are given by :

$$E_V = -a_v(1 - k_v I^2)A, \quad (6)$$

$$E_S = a_s(1 - k_s I^2)A^{2/3}(S / 4\pi R_0^2), \quad (7)$$

$$E_C = 0.6e^2(Z^2 / R_0) \times 0.5 \int (V(\theta) / V_0)(R(\theta) / R_0)^3 \sin\theta d\theta, \quad (8)$$

where A , Z and $I = (N - Z)/A$ are the mass, charge and relative neutron excess of the compound nucleus. $V(\theta)$ is the electrostatic potential at the surface of the shape and V_0 the surface potential of the sphere. The volume and surface coefficients a_v , a_s and the effective sharp radius R_0 have been chosen as :

$$a_v(T) = 15.494(1 + 0.00337T^2) \text{ MeV}, \quad (9)$$

$$a_s(T) = 17.9439(1 + 1.5T / 17)(1 - T / 17)^{3/2} \text{ MeV}, \quad (10)$$

$$R_0(T) = (1.28A^{1/3} - 0.76 + 0.8A^{-1/3})(1 + 0.0007T^2) \text{ fm}. \quad (11)$$

This later formula proposed in Ref [27] allows to reproduce the small increase of the ratio $r_0 = R_0 / A^{1/3}$ with the mass ; for example, $r_0 = 1.11 \text{ fm}$ for ^{20}Ne and $r_0 = 1.18 \text{ fm}$ for ^{240}Pu . In these reactions at energies around the Coulomb barrier, the temperature dependence has been neglected.

For comparison, the set of parameters used in the original version of the liquid drop model [28] was : $a_s = 17.9439 \text{ MeV}$ and $r_0 = 1.2249 \text{ fm}$. The potential defined by Krappe, Nix and Sierk [17] assumes $a_s = 21.7 \text{ MeV}$ and $r_0 = 1.18 \text{ fm}$ while the recent version of the Thomas-Fermi model [29] supposes $a_s = 18.63 \text{ MeV}$ and $r_0 = 1.14 \text{ fm}$.

The surface and volume asymmetry coefficients take on the values :

$$k_s = 2.6 \quad \text{and} \quad k_v = 1.8. \quad (12)$$

When the two nuclei are separated :

$$E_V = -a_v \left[(1 - k_v I_1^2) A_1 + (1 - k_v I_2^2) A_2 \right], \quad (13)$$

$$E_S = a_s \left[(1 - k_s I_1^2) A_1^{2/3} + (1 - k_s I_2^2) A_2^{2/3} \right], \quad (14)$$

$$E_c = \frac{3}{5} e^2 Z_1^2 / R_1 + \frac{3}{5} e^2 Z_2^2 / R_2 + e^2 Z_1 Z_2 / r \quad (15)$$

where A_i , Z_i , R_i and I_i are the masses, charges, radii and relative neutron excesses of the two nuclei and r the distance between their mass centres.

The discontinuity of a few MeV which appears at the contact point when Z_1/A_1 and Z_2/A_2 are very different has been removed linearly from the contact point to the sphere since it is due to the progressive rearrangement of the nuclear matter.

The surface energy E_S takes only into account the effects of the surface tension forces in an half space and does not include the contribution due to the attractive nuclear forces between the surfaces in regard in the neck or the gap between the nascent fragments or the colliding nuclei. The nuclear proximity energy term E_N allows to take into account these additional surface effects when crevices appear in the deformation path [27, 30, 9].

$$E_N = 2\gamma \int_{h_{min}}^{h_{max}} \phi(D/b) 2\pi h dh. \quad (16)$$

h is the ring radius in the plane perpendicular to the deformation axis and D the distance between the infinitesimal surfaces (see fig. 2). b is the surface width fixed at the standard value of 0.99 fm. The ϕ function is taken with the parametrisation of Feldmeier [30]. The surface parameter γ is given by a geometric mean between the surface parameters of the two nuclei :

$$\gamma = 0.9517 \sqrt{(1 - k_s I_1^2)(1 - k_s I_2^2)} \text{ MeV. fm}^{-2}. \quad (17)$$

In this generalized liquid drop model, the surface diffuseness is not considered in the surface energy term and the proximity energy vanishes when there is no neck as for ellipsoids for example.

Fig. 2. Area of the proximity force influence and definition of D and h .

