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A NEW TECHNIQUE IN THE STATISTICAL MODEL OF PARTICLE PRODUCTION

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A B S T R A C T

A new method is introduced to account for conservation laws in the statistical model of particle production. The approach becomes exact in the limit of high multiplicities; it leads to simple analytical expressions which allow immediate estimates of the influence of the various conservation laws under different circumstances. Momentum conservation is shown to have little influence on the energy spectra of the emitted particles; its consequences for the angular correlation of particles are studied. The statistical weight factors, which result from the coupling of the isospin vectors of the emitted particles with isospin conservation imposed, are shown to be well represented by semi-classical expressions; the dependence of the weight factors on total isospin and multiplicity becomes obvious. The general effects of angular momentum conservation in the statistical model are studied; the angular distributions are forward - backward symmetric in the centre of mass system, a prediction specific to the model and a suitable test to its validity. In the classical approximation the angular distribution is forward-backward peaked, though not more than a limiting distribution $1/\sin\theta$, where θ is the angle of the particle with the beam direction. Angular momentum conservation leaves energy spectra nearly unchanged, while multiplicities are strongly affected by it; the effective interaction volumes in nucleon - antinucleon annihilation and nucleon - nucleon collisions are shown to be approximately the same, when this effect is taken into account.

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1. Introduction

It is a standard procedure in strong interaction studies to use the predictions of the statistical model (SM) as a reference to which experimental results are compared. When this model was first introduced by Fermi ¹⁾, it was extremely simple : momentum and angular momentum effects were neglected, and it was mainly applied to "low energy" reactions, which have low multiplicities. At present, the SM is applied to energies at which the number of energetically possible reactions is very large; furthermore, the number of reactions increases very rapidly with energy, so that for this reason alone the calculations tend to become very laborious. In addition, momentum conservation is included as a rule and great progress has been made towards the incorporation of angular momentum conservation in the last year ^{2),3)}. This increases the labour even further and it has become necessary to turn to high speed computers to obtain results in reasonable time. In the process the original simplicity of the SM has largely been lost. It is thus most desirable to find an approximation technique which at least partly enables us to restore simplicity and which will keep the computing to within reasonable limits.

A satisfactory approximation should aim at fulfilling the following criteria :

- 1) it should have a simple physical interpretation, so that its region of validity can be determined from physical considerations.
- 2) its predictions should be particularly good for large multiplicities, as these are handled only with difficulty by present methods.
- 3) the structure of its results should be such as to facilitate the insight into the dependence of cross-sections on the variables and to permit rapid qualitative estimates.
- 4) it should be possible to use it to generalise the SM to include effects only little discussed so far, angular momentum effects for example.

In this paper we present an approximation, the random coupling approximation (RCA), which seems to meet all of these requirements. The method is sufficiently flexible, that it can be applied with equal ease to momentum phase space integrals, to the statistical weights due to isospin conservation and to angular momentum effects. One of the main problems of the SM is the evaluation of the momentum phase space of n particles with given total energy E and given total momentum $\vec{P}$. We will therefore outline the principal idea of the RCA for this case.

The method of random coupling simply observes that the net effect of momentum conservation is equivalent to the requirement that the n momentum vectors add randomly to the given total momentum $\vec{P}$; this momentum is obtained with a certain 'a priori' probability, that is described to a good approximation by a Gaussian distribution. As the 'a priori' probability singles out no direction in space, the problem now depends only on the absolute values of the momenta, which immediately cuts down the number of integrations from $3n$ to n . The detailed application of the RCA to momentum phase space is studied in section 2. As a byproduct of that discussion and as a non-trivial application we will estimate the angular correlation of two particles in nucleon-antinucleon annihilation.

The statistical weight factors due to isospin conservation are essentially nothing but the 'a priori' probability that n isospin vectors couple to given total isospin T and a given projection of it, T_z . The exact formal solution can be given by Clebsch-Gordan techniques ^{4) -6)} or by projection operator technique ⁷⁾; while this is sufficient for numerical calculations of statistical weights, it does not readily provide insight into their behaviour.

Starting from a semi-classical picture we obtain in section 3 the 'a priori' probability for random coupling of the n isospin vectors in complete analogy to the corresponding problem for momentum vectors. The semi-classical result yields a very simple analytical form which reproduces the exact solution excellently. It should be advantageous for application to problems, for which numerical values of statistical weights are not available.

The conservation of total angular momentum and its projection on the z-axis imposes restrictions on the phase space available for the emitted particles. Until recently, it was customary to assume that the restrictions imposed by angular momentum conservation are negligible. Cook ²⁾ and Koba ³⁾ have recently taken it formally into account by Clebsch-Gordan techniques, but the expressions obtained are still rather awkward for practical use. Because the angular momenta in high energy processes are quite large, it should be very appropriate to apply classical arguments to the angular momentum conservation, in particular as this is known to work in compound nuclear processes, even for relatively low angular momenta ^{8),9)}. In section 4 we take angular momentum conservation into account by considering it as due to the 'a priori' probability that the n angular momenta couple to given total angular momentum. The influence of angular momentum on branching ratios and the typical angular distributions becomes qualitatively very simple. The most important feature of a statistical angular distribution is its forward - backward symmetry in the CM, which is due to the fact that interference between different partial waves has no place in a statistical description. Furthermore, we discuss the limiting cases of the angular distribution in the SM and show that the emitted particles must be expected to be forward-backward peaked. The multiplicity depends strongly on angular momentum. We show that this influence can be approximately represented by a scale factor in the interaction volume. When angular momentum conservation is included the interaction volumes of nucleon-antinucleon annihilation and multiple production processes become the same within a factor 2.