The rotational energy has been determined within the rigid body ansatz. Indeed, it has been shown that corrective terms arising from the orbital motion and the spin degrees of freedom roughly cancel each other, particularly at large deformations.

$$E_{Rot} = \frac{\hbar^2 l(l+1)}{2I_{\perp}}. \quad (18)$$

As an example, in fig. 3 are given separately the different varying contributions to the potential energy together with the symmetric deformation energy for the ^{160}Dy nucleus. The volume energy is constant during the deformation process. The slope of the E_s and E_N curves changes drastically at the contact point since the surface is constant after the separation of the two spherical fragments and since the nuclear attraction is greatest at the contact point. Nevertheless, the total energy varies gently even around the contact point.

Fig. 3. Variation of the surface energy E_s , the Coulomb energy E_c , the nuclear proximity energy E_N and the deformation energy E (in MeV) for the ^{160}Dy nucleus plotted against the reduced distance between the mass centres. The dashed line corresponds to the contact point between the two equal spherical nuclei.

4. Analytic expressions describing the fusion barrier heights and positions

The main ingredients allowing to determine the fusion cross sections are the fusion barrier height $E_{0,\text{fus}}$ and position $R_{0,\text{fus}}$ (distance between the mass centres of the projectile and target) as well as the dissipated energy due to friction forces [21-22, 9]. For example, the Wong's formula [31] using only the s -wave parameters $E_{0,\text{fus}}$ and $R_{0,\text{fus}}$ gives good results around the Coulomb barrier for light and medium systems. In their detailed report, Vaz et al [19] have shown that the Yukawa-plus-exponential folding potential of Krappe et al [17]

and the nuclear potential of Ngô and Ngô [18] based on the energy density formalism lead to a very good agreement with experimental data. Later on, our generalized liquid drop model has also been able to reproduce the experimental results with the same accuracy [9, 10, 20].

Vaz et al have provided formulas for $E_{0,\text{fus}}$ and $R_{0,\text{fus}}$ with two coefficients adjusted to the empirical data. An extended version of these formulas with three adjusted parameters is proposed in equations (19) and (20), respectively for the fusion barrier height $E_{0,\text{fus}}$ (in MeV) and fusion barrier radius $R_{0,\text{fus}}$ (in fm) as functions of the mass and charge numbers of the two colliding nuclei. The fitting procedure has been realized with the theoretical data calculated from our generalized liquid drop model.

$$E_{0,\text{fus}}(A_1, Z_1, A_2, Z_2) = \frac{Z_1 Z_2}{(A_1^{1/3} + A_2^{1/3})(1.821 - 0.1411 \ln(Z_1 Z_2) + 1.231 \times 10^{-4} Z_1 Z_2)} , \quad (19)$$

$$R_{0,\text{fus}}(A_1, Z_1, A_2, Z_2) = (A_1^{1/3} + A_2^{1/3})(1.908 - 0.0857 \ln(Z_1 Z_2) + 3.94 / Z_1 Z_2) . \quad (20)$$

The theoretical fusion barrier height $E_{0,\text{fus}}$ and radius $R_{0,\text{fus}}$ may be also approximated within the two following polynomial developments :

$$\begin{aligned} E_{0,\text{fus}}(A_1, Z_1, A_2, Z_2) = & 15.5 + 0.10146 Z_1 Z_2 - 2.097(Z_1 + Z_2) - 0.01263(Z_1^2 + Z_2^2) \\ & + 1.3376 \times 10^{-4}(Z_1^3 + Z_2^3) - 1.36 \times 10^{-7}(Z_1^4 + Z_2^4) - 12.751(A_1^{1/3} + A_2^{1/3}) \\ & + 0.58494(Z_1 A_1^{1/3} + Z_2 A_2^{1/3}) + 0.93254(Z_2 A_1^{1/3} + Z_1 A_2^{1/3}) - 4.141 \times 10^{-5} Z_1 Z_2 (A_1^{1/3} + A_2^{1/3}) \\ & - 0.14869(Z_1 + Z_2) A_1^{1/3} A_2^{1/3} + 11.678(A_1^{2/3} + A_2^{2/3}) - 3.421 \times 10^{-4}(Z_1^2 A_1^{2/3} + Z_2^2 A_2^{2/3}) \\ & + 3.4075 \times 10^{-4}(Z_2^2 A_1^{2/3} + Z_1^2 A_2^{2/3}) - 4.40754 A_1^{1/3} A_2^{1/3} - 6.3306 \times 10^{-4} Z_1 Z_2 A_1^{1/3} A_2^{1/3} \\ & - 3.09836(A_1 + A_2) + 0.255454(A_1^{4/3} + A_2^{4/3}) ; \end{aligned} \quad (21)$$

$$\begin{aligned} R_{0,\text{fus}}(A_1, Z_1, A_2, Z_2) = & 6.965 + 8.43 \times 10^{-4} Z_1 Z_2 + 6.68 \times 10^{-7} Z_1^2 Z_2^2 - 0.02752(Z_1 + Z_2) \\ & + 2.814 \times 10^{-4}(Z_1^2 + Z_2^2) + 6.325 \times 10^{-6}(Z_1^3 + Z_2^3) + 5.103 \times 10^{-4} A_1 A_2 \\ & - 2.642 \times 10^{-7} Z_1 Z_2 A_1 A_2 + 2.9916 \times 10^{-8} A_1^2 A_2^2 + 0.05757(A_1 + A_2) \\ & - 3.02 \times 10^{-4}(Z_1 A_1 + Z_2 A_2) - 7.62 \times 10^{-4}(Z_1 A_2 + Z_2 A_1) + 5.541 \times 10^{-6} Z_1 Z_2 (A_1 + A_2) \\ & - 1.871 \times 10^{-6} A_1 A_2 (Z_1 + Z_2) - 2.514 \times 10^{-4}(A_1^2 + A_2^2) - 1.474 \times 10^{-8}(Z_1^2 A_1^2 + Z_2^2 A_2^2) \\ & - 3.143 \times 10^{-8}(Z_1^2 A_2^2 + Z_2^2 A_1^2) + 9.832 \times 10^{-7}(A_1^3 + A_2^3) . \end{aligned} \quad (22)$$

In table 1, theoretical fusion barrier characteristics and empirical data are compared in the whole mass and asymmetry range. The polynomial estimator (21) is more precise than the modified formula (19) to calculate the fusion barrier height while the fusion barrier position is determined with the same accuracy with the extended formula (20) as with the polynomial expansion (22). The polynomial formulas reproduce precisely the values obtained with the generalized liquid drop model. These different formulas are valid for $Z_1 Z_2 \leq 2200$.

Table 1

Comparison of empirical and theoretical fusion barrier heights ($E_{0,\text{fus}}$ in MeV) and positions ($R_{0,\text{fus}}$ in fm). The references of the empirical data (second and 6th columns) are given in [9,10,20]. The results obtained within our generalized liquid drop model are given in the third and seventh columns. The fourth and eighth columns correspond respectively to the formulas (19) and (20) while the fifth and last columns give the values calculated with the polynomials (21) and (22).

Reaction	$E_{0,\text{fus}}$ (Emp)	$E_{0,\text{fus}}$ (GLDM)	$E_{0,\text{fus}}$ expression (19)	$E_{0,\text{fus}}$ polynomial (21)	$R_{0,\text{fus}}$ (Emp)	$R_{0,\text{fus}}$ (GLDM)	$R_{0,\text{fus}}$ expression (20)	$R_{0,\text{fus}}$ polynomial (22)
$^{12}\text{C} + ^{10}\text{B}$	5.24	5.03	5.02	5.02	7.45	7.81	7.77	7.85
$^4\text{He} + ^{44}\text{Ca}$	6.37	6.20	5.99	6.49	8.25	8.51	8.65	8.60
$^4\text{He} + ^{59}\text{Co}$	8.26	8.10	7.79	8.43	8.63	8.83	8.98	8.93
$^{12}\text{C} + ^{27}\text{Al}$	12.29	12.24	12.13	12.22	8.36	8.40	8.38	8.42
$^{32}\text{S} + ^{24}\text{Mg}$	27.93	28.48	28.73	28.52	9.2	8.94	8.95	8.95
$^{32}\text{S} + ^{27}\text{Al}$	29.89	30.47	30.81	30.55	9.32	9.07	9.07	9.05
$^{12}\text{C} + ^{110}\text{Pd}$	35.1	36.59	36.70	36.59	-	10.11	10.20	10.11
$^{16}\text{O} + ^{144}\text{Sm}$	59.9	62.29	63.51	62.34	-	10.66	10.74	10.66
$^{40}\text{Ca} + ^{58}\text{Ni}$	73.36	74.72	77.03	74.62	10.20	9.98	10.01	10.00
$^{35}\text{Cl} + ^{90}\text{Zr}$	84.87	86.50	89.10	86.40	10.74	10.51	10.50	10.50
$^{16}\text{O} + ^{238}\text{U}$	83.2	84.70	86.14	85.79	-	11.67	11.75	11.59
$^{40}\text{Ar} + ^{112}\text{Sn}$	109.1	109.5	112.4	109.6	-	10.97	10.96	10.97
$^{86}\text{Kr} + ^{62}\text{Ni}$	119.3	121.4	124.2	121.4	-	11.11	11.04	11.13
$^{40}\text{Ar} + ^{148}\text{Sm}$	129.4	130.3	132.3	130.3	-	11.44	11.41	11.44
$^{86}\text{Kr} + ^{70}\text{Ge}$	136.5	137.2	139.4	137.2	-	11.19	11.16	11.23
$^{40}\text{Ar} + ^{165}\text{Ho}$	141.4	138.4	139.9	138.3	-	11.65	11.65	11.66
$^{37}\text{Cl} + ^{209}\text{Bi}$	156.2	157.2	156.7	157.2	-	11.99	11.95	11.98
$^{81}\text{Br} + ^{104}\text{Ru}$	174	176.2	174.9	176.2	-	11.61	11.57	11.64
$^{90}\text{Zr} + ^{96}\text{Zr}$	184.5	182.8	180.8	182.9	-	11.60	11.58	11.62
$^{40}\text{Ar} + ^{238}\text{U}$	185	179.3	175.9	179.3	-	12.30	12.26	12.30
$^{100}\text{Mo} + ^{90}\text{Zr}$	195.9	191.2	187.9	191.2	-	11.66	11.62	11.64
$^{54}\text{Cr} + ^{196}\text{Os}$	203.5	198.8	192.9	198.8	-	12.20	12.14	12.26
$^{86}\text{Kr} + ^{123}\text{Sb}$	209.4	204.1	198.2	204.1	-	11.95	11.89	11.96
$^{48}\text{Ca} + ^{248}\text{Cm}$	210	203.1	195.4	203.1	-	12.60	12.52	12.58
$^{124}\text{Sn} + ^{94}\text{Zr}$	219.5	219.9	210.9	219.9	-	12.06	12.00	12.05
$^{58}\text{Fe} + ^{209}\text{Bi}$	236.5	232.1	219.3	232.3	-	12.27	12.28	12.23
$^{76}\text{Ge} + ^{170}\text{Er}$	237	234.6	221.6	234.7	-	12.29	12.23	12.27