2. Momentum phase space

The SM is a phase space description of the strong interaction reactions in which all appropriate quantum numbers have to be imposed. The phase space is strongly dominated by the accessible volume in momentum phase space; it has been customary to disregard the influence of angular momentum as this greatly complicates the problem. We return to this point in section 4; we will here study the momentum phase space integrals, which occur in the ordinary treatments. The momentum integrals are of the form

$$\mathcal{S}_n^*(E, \vec{P}) = \int \dots \int d\vec{p}_1 \dots d\vec{p}_n f(p_1, \dots, p_n) \delta^{(3)}(\sum_i \vec{p}_i - \vec{P}) (\sum_i \mathcal{E}_i - E) \quad (1)$$

In Eq. (1) we denote by $\vec{p}_i$ the momentum of the i^{th} particle, by $\mathcal{E}_i = \sqrt{p_i^2 + m_i^2}$ its energy, by $\vec{P}$ and E the total momentum and energy of the system. The weight function $f(p_1, \dots, p_n)$ takes into account the exact version of the SM, which is considered. It is unity for the non-covariant formulation of the SM^{1), 10)}, and equals $\prod_{i=1}^n (p_i^2 + m_i^2)^{-\frac{1}{2}}$ for the covariant formulation¹¹⁾. We will not specify it any further, but note that it depends only on the absolute value of the particle momenta, not on their direction. The exact form of the weight function is completely irrelevant for the further discussion, which deals with effects due to the angular dependence of the momenta.

In order to obtain a simple approximation to Eq. (1) we first observe that it depends on the direction of the particle momenta $\vec{p}_i$ only through the volume elements $d\vec{p}_i$ and through the momentum conserving factor $\delta^{(3)}(\sum_i \vec{p}_i - \vec{P})$. Therefore, if we keep the lengths of the momentum vector, p_i , fixed and for a moment neglect momentum conservation, the angular integration yields a factor $(4\pi)^n$. The momentum vectors couple randomly in space to a wide range of different total momenta $\vec{P}$. The effect of the momentum conservation is simply to single out just those combinations of the momentum vectors $\vec{p}_i$, which couple to the given total momentum $\vec{P}$. To account for momentum conservation, we should therefore multiply $(4\pi)^n$ by the 'a priori' probability $P(p_1, \dots, p_n; \vec{P})$ that n momentum vectors of given lengths p_i add randomly to the total momentum $\vec{P}$.

This factor is identically

$$P(p_1, \dots, p_n; \vec{P}) = \frac{1}{(4\pi)^n} \dots d\vec{n}_1 \dots d\vec{n}_n \delta^{(3)}(\sum_i p_i \vec{n}_i - \vec{P}) \quad (2)$$

where $\vec{n}_i$ are unit vectors. We notice in passing that the integration of the 'a priori' probability over all $\vec{P}$ in Eq. (2) yields unity as should be. We can thus identically rewrite Eq. (1) in terms of Eq. (2) as

$$\mathcal{S}_n^*(E, \vec{P}) = (4\pi)^n \int_0^\infty \dots \int_0^\infty p_1^2 dp_1 \dots p_n^2 dp_n f(p_1, \dots, p_n) P(p_1, \dots, p_n; \vec{P}) \times \delta(\sum_i \varepsilon_i - E) \quad (3)$$

The importance of identically expressing Eq. (1) as Eq. (3) in terms of the 'a priori' probability is that this probability depends only on the lengths of the momentum vectors, p_i . If we therefore can find a reasonably simple expression for $P(p_1, \dots, p_n; \vec{P})$ the number of integrations necessary to evaluate the phase space has been reduced from $3n$ to n .

The second observation is that the total momentum $\vec{P}$ in nearly all practical applications is quite small; in most problems it will be the momentum of one or two of the particles only, and when we are interested in the phase space in the CM, we have even $\vec{P} \equiv 0$. Under these circumstances an asymptotic expression for $P(p_1, \dots, p_n; \vec{P})$ valid for large n is readily obtained by the following statistical consideration :

Suppose we have n independent quantities $x_1, \dots, x_n$ which have probability distributions so that their mean values

$$\langle x_i \rangle = 0 \quad (4)$$

for all of them. The sum $X = \sum_i x_i$ will then have a distribution which tends very rapidly towards a Gaussian with increasing n (the central limit theorem):

6.

$$P_x(X) \simeq \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-X^2/2\sigma^2} \quad (5)$$

The mean square deviation of X from $0, \sigma^2$, is given by

$$\sigma^2 = \sum_i \langle x_i^2 \rangle. \quad (6)$$

We apply the central limit theorem to the projections of the momentum vectors p_i on some axis, p_{ix} . As the direction of p_i is considered to be random in our approximation, either sign is equally probable; thus, $\langle p_{ix} \rangle = 0$. Furthermore, no axis in space is preferred, and we have therefore

$$\langle p_{ix}^2 \rangle = \frac{1}{3} \langle p_{ix}^2 + p_{iy}^2 + p_{iz}^2 \rangle = p_i^2/3.$$

The sum of the projections is obviously the total momentum component P_x along this axis.

$$P_x = \sum_i p_{ix} \quad (7)$$

Applying Eqs. (5) and (6) we have immediately

$$P_x(p_1, \dots, p_n; P_x) \simeq \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-P_x^2/2\sigma^2} \quad (8)$$

with σ^2 given by

$$\sigma^2 = \frac{1}{3} \sum_i p_i^2 \quad (9)$$

We could have used any axis in space for this argument, as the prescribed length p_i of each random vector does not impose any restriction on the sign of a projection. The probability of finding a total momentum $\vec{P}$ is the product of the probabilities in three perpendicular directions. Using the isotropy of the problem :

$$P(p_1, \dots, p_n; \vec{P}) \simeq \frac{1}{(2\pi\sigma^2)^{3/2}} e^{-\frac{(P_x^2 + P_y^2 + P_z^2)/2\sigma^2}{}} = \frac{1}{(2\pi\sigma^2)^{3/2}} e^{-\frac{\vec{P}^2/2\sigma^2}{}} \quad (10)$$