The following simpler formulas (23) and (24) give the theoretical fusion barrier heights and positions for identical target and projectile as functions of the mass and charge numbers of one of the colliding nuclei. They reproduce slightly better the theoretical predictions for the light nuclei (see table 2).

$$E_{0, \text{sym fus}}(A_1, Z_1) = -0.44 + 0.09A_1 + 0.007718A_1^2 - 0.0012081A_1^3 - 5.7291 \times 10^{-6} A_1^4 + 0.02Z_1 \\ - 0.06817A_1Z_1 + 0.00820466A_1^2Z_1 + 7.77506 \times 10^{-5} A_1^3Z_1 + 0.25838Z_1^2 - 0.017856Z_1^2A_1 \\ - 3.50134 \times 10^{-4} A_1^2Z_1^2 + 0.01121Z_1^3 + 6.4642 \times 10^{-4} A_1Z_1^3 - 4.12116 \times 10^{-4} Z_1^4; \quad (23)$$

$$R_{0, \text{sym fus}}(A_1, Z_1) = 7.17 + 0.1521A_1 - 0.004704A_1^2 - 6.478 \times 10^{-5} A_1^3 - 0.16947Z_1 \\ + 0.016742A_1Z_1 + 5.5422 \times 10^{-4} A_1^2Z_1 - 0.0153Z_1^2 - 0.00145874Z_1^2A_1 + 0.0012094Z_1^3. \quad (24)$$

Table 2

Comparison of empirical and theoretical symmetric fusion barrier heights ($E_{0,\text{fus}}$ in MeV) and positions ($R_{0,\text{fus}}$ in fm). The references of the empirical data (2nd and 7th columns) are given in [9,10,20,32-33]. The results obtained within our generalized liquid drop model are given in the 3rd and 8th columns. The 4th and 9th columns correspond respectively to the improved formulas (19) and (20). The results derived from the polynomial expressions (21) and (22) are given in the 5th and 10th columns while the 6th and last columns give the values calculated with the simpler formulas (23) and (24).