Substitution of Eqs. (9) and (10) into Eq. (3) gives

$$\mathcal{S}_n^*(E, \vec{P}) \simeq (4\pi)^n \int_0^\infty \dots \int_0^\infty p_1^2 dp_1 \dots p_n^2 dp_n \frac{f(p_1, \dots, p_n)}{\left[2\pi \cdot \frac{1}{3}(\sum_i p_i^2)\right]^{3/2}} \quad (11)$$

$$\exp\left\{-\frac{\vec{P}^2/2}{3}(\sum_i p_i^2)\right\} \delta(\sum_i \varepsilon_i - E)$$

The evaluation of Eq. (11) is quite straightforward by ordinary computer methods and we have thus obtained a simple approximation to the momentum phase space, $\mathcal{S}_n^*(E, \vec{P})$. It is useful, however, to make further approximations of Eq. (11) in order to get a good qualitative understanding of the influence of momentum conservation and to be able to evaluate $\mathcal{S}_n^*(E, \vec{P})$ simply without the use of a computer.

We observe that the probability function depends on the momenta p_i only through the quantity $\sum_i p_i^2$. This implies that the dependence of $P(p_1, \dots, p_n; \vec{P})$ on any individual one of the p_i is quite weak; the sum $\sum_i p_i^2$ is nearly a constant over the range of combinations p_i , which gives the main contribution to the phase space integral (11). We will therefore only commit small errors in replacing the sum by its mean value. The probability function will then not depend on momenta and can be taken outside the integration signs. We denote the phase space without momentum restrictions by $N_n^*(E)$

$$N_n^*(E) = (4\pi)^n \int_0^\infty \dots \int_0^\infty p_1^2 dp_1 \dots p_n^2 dp_n f(p_1, \dots, p_n) \delta(\sum_i \varepsilon_i - E) \quad (12)$$

According to the argument above for the near constancy of $\sum_i p_i^2$, we write Eq. (11) as

$$g_n^*(E, \vec{P}) = \left\langle \frac{\exp\left\{-\vec{P}^2/2\left(\sum_i p_i^2\right)\right\}}{\left[2\pi \cdot \frac{1}{3}\left(\sum_i p_i^2\right)\right]^{3/2}} \right\rangle N_n^*(E) \simeq \frac{\exp\left\{-\vec{P}^2/2\left\langle\sum_i p_i^2\right\rangle\right\}}{\left[2\pi \cdot \frac{1}{3}\left\langle\sum_i p_i^2\right\rangle\right]^{3/2}} N_n^*(E) \quad (13)$$

In the last step in Eq. (13) we have everywhere replaced the sum $\sum_i p_i^2$ by a mean value which is defined by

$$\left\langle\sum_i p_i^2\right\rangle N_n^*(E) \equiv \sum_i (4\pi)^n \int_0^\infty \dots \int_0^\infty p_1^2 dp_1 \dots p_i^4 dp_i \dots p_n^2 dp_n f(p_1, \dots, p_n) \times \delta\left(\sum_i \epsilon_i - E\right) \quad (14)$$

The expression for $g_n^*(E, \vec{P})$ which is approximately obtained in Eqs. (13) and (14) becomes exact in the limit of large multiplicities. It must, however, be emphasized that Eq. (11) is more appropriate for a computer evaluation of $g_n^*(E, \vec{P})$.

Eq. (13) permits immediately the following statement about the covariant and non-covariant S.M. : we must expect that $\left\langle\sum_i p_i^2\right\rangle$ is nearly the same for the two models. Therefore the ratio between two predicted cross-sections, based on the two models, will be nearly independent of momentum conservation. Only when absolute branching ratios are calculated is it necessary to include momentum conservation.

To estimate the error which typically is associated with the RCA we will compare the approximate evaluation of $g_n^*(E, \vec{P})$ to limiting cases which can be exactly evaluated. Two convenient cases of this kind are provided by the non-covariant model in the limit when either all the emitted particles are extremely relativistic (E.R.) or when they are non-relativistic (N.R.). It will be shown that the errors introduced are of order $1/n$ and that they thus vanish as the multiplicity becomes large.

The limiting cases of the non-covariant model with momentum conservation have the exact phase space integrals :

$$\mathcal{S}_n^{*(E, \vec{P})}_{NR} = \frac{(2\pi)^{\frac{3n-3}{2}} \left(\prod_{i=1}^n m_i^{3/2} \right) \left[T - \frac{\vec{P}^2}{2(\sum_{i=1}^n m_i)} \right]^{\frac{3n-5}{2}}}{(\sum_{i=1}^n m_i)^{3/2} \Gamma(\frac{3n-3}{2})} ; \quad n \geq 2 \quad (15)$$

where $T = E - \sum_{i=1}^n m_i$ is the total kinetic energy of the system;

$$\mathcal{S}_n^{*(E, \vec{P})}_{ER} = \left(\frac{\pi}{2}\right)^{n-1} \frac{1}{(2n-1)!} \frac{1}{(2n-2)!} \frac{d^n}{dE^n} (E^2 - \vec{P}^2)^{2n-2} ; \quad n \geq 2 \quad *) \quad (16)$$

We will also need the corresponding phase space without momentum conservation

$$N_n^{*(E)}_{NR} = \frac{(2\pi)^{\frac{3n}{2}} \left(\prod_{i=1}^n m_i^{3/2} \right) T^{\frac{3n-1}{2}}}{\Gamma(\frac{3n}{2})} \quad 1) \quad (17)$$

and

$$N_n^{*(E)}_{ER} = (8\pi)^n \frac{E^{3n-1}}{(3n-1)!} \quad 1) \quad (18)$$

The approximate results are expected to be most accurate for relatively small $|\vec{P}|$; we will therefore study the accuracy of the first coefficients in a series expansion in $\vec{P}^2$. Eq. (13) suggests that we expand

*) Eqs. (15) and (16) are given by Lepore and Stuart ¹²⁾ for $\vec{P}=0$. Eq. (15) is directly obtained from their result by subtracting the energy of CM motion from the total kinetic energy T ; Eq. (16) is derived by a generalisation of the technique used by Lepore and Stuart.

$$\frac{g_{n(E, \vec{P})}^*}{N_n^*(E)} = A \left[1 - \frac{\vec{P}^2}{2S_2^2} + \frac{1}{2!} \left(\frac{\vec{P}^2}{2S_4^2} \right)^2 - \dots \right] \quad (19)$$

Our approximation implies

$$A \simeq \frac{1}{(2\pi)^{3/2}}; \quad S_2^2 \simeq S_4^2 \simeq \dots \simeq \sigma^2 \quad (20)$$

as is immediately evident from Eq. (13).

From Eqs. (15) and (17) we have

$$A_{NR} = \frac{1}{(2\pi)^{3/2}} \frac{1}{(T \sum_i m_i)^{3/2}} \frac{\Gamma(\frac{3n}{2})}{\Gamma(\frac{3n-3}{2})} \quad (21)$$

$$(S_2^2)_{NR} = \frac{2}{3n-5} T(\sum_i m_i)$$

and similarly from Eqs. (16) and (18)

$$A_{ER} = \frac{2}{\pi} \frac{(3n-1)(3n-2)(3n-3)}{(2n-1)} 2^{-4n} \binom{4n-4}{2n-2} E^{-3} \quad (22)$$

$$(S_2^2)_{ER} = \frac{4n-5}{(3n-4)(3n-5)} E^2;$$

To compare the accuracy to which (20) and (22) are approximated, we calculate σ^2 and use Eq. (20) :

$$\sigma_{NR}^2 = \frac{1}{3} \langle \sum_i p_i^2 \rangle_{NR} = \frac{2}{3} \frac{(\sum_i m_i)}{n} T \quad (23)$$

$$\sigma_{ER}^2 = \frac{1}{3} \langle \sum_i p_i^2 \rangle_{ER} = \frac{4}{3} \frac{E^2}{(3n+1)}$$

which obtains from the definition of $\langle \sum_i p_i^2 \rangle$ in Eq. (14) after elementary integrations.

From Eq. (23) together with Eq. (21) we obtain the ratio of the exact coefficients to the approximate random coupling coefficients :

$$\begin{aligned} (2\pi\sigma_{\text{NR}}^2)^{3/2} A_{\text{NR}} &= 1 - \frac{5}{4} \frac{1}{n} + O\left(\frac{1}{n^2}\right) \\ \frac{(s_2^2)_{\text{NR}}}{\sigma_{\text{NR}}^2} &= 1 + \frac{5}{3} \frac{1}{n} + O\left(\frac{1}{n^2}\right) \end{aligned} \quad (24)$$

The same procedure applied to the relativistic limit, Eqs. (21) and (23), gives

$$\begin{aligned} (2\pi\sigma_{\text{ER}}^2)^{3/2} A_{\text{ER}} &= 1 - \frac{33}{16} \frac{1}{n} + O\left(\frac{1}{n^2}\right) \\ \frac{(s_2^2)_{\text{ER}}}{\sigma_{\text{ER}}^2} &= 1 + \frac{35}{12} \frac{1}{n} + O\left(\frac{1}{n^2}\right) \end{aligned} \quad (25)$$

It is thus clear from Eqs. (24) and (25) that the errors introduced by our approximation is of the order $1/n$; the approximation becomes increasingly exact as $n \rightarrow \infty$. The approximation is therefore most useful in the case of high multiplicities, i.e. when computer calculations tend to become prohibitively long.

The approximation tested in Eqs. (24) and (25) is in general the joint effect of introducing a probability distribution instead of using exact momentum conservation [Eq. (11)] and that of replacing $\sum_i p_i^2$ by its mean value [Eq. (13)]. It is therefore interesting to notice that for non-relativistic particles of equal mass $\sum_i p_i^2 = 2mT$, i.e. no approximation is involved in assuming the sum to be a constant. In this case the effect of the first approximation is tested separately, and for this case Eq. (24) gives the accuracy of Eq. (11) directly. A comparison to Eq. (25) which involves the additional approximation of Eq. (13) seems to indicate that it approximately doubles the error, if conditions are not particularly favourable.

The main advantage of the RCA is that it permits very rapid, though approximate, calculation of the phase space $\mathcal{S}_n^*(E, \vec{P})$ and that it clarifies the extent to which it is important to include momentum conservation. With the exception of very small multiplicities, the phase space is nearly always sufficiently well represented by this method, in particular when it is realised that the inclusion of angular momentum conservation introduces corrections to the relative branching ratios in the SM which are at best of the same order as the errors in our approximation and usually considerably larger (see section 4). The influence of angular momentum is usually ignored; it is thus by no means evident that a very exact evaluation of the momentum phase space provides any quantitative advantage over the RCA.