Reaction	$E_{0,\text{fus}}$ (Emp)	$E_{0,\text{fus}}$ (GLDM)	$E_{0,\text{fus}}$ formula (19)	$E_{0,\text{fus}}$ polynomial (21)	$E_{0,\text{fus}}$ formula (23)	$R_{0,\text{fus}}$ (Emp)	$R_{0,\text{fus}}$ (GLDM)	$R_{0,\text{fus}}$ formula (20)	$R_{0,\text{fus}}$ polynomial (22)	$R_{0,\text{fus}}$ formula (24)
$^{14}\text{N} + ^{14}\text{N}$	8.17	8.01	7.96	7.91	8.01	7.85	8.03	7.98	8.07	8.03
$^{24}\text{Mg} + ^{24}\text{Mg}$	21.53	21.88	21.94	21.91	21.87	8.37	8.70	8.71	8.72	8.68
$^{28}\text{Si} + ^{28}\text{Si}$	28.95	29.04	29.33	29.09	29.04	8.25	8.96	8.96	8.95	8.91
$^{30}\text{Si} + ^{30}\text{Si}$	28.1	28.40	28.67	28.45	28.38	8.6	9.18	9.17	9.13	9.17
$^{58}\text{Ni} + ^{58}\text{Ni}$	97.9	100.6	103.6	100.5	100.6	10.2	10.37	10.39	10.39	10.40
$^{64}\text{Ni} + ^{64}\text{Ni}$	93.5	97.32	100.3	97.34	97.39	-	10.77	10.73	10.80	10.83
$^{74}\text{Ge} + ^{74}\text{Ge}$	120.8	123.1	125.9	123.0	123.1	-	11.13	11.06	11.16	11.13
$^{90}\text{Zr} + ^{90}\text{Zr}$	182	184.8	182.7	184.8	184.7	-	11.48	11.46	11.45	11.46
$^{96}\text{Zr} + ^{96}\text{Zr}$	181.4	180.9	178.8	180.8	180.8	-	11.79	11.70	11.83	11.78
$^{100}\text{Mo} + ^{100}\text{Mo}$	207.2	197.8	193.2	197.8	197.7	-	11.84	11.79	11.85	11.82

5. Analytic expressions describing the fission barrier heights and profiles

The height of the macroscopic symmetric fission barriers in the fusion-like fission valley within the above described potential is reproduced, with an error generally less than 0.5 MeV, by the polynomial :

$$E_{\text{sym fis}}(A, Z) = 10.68 + 0.05884A - 0.0808188A^2 + 4.7769 \times 10^{-5} A^3 - 7.83427 \times 10^{-6} A^4 \\ + 2.20025Z + 0.388702AZ + 0.0016947A^2Z + 7.59413 \times 10^{-5} A^3Z - 0.517103Z^2 \\ - 0.00894564AZ^2 - 2.89901 \times 10^{-4} A^2Z^2 + 0.0114241Z^3 + 5.12411 \times 10^{-4} AZ^3 \\ - 3.51152 \times 10^{-4} Z^4. \quad (25)$$

A and Z are the mass and charge of the fissioning nucleus. In table 3, experimental fission barrier heights are compared with the macroscopic-microscopic and pure macroscopic fission barrier heights given by the GLDM and with the results obtained using the polynomial expression (25). The shell effects have been calculated as in the Droplet Model of Myers [36] with slightly different parameters [11, 35]. The theoretical predictions for this peculiar deformation valley agree with experimental data and, consequently, the polynomial (25) allows to obtain rapidly a realistic value of the macroscopic fission barrier height.

Table 3

Comparison of experimental and calculated fission barrier heights. The lower and higher values found experimentally [34] are given. The third and fourth columns correspond respectively to the predictions within the generalized liquid drop model with and without including the shell effects [11, 35]. The last column gives the macroscopic fission barrier heights obtained with the polynomial (25).

Compound nucleus	Experimental barrier (MeV)	Theoretical barriers (MeV)		
		GLDM with shell effects	GLDM without shell effects	Polynomial (25)
¹⁰⁹ Cd	34.0	40.8	41.8	41.8
¹⁴⁹ Eu	32.5 ± 2	33.7	34.2	34.0
¹⁵² Tb	29.0 ± 1	32.0	32.2	32.1
¹⁵⁷ Ho	26.5 ± 2	31.3	30.7	30.5
¹⁷³ Lu	27.1-31.5	30.0	28.3	28.2
¹⁷⁹ Ta	25.4-29.2	27.9	26.6	26.5
¹⁸⁰ W	23.9-28.7	26.7	25.5	25.4
¹⁸⁵ Re	24.0-26.4	25.5	24.8	24.8
¹⁹¹ Ir	20.6-23.7	22.7	23.0	22.9
¹⁹⁵ Au	18.6-21.6	21.1	20.9	20.8
²⁰¹ Tl	19.5-23.5	22.8	18.9	18.8
²⁰⁶ Pb	26.0-26.8	25.4	17.5	18.0
²⁰⁹ Bi	21.9-25.5	25.0	16.9	16.9
²¹³ At	14.3-17.2	19.5	14.7	14.6
²¹⁶ Rn	13.1 ± 1	16.1	13.6	13.4
²²⁶ Ac	8.0	9.0	10.2	9.7
²³² Th	6.0	8.3	9.2	8.1