As a simple, but non-trivial, application of the RCA we finally calculate qualitatively the angular correlation between two non-identical particles due to momentum conservation [cf. ¹³⁾]. We neglect the influence of angular momentum conservation, so that the result applies best to the case of nucleon-antinucleon annihilation for which this correction is rather small ^{*)}. For illustration we calculate the mean angular correlation between two particles in the extreme relativistic limit for the non-covariant model. The angular correlation of particles 1 and 2 is determined by the phase space available for the $(n-2)$ remaining particles, which have an energy $E - \mathcal{E}_1 - \mathcal{E}_2$ and a momentum $-(\vec{p}_1 + \vec{p}_2)$. This phase space is thus $\mathcal{S}_{n-2}^*(E - \mathcal{E}_1 - \mathcal{E}_2; -(\vec{p}_1 + \vec{p}_2))$ in the CM of all the particles. To obtain the mean angular correlation $\langle W(\vec{p}_1, \vec{p}_2) \rangle$ to a good approximation we simply replace the energies $\mathcal{E}_1 = p_1$ and $\mathcal{E}_2 = p_2$ by their mean values $\frac{E}{n}$. We furthermore observe, that only the factor which imposes restrictions on the length of the total momentum will contribute to the angular correlation. Eq. (13) gives thus

$$\langle W(\vec{p}_1, \vec{p}_2) \rangle \propto \langle e^{-\frac{(\vec{p}_1 + \vec{p}_2)^2}{2\sigma^2}} \rangle \propto e^{-\frac{\langle p_1 \rangle \langle p_2 \rangle \cos \theta}{\sigma^2}} \quad (26)$$

where θ is the angle between $\vec{p}_1$ and $\vec{p}_2$.

^{*)} The consideration can, however, be generalised to include angular momentum conservation.

The value of σ^2 is obtained from Eq. (23), and has to be taken for $(n-2)$ particles with a total energy $\frac{n-2}{n}E$. Neglecting unity as compared to $3n$ we obtain

$$\sigma^2 \simeq \frac{4}{9} \cdot \frac{n-2}{n^2} E^2 \quad (27)$$

This yields immediately on substitution into Eq. (26) the approximate and normalised correlation

$$\langle W(\vec{p}_1, \vec{p}_2) \rangle \simeq \frac{9}{4(n-2)} e^{-\frac{9}{4} \frac{\cos \theta}{(n-2)}}; \quad n \gg 1 \quad (28)$$

which is in a convenient form for crude estimates. As expected, the angular correlation is an anticorrelation of $\vec{p}_1$ and $\vec{p}_2$, which disappears with increasing multiplicity.

3. Statistical weight factors

Since the branching ratios in the SM are considered directly proportional to the available phase space for the reaction products, we have to calculate the accessible volume in the discrete spin-isospin phase space in addition to that of configuration-momentum phase space.

Suppose n distinguishable particles to be emitted and denote their isospins by t_i , their spins by s_i . There is an 'a priori' statistical weight factor g_i associated with every particle. If the experiment does not ask the charge of the particle, $g_i = (2t_i+1)(2s_i+1)$, which corresponds to the different discrete values of the z-components of spin and isospin. If the charge of the particle is determined, the z-component of the isospin is known, and the weight factor is $g_i = 1 \cdot (2s_i+1)$. We have assumed that the polarisation of the emitted particles is not studied. Therefore, if no conservation laws impose restrictions on the discrete phase space available, this is simply $\prod_{i=1}^n g_i$. We have furthermore to take into account that not all particles are distinguishable in practice; all the permutations of indistinguishable particles correspond to the same configuration and must be counted once only. Members of different isospin multiplets are always distinguishable. If the experiment measures only the total production of an isospin multiplet, all the particles belonging to the multiplet are indistinguishable; on the other hand, if the experiment studies the charge distribution inside the multiplet, particles of different charge become distinguishable, while those of same charge remain indistinguishable. To count permutations of indistinguishable particles once only, we have to multiply the spin-isospin phase space factor by $\frac{1}{\nu_1! \dots \nu_a!}$, where ν_j denotes the number of indistinguishable particles of kind j . Before the restrictions due to conservation laws are accounted for the statistical weight factor M_n is thus

$$M_n = \frac{\prod_{i=1}^n g_i}{\prod_{j=1}^a \nu_j!} \quad (29)$$

The requirement that total angular momentum and total isospin are conserved restricts the phase space volume M_n . The effect of angular momentum conservation on Eq. (29) is nearly negligible. This is because the intrinsic spins $\vec{s}_i$ are much smaller than the orbital angular momenta $\vec{l}_i$ with which they must couple to give the total angular momentum $\vec{I}$. The conservation of angular momentum will therefore mainly effect the orbital angular momenta (see section 4), while the sum $\sum_i \vec{s}_i = \vec{S}$ will be permitted to take any value within its possible range nearly unrestricted. In other words : the orbital angular momentum of the system acts as a reservoir, which can accommodate the small angular momentum of the total spin $\vec{S}$ without difficulty.

The situation is very different for the isospins. The two initially colliding particles specify both the total isospin T of the system and its projection T_z . The isospins of the emerging particles necessarily couple to these values. Out of the unrestricted statistical weight M_n we have therefore to select exactly those combinations of $\vec{t}_i$, which give T and T_z . The correct statistical weight factor $L_n(T, T_z)$ is obtained from Eq. (29) by multiplying with the 'a priori' probability $P(T, T_z)$ that the n isospin spin vectors $\vec{t}_i$ couple to T and T_z .

$$L_n(T, T_z) \equiv P(T, T_z) M_n = P(T, T_z) \frac{\prod_{i=1}^n g_i}{\prod_{j=1}^a \nu_j!} \quad (30)$$

To obtain the 'a priori' probability $P(T, T_z)$ we will consider the isospin vectors $\vec{t}_i$ to be semi-classical, so that they couple like ordinary vectors in space. The 'a priori' probability can then be obtained considering the random coupling of vectors in analogy with the case of momentum vectors in the preceding section. We will later compare our results in special cases to numerical quantal results, which show that this approximation is excellent.

We have to distinguish clearly between two different cases : a) no charges of the emitted particles are specified, so that there is no further condition on the $\vec{t}_i$. b) the charges of the particles are known, thus the projections t_{iz} are given. We are then free to couple randomly only that part of $\vec{t}_i$ which is perpendicular to the z-axis. In addition to these two cases, one may occasionally deal with mixed cases, in which the charges of certain particles only are specified. For example, only the pion charges may be determined, and the rest left unspecified.

We will first treat case a) when the isospin vectors couple completely randomly in space. Eqs. (9) and (10) for the asymptotic expression of the 'a priori' probability of randomly coupling momentum vectors can be used immediately for the coupling of n vectors $\vec{t}_i$ to a resultant $\vec{T}$.

$$P(\vec{T}) \simeq \frac{1}{(2\pi\sigma^2)^{3/2}} e^{-\vec{T}^2/2\sigma^2} \quad (31)$$

$$\text{with } \sigma^2 = \frac{1}{3} \sum_i \vec{t}_i^2.$$

In the case of isospins we are not interested in the 'a priori' probability of a $\vec{T}$ with a specific direction, but in that for which the absolute value of the isospin is T and its projection T_z . Consequently we integrate Eq. (31) over T_x and T_y with the restriction $T_x^2 + T_y^2 = \vec{T}^2 - T_z^2 = \text{const.}$

$$P(T, T_z) = \iint_{T_x^2 + T_y^2 = \vec{T}^2 - T_z^2} dT_x dT_y P(\vec{T}) = \pi^{1/2} P(\vec{T}) \simeq \pi^{-1/2} \frac{2T}{(2\sigma^2)^{3/2}} e^{-\vec{T}^2/2\sigma^2} \quad (32)$$

This simple result, which we have obtained by purely classical reasoning, is generalised to a semi-classical expression by observing Kramers' rules of replacing T by $T+\frac{1}{2}$ and $\vec{T}^2$ by $T(T+1)$. Correspondingly, we have to replace $\vec{t}_i^2$ by $t_i(t_i+1)$ for the individual isospin vectors in the expression for σ^2 . This gives

$$P(T, T_z) \simeq \pi^{-\frac{1}{2}} \frac{2T+1}{(2\sigma^2)^{3/2}} e^{-T(T+1)/2\sigma^2} \quad (33)$$

with $\sigma^2 = \frac{1}{3} \sum_i t_i(t_i+1)$.

In complete analogy with the results of section 2 we expect this semi-classical expression to be correct to order $1/n$.

The 'a priori' probability of Eq. (33) is the one of greatest practical importance, as it corresponds to the case when the charge distribution among the emerging particles is left unspecified. It is sometimes of interest to study the charge distribution (case b); for example, we may want to consider the emission of n_+ , n_0 and n_- pions of positive, neutral and negative charges. In this case the projection t_{iz} of the isospin $\vec{t}_i$ is determined. The derivation of the 'a priori' probability of a certain T and T_z is similar to the preceding, with the difference that the isospins are free to couple randomly only in the plane perpendicular to the z -axis, thus in two dimensions only. The length of a vector which couples in this plane is $t_{i\perp} = \sqrt{t_i^2 - t_{iz}^2}$. Furthermore, we obviously must have

$$T_z = \sum_i t_{iz} \quad (34)$$

The probability $P(T, T_z)$ is thus

$$P(T, T_z) \simeq \iint_{T_x^2 + T_y^2 = \vec{T}^2 - T_z^2} \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} e^{-T_x^2/2\sigma^2} \cdot \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} e^{-T_y^2/2\sigma^2} dT_x dT_y = \quad (35)$$

$$\frac{1}{2\pi\sigma^2} \iint_{\vec{T}_\perp^2 = \vec{T}^2 - T_z^2} e^{-\vec{T}_\perp^2/2\sigma^2} dT_x dT_y = \frac{2T}{2\sigma^2} e^{-(\vec{T}^2 - T_z^2)/2\sigma^2}$$

valid for $T \gg |T_z|$ and with Eq. (34) fulfilled. In the derivation of Eq. (35) we have made use of the relation $d(T_\perp^2) = d(T^2) = 2TdT$. The quantity σ^2 applies this time to two dimensions and as none of the x - or y -direction are specified

$$\sigma^2 = \langle \sum_i t_{ix}^2 \rangle = \frac{1}{2} \langle \sum_i (t_{ix}^2 + t_{iy}^2) \rangle = \frac{1}{2} \sum_i (\vec{t}_i^2 - t_{iz}^2) \quad (36)$$

The corresponding semi-classical result again follows immediately from Kramers' rules and is

$$P(T, T_z) \simeq \frac{(2T+1)}{2\sigma^2} e^{-\{T(T+1) - T_z^2\}/2\sigma^2}; \quad T \gg |T_z|$$

with

$$\sigma^2 = \frac{1}{2} \sum_i \{t_i(t_i+1) - t_{iz}^2\} \quad (37)$$

This technique can also be applied to more complex cases. Consider for example the case when the charges of n_1 of the particles of isospin $\vec{t}_1$ are measured (e.g. the nucleon charges may be studied), while the charge distribution among the rest of the particles, n_2 , is unspecified. The isospins of the latter particles are denoted by $\vec{t}_k$. Thus

$$\begin{aligned}\vec{T} &= \sum_i \vec{t}_i + \sum_k \vec{t}_k ; \\ T_z &= \sum_i t_{iz} + \sum_k t_{kz} = T_{1z} + T_{2z}\end{aligned}\quad (38)$$