Fig. 4 visualizes the symmetric fission barrier heights calculated from (25). The table of isotopes with its central ridge corresponding to the β -stability valley is correctly generated. The formula (26) reproduces, with a maximal error of 0.25 MeV, the macroscopic symmetric fission barrier height of these β -stable nuclei, for $A \leq 250$ (see fig. 5).

$$E_{sym\,fis}(A_\beta) = 7.243 + 1.4954A_\beta - 0.0283A_\beta^2 + 3.56985 \times 10^{-4}A_\beta^3 - 3.0447 \times 10^{-6}A_\beta^4 + 1.55496 \times 10^{-8}A_\beta^5 - 4.26757 \times 10^{-11}A_\beta^6 + 4.81912 \times 10^{-14}A_\beta^7. \quad (26)$$

The β -stability line has been described by the relation :

$$Z = \frac{1}{2} \left(A - \frac{0.4A^2}{200 + A} \right). \quad (27)$$

Fig. 4. Symmetric fission barrier height as functions of the mass and charge numbers in the fusion-like fission valley.

Fig. 5. Symmetric fission barrier height as a function of the mass for β -stable nuclei. The dots correspond to values given by the GLDM and the curve to the polynomial (26).

New data on asymmetric and very asymmetric fission barriers for light and medium nuclei are now available [5-6, 37]. They form a severe test for the different theoretical models. It has been shown [12-38] that asymmetric fission through compact quasi-molecular

shapes allows to reproduce these new experimental results. The formula (28) gives the asymmetric fission barrier heights obtained with the GLDM with a good accuracy when $Z \leq 92$. For a fixed asymmetry $\alpha = 100(A_1 - A_2)/A$, $E_{fis}(A, Z, \alpha)$ is the lowest barrier height when looking through all the different possible combinations of Z_1 and $Z_2 = Z - Z_1$.

$$\begin{aligned}
E_{fis}(A, Z, \alpha) = & 12.66 + 0.62955A - 0.0400035A^2 - 2.7264 \times 10^{-5}A^3 - 3.807 \times 10^{-8}A^4 \\
& + 0.631Z + 0.166322AZ + 7.0744 \times 10^{-4}A^2Z - 0.212091Z^2 - 0.00249441AZ^2 \\
& + 0.00260072Z^3 + 9.04 \times 10^{-8}Z^4 + 0.2559\alpha - 3.251 \times 10^{-4}A\alpha + 3.36264 \times 10^{-5}A^2\alpha \\
& - 0.008697Z\alpha - 1.7944 \times 10^{-4}AZ\alpha + 3.4154 \times 10^{-4}Z^2\alpha - 0.00952\alpha^2 - 7.8256 \times 10^{-6}A\alpha^2 \\
& + 8.208 \times 10^{-5}Z\alpha^2 + 1.5792 \times 10^{-4}\alpha^3 - 1.2478 \times 10^{-6}\alpha^4.
\end{aligned} \tag{28}$$

The formula (29) provides the theoretical asymmetric fission barrier heights for β -stable nuclei. The two-dimensional surface generated by this expression is displayed in fig. 6. The topology of the Businaro-Gallone picture appears [39, 13]. For light nuclei, the barrier height decreases monotonously with asymmetry while, for heavy nuclei, the barrier height increases with asymmetry till a maximum and then drops again for the most asymmetric decays. The Businaro-Gallone plateau where the exit channel probability is rather insensitive to the mass-asymmetry coordinate appears for masses between 80 and 145 in good agreement with the experimental estimations [40] which locate the BG point between 85 and 145.

Fig. 6. Fission barrier height as functions of the mass and asymmetry for β -stable nuclei.

$$\begin{aligned}
E_{fis}(A_\beta, \alpha) = & 18.11 + 0.7369A_\beta - 0.0063392A_\beta^2 + 1.74384 \times 10^{-5} A_\beta^3 - 1.97978 \times 10^{-8} A_\beta^4 \\
& + 0.255\alpha - 0.001802A_\beta\alpha + 1.3877 \times 10^{-5} A_\beta^2\alpha - 4.57229 \times 10^{-8} A_\beta^3\alpha - 0.013774\alpha^2 \\
& + 1.6865 \times 10^{-5} A_\beta\alpha^2 + 1.8094 \times 10^{-7} A_\beta^2\alpha^2 + 2.2493 \times 10^{-4} \alpha^3 - 3.047 \times 10^{-7} A_\beta\alpha^3 \\
& - 1.381 \times 10^{-6} \alpha^4.
\end{aligned} \tag{29}$$

The discovery of superdeformed states at high angular momentum and the open question of the existence of hyperdeformed rotating states has renewed interest in investigating the stability of a nucleus against rotation. It has been shown [16, 41] that the evolution of rotating nuclei in the fusion-like deformation valley is compatible with the available data on super and hyperdeformed states. The formula (30) valid for $115 < A_\beta < 250$ and $L < 120$ gives the symmetric fission barrier height for β -stable nuclei as a function of the angular momentum. The figure 7 displays the global behaviour.

$$\begin{aligned}
E_{sym fis}(A_\beta, L) = & 0.83216A_\beta - 0.0038734A_\beta^2 - 4.89 \times 10^{-6} A_\beta^3 + 2.924 \times 10^{-8} A_\beta^4 + 0.137L \\
& - 0.0063765A_\beta L + 5.4417 \times 10^{-5} A_\beta^2 L - 1.1886 \times 10^{-7} A_\beta^3 L - 0.01476L^2 + 1.932 \times 10^{-5} A_\beta L^2 \\
& + 8.64 \times 10^{-8} A_\beta^2 L^2 + 1.394 \times 10^{-4} L^3 - 3.2938 \times 10^{-7} A_\beta L^3 - 2.293 \times 10^{-7} L^4.
\end{aligned} \tag{30}$$

Fig. 7. Symmetric fission barrier height as functions of mass and angular momentum ($\hbar$ unit) for β -stable nuclei.

The knowledge of the whole fission barrier is needed to investigate the fragment kinetic energies, the superdeformed and hyperdeformed states, the double humped barriers and the half-lives of compound nuclei. The fission barrier profile varies strongly with the mass particularly before reaching the saddle-point and, consequently, three formulas (31), (32) and (33) depending on the mass and deformation range are proposed to simulate the symmetric fission barriers for beta-stable nuclei given by the GLDM within the compact quasi-molecular shapes. The selected deformation parameter r (see equation 34) is the

distance between the centres of mass of each nascent fragment. c is the semi-elongation along the fission axis. The meaning of r is poor for one-body shapes. It is calculated assuming that the future fragments are portion of lemniscatoids separated by a plane perpendicular to the fission axis. For the initial spherical compound nucleus, $r = 0.75R_0$ and at the contact point between the spherical fragments, $r = 1.5874R_0$. In this deformation valley, the later two separated sphere configuration plays the major role and, then, r is simply the distance between the mass centres. δ is $100r/R_0$ and $\Delta = \delta - 75$. The formulas (31) and (32) are valid for $\delta < 200$ and respectively for $A < 210$ and $A \geq 210$ while the formula (33) is valid for all masses and larger deformations : $200 \leq \delta \leq 280$. For still more separated fragments only the Coulomb repulsion takes place, the proximity energy being negligible.

$$\begin{aligned}
E_{pot}(A_\beta, \Delta) = & 0.0112\Delta + 0.007487A_\beta\Delta - 6.988 \times 10^{-5} A_\beta^2\Delta + 3.302 \times 10^{-7} A_\beta^3\Delta \\
& - 5.8595 \times 10^{-10} A_\beta^4\Delta - 0.00181\Delta^2 - 3.2816 \times 10^{-5} A_\beta\Delta^2 - 2.2584 \times 10^{-7} A_\beta^2\Delta^2 \\
& - 4.74 \times 10^{-13} A_\beta^3\Delta^2 + 4.336 \times 10^{-5} \Delta^3 + 1.05 \times 10^{-6} A_\beta\Delta^3 + 7.839 \times 10^{-10} A_\beta^2\Delta^3 \\
& - 2.2061 \times 10^{-7} \Delta^4 - 5.6228 \times 10^{-9} A_\beta\Delta^4 + 1.02 \times 10^{-10} \Delta^5,
\end{aligned} \tag{31}$$

$$\begin{aligned}
E_{pot}(A_\beta, \Delta) = & -0.00255\Delta - 0.157745A_\beta\Delta + 0.00189569A_\beta^2\Delta - 7.5577 \times 10^{-6} A_\beta^3\Delta \\
& + 1.00109 \times 10^{-8} A_\beta^4\Delta + 0.564875\Delta^2 - 0.00635118A_\beta\Delta^2 + 2.43282 \times 10^{-5} A_\beta^2\Delta^2 \\
& - 3.15794 \times 10^{-8} A_\beta^3\Delta^2 - 0.00115226\Delta^3 + 5.28842 \times 10^{-6} A_\beta\Delta^3 - 8.03767 \times 10^{-9} A_\beta^2\Delta^3 \\
& + 6.35737 \times 10^{-6} \Delta^4 - 4.62167 \times 10^{-9} A_\beta\Delta^4 - 2.34495 \times 10^{-8} \Delta^5,
\end{aligned} \tag{32}$$