Here T, T_z and T_{1z} are known quantities. We can regard the $\vec{t}_k$ as coupling randomly in space, while the $\vec{t}_i$ only couple their component perpendicular to the z -axis randomly. By reasoning in analogy with the preceding we find the semi-classical probability, valid asymptotically for n_1 and n_2 both large :

$$P(T, T_z, T_{1z}) \simeq \frac{1}{2} \frac{(2T+1)}{(\sigma_1^2 + \sigma_2^2)} \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(T_z - T_{1z})^2}{2\sigma_2^2}} e^{-\frac{T(T+1) - T_z^2}{2(\sigma_1^2 + \sigma_2^2)}} ; T \gg |T_z| \quad (39)$$

with

$$\begin{aligned}\sigma_1^2 &= \frac{1}{2} \sum_i \{t_i(t_{i+1}) - t_{iz}^2\} \\ \sigma_2^2 &= \frac{1}{3} \sum_k t_k(t_k+1)\end{aligned}$$

Eq. (39) is expected to be correct to order $1/n_1$ and $1/n_2$ *).

The exact 'a priori' probabilities can in principle be calculated by application of Racah techniques ⁴⁾⁻⁶⁾ or by the use of projection operators ⁷⁾, but in most cases the expressions do not lend themselves to an easy comparison with our approximate values.

*) In particular, Eq. (39) does not apply to $n_1=0$ or $n_2=0$ as is evident from comparison to Eqs. (33) and (37).

An exception is the statistical factor for the coupling of n pions ($t_i=1$), of which n_+ , n_0 and n_- have positive, neutral and negative charge. For $T=0$ Cerulus ⁷⁾ obtains the exact 'a priori' probability ($n_+=n_-$)

$$P_{n_+ n_0 n_-}^{T=0} = \begin{cases} 2^{-2n_+} \sum_{q=0}^{n_+} \binom{2n_+}{2q} \frac{1}{n_0+2q+1} \approx \frac{1}{2n_0+n_++n_-+1} & \text{for } n_0 \text{ even} \\ 2^{-2n_+} \sum_{q=0}^{(n_+-1)} \binom{2n_+}{2q+1} \frac{1}{n_0+2q+2} \approx \frac{1}{2n_0+n_++n_0+1} & \text{for } n_0 \text{ odd} \end{cases} \quad (40)$$

where the approximate value is correct to order $1/n$. Eq. (37) gives the approximate value for $T=0$ as $1/2\sigma^2$ with

$$\sigma^2 = \frac{1}{2} \left[n_+(1.2-1^2) + n_0(1.2-0^2) + n_-(1.2-1^2) \right] = \frac{n_++2n_0+n_-}{2}$$

Our approximation thus gives the result $1/(2n_0+n_++n_-)$, which is identical with the result of Eq. (40) to order $1/n$.

Tabulated values for the 'a priori' probability exist for special cases and small number of particles ⁴⁾⁻⁶⁾. They provide a further check on our approximation. In Figs. (1)-(2) we have studied the exact and approximate values of the 'a priori' probability. All the comparisons have been made after division by the trivial phase space factor $(2T+1)$. We have used $T(T+1)$ as variable and a logarithmic scale, so that the random coupling approximation yields a straight line. From the figures it is obvious that the variables used are the natural ones, as the exact solutions approximate a straight line very well. The quantitative agreement is surprisingly good even when the number of particles is very small.

The approximate formulae for statistical weights are not in any way intended to compete in accuracy with the exact expressions obtained by conventional techniques. They are intended to bring out the asymptotic limits for the 'a priori' probabilities and to give a simple, qualitative insight into their dependence on various quantities. In addition, they should be quite useful for orientation about the magnitude of the weight factors. We want finally to emphasize that our semi-classical approximation is most accurate for low values of T and T_z , contrary to what might be thought of a classical approach.

4. Angular momentum effects

Since the SM makes the assumption that strong interaction reactions are governed by the phase space available to the reaction products, any calculation of the phase space must account for all restrictions imposed on it. In particular, in the CM system not only the total momentum, $\vec{P}=0$, is a good quantum number, but also the total angular momentum of the system. If we adopt a purely classical view, we have thus to impose the condition on the configuration momentum phase space that the sum of the individual particle angular momenta, $\vec{l}_i = \vec{r}_i \times \vec{p}_i$, add to the total angular momentum $\vec{I}$. The accessible volume in the classical phase space including momentum and angular momentum conservation is simply

$$R_n(E, \vec{P}=0, \vec{I}) = \int \dots \int d\vec{p}_1 \dots d\vec{p}_n d\vec{r}_1 \dots d\vec{r}_n f(p_1, \dots, p_n) \delta^{(3)}(\sum_i \vec{p}_i) \delta(\sum_i \varepsilon_i - E) \delta^{(3)}(\sum_i \vec{r}_i \times \vec{p}_i - \vec{I}) \quad (41)$$

The integrals over the configuration space variables, $\vec{r}_i$, are over the interaction volume Ω . It is obvious that the introduction of angular momentum conservation must effect branching ratios; it should, however, also be noticed that the classical expression (41) necessarily implies a unique angular distribution of the products, since the direction of a $\vec{p}_i$ is not independent of that of $\vec{I}$ because of the angular momentum conservation ^{*)}. There is therefore no ground whatsoever for the belief that statistical angular distributions have to be isotropic. Isotropy will only result if the total angular momentum $\vec{I}$ takes on any direction in space with equal probability. Since it is perpendicular to the beam, we do not have isotropy.