$$\begin{aligned}
E_{pot}(A_\beta, \delta) = & 0.00015A_\beta + 0.008308A_\beta^2 + 3.261 \times 10^{-5} A_\beta^3 - 1.35614 \times 10^{-7} A_\beta^4 \\
& + 1.55233 \times 10^{-10} A_\beta^5 + 0.009524A_\beta\delta - 1.75779 \times 10^{-4} A_\beta^2\delta + 1.29808 \times 10^{-7} A_\beta^3\delta \\
& - 3.345 \times 10^{-11} A_\beta^4\delta + 5.43176 \times 10^{-4} \delta^2 - 1.38346 \times 10^{-5} A_\beta\delta^2 + 4.66506 \times 10^{-7} A_\beta^2\delta^2 \\
& - 1.48275 \times 10^{-10} A_\beta^3\delta^2 - 2.4682 \times 10^{-6} \delta^3 - 5.07973 \times 10^{-8} A_\beta\delta^3 - 4.13955 \times 10^{-10} A_\beta^2\delta^3 \\
& + 5.0055 \times 10^{-9} \delta^4 + 9.675 \times 10^{-11} A_\beta\delta^4 - 4.07 \times 10^{-12} \delta^5,
\end{aligned} \tag{33}$$

$$r = 2 \frac{\int_0^c x d\tau}{\int_0^c d\tau}. \tag{34}$$

The resulting two-dimensional surface is displayed in figure 8. The endothermic and exothermic fission appears for light and heavy nuclei. For heavy nuclei, the profile of the deformation energy is relatively flat till the bottom of the scission barrier ; a second deformed macroscopic minima even appearing for the heaviest nuclei [11, 35].

Fig. 8. Symmetric fission barrier as functions of the mass and deformation for β -stable nuclei.

6. Analytic expressions describing the proximity energy

In the studied deformation path, the balance between the Coulomb and proximity forces governs the profile of the potential barrier around the top since this one corresponds to two-separated spheres. The determination of the Coulomb energy is then easy while the effects of the proximity forces are less known and have been calculated within different empirical nuclear potentials [17, 18, 27, 30, 9]. The formula (35) reproduces the proximity energy of the generalized liquid drop model for symmetric deformation ($95 < \delta < 260$) and β -stable nuclei.

$$\begin{aligned}
 E_{prox}(A_\beta, \delta) = & -175.6 + 71.762A_\beta^{1/3} - 6.83A_\beta^{1/3} \exp[-|\delta - 158.7|/18] - 4.807A_\beta^{2/3} \\
 & + 2.97765\delta - 1.16591A_\beta^{1/3}\delta + 0.0675251A_\beta^{2/3}\delta - 0.014766\delta^2 + 0.0053A_\beta^{1/3}\delta^2 \\
 & - 2.0914 \times 10^{-4} A_\beta^{2/3}\delta^2 + 2.1288 \times 10^{-5} \delta^3 - 6.529 \times 10^{-6} A_\beta^{1/3}\delta^3.
 \end{aligned} \tag{35}$$

The figure 9 visualizes the variation of E_{prox} with A_β and δ . The profile of the nuclear proximity potential is asymmetric in δ for a given mass and the tail of the potential extends to large deformations. The minimum of E_{prox} at the contact point varies smoothly with the mass. It is given by the formula (36).

$$E_{prox,contact}(A_\beta) = 17.43 - 13.363A_\beta^{1/3} + 0.547A_\beta^{2/3}. \tag{36}$$

Fig. 9. Proximity energy as functions of the mass and deformation for β -stable nuclei.

7. Conclusion

Recent studies have shown that the experimental data on heavy-ion collisions, fusion, fission and cluster radioactivity seem compatible with entrance and exit channels through compact quasi-molecular shapes when the nucleus energy is determined within a generalized liquid drop model including an accurate radius, a proximity energy term, the mass asymmetry and the temperature.

The main characteristics of this deformation path through necked one-body shapes with quasi-spherical ends and two separated spheres have been fitted from calculations within this generalized liquid drop model, mainly by polynomials. Different analytic expressions are proposed to calculate rapidly the fusion barrier height and the symmetric and asymmetric macroscopic fission barrier height, the fusion barrier radius, the symmetric fission barriers of β -stable nuclei and the proximity energy in this deformation path. Their predictions have been compared in different tables with the model data and the experimental results. A simple fortran code *fisfus.ana* is available by *e-mail*.

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