^{*)} Cook ²⁾ and Koba ³⁾ have considered some aspects of angular momentum conservation effects on branching ratios from a quantum mechanical viewpoint but neglecting momentum conservation. Their results are rather involved and the structure of their solutions is not evident.

While the introduction of angular momentum conservation poses no problem of principle for the classical system, Eq. (41) cannot be taken over identically to quantum mechanics since the particle momentum $\vec{p}_i$ does not commute with the particle angular momentum $\vec{l}_i$. We will therefore briefly analyse the approximations which go into Eq. (41) and at the same time deduce general properties of angular distributions in statistical processes.

The total angular momentum I of the system is a good quantum number. The two initially colliding particles are generally not in a state of definite total angular momentum, but we can easily analyse the process in terms of partial waves. The matrix element for any reaction will thus be a sum of coherent contributions from different I . Since the corresponding cross-section is proportional to the matrix element squared, there will occur interference terms between different partial waves ^{*)}. At this point, we invoke the basic assumption of the SM, namely that the formation process is unrelated to final products except through conservation of exact quantum numbers. Since phase relations between the matrix elements of different channels carry information about the formation process we are therefore forced to make the statistical assumption that the matrix elements of different channels have uncorrelated, random phases; this is the natural assumption of any SM and seems an implicit assumption in previous work [e.g. ¹⁰⁾] since any systematic constructive or destructive interference destroys the possibility of considering branching ratios proportional to phase space on the average. While we here only need the randomness of phases, it should be observed that it is a further consequence of the statistical assumption that the matrix elements have Gaussian probability distributions, if the assumption of randomness is to be invariant with respect to rotations in the Hilbert space of state vectors ^{**)}.

*) There are some cases in which no interference between partial waves occurs, notably for the multiplicities. Interference is, however, important in the case of energy distributions and angular distributions.

**) See ref. 9) section 11.

We now apply the random phase assumption to the partial waves. When we take the average over a large number of events the interference between different partial waves, I and I' , will drop out since it is random; the contributions from different angular momenta are completely incoherent on the average. This has a most important consequence for the angular distribution of secondary particles : the average angular distribution will have forward-backward symmetry in CM with respect to the beam direction even when the colliding particles are not identical This result is easily understood as follows : since only exact quantum numbers determine the final products and their properties it is in particular irrelevant how the total angular momentum I and its projection I_z have been achieved. These two quantum numbers will be the same, if the two colliding particles are interchanged and the reaction products must thus have a forward-backward symmetry in their distribution. This prediction is very general and is independent of parameters of the model; it applies equally well if final state interactions are included, since it is, in a sense, a statement about the nature of the intermediate, strongly interacting, stage of the process, in which the formation process becomes independent of the emission process. The general nature of this property of the angular distribution makes it very suitable as a test for separating statistical from non-statistical reactions. Any process with an average non-symmetrical angular distribution must be non-statistical, while the reverse is not necessarily true ^{*)}. The occurrence of random phases in the SM has the further consequence that non statistical and statistical processes become incoherent on the average and thus will be purely additive.

In order to obtain more precise statements about the influence of angular momentum conservation, we observe that the mean momentum of the emitted particles in most cases is fairly large. This suggests that we neglect the non-commutativity of $\vec{r}_i$ and $\vec{p}_i$ in calculating the phase space. Crude estimates indicate that $\langle r_i p_i \rangle \sim 3-5$, which implies that such an approximation hardly will affect any qualitative aspects of the problem. If we drop interference terms as before we

*) One expects naïvely that low multiplicity events tend to retain more memory of the formation process than high multiplicity events. As an example : $K^- + p \rightarrow \sum + \pi$ has an asymmetrical angular distribution which is unexplainable on the basis of the SM.

are immediately led to Eq. (41) for the description of the effects of angular momentum. We will for the moment treat $\vec{I}$ as having a specific direction and later take into account that only its magnitude and projection are known.

In the spirit of the classical approximation we account for the intrinsic spins of the particles only by their statistical weight factors, which is to say that the particle orbital angular momenta are large compared to their intrinsic spins. For simplicity we will first discuss the case of a spherically symmetric interaction volume Ω_0 ; this brings out all the qualitative aspects of the problem, but since the interaction volume usually is taken to be Lorentz contracted, we will for the completeness discuss the case of cylindrical symmetry around the beam direction later.

We start from the classical expression for the available phase space, Eq. (41). In analogy with section 2, we observe that the delta functions of momentum and angular momentum conservation just single out those combinations of $\vec{r}_i$ and $\vec{p}_i$ which happen to give the correct value of $\vec{P}=0$ and $\vec{I}$. We therefore use the same technique as in section 2 and consider Eq. (41) for fixed absolute values of $\vec{r}_i$ and $\vec{p}_i$, but carry out the angular integrations over these variables. Without delta function restrictions the angular integration over the $2n$ solid angles gives $(4\pi)^{2n}$; the delta functions reduce this value by the 'a priori' probability $P(p_1, \dots, p_n; r_1, \dots, r_n; \vec{P}=0; \vec{I})$ that the momenta and angular momenta of the particles add to $\vec{P}=0$ and $\vec{I}$. The phase space is thus identically

$$R_n(E, \vec{P}=0, \vec{I}) \equiv (4\pi)^{2n} \int \dots \int p_1^2 dp_1 \dots p_n^2 dp_n \cdot r_1^2 dr_1 \dots r_n^2 dr_n f(p_1, \dots, p_n) \delta(\sum_i \epsilon_i - E) P(p_1, \dots, p_n; r_1, \dots, r_n; \vec{P}=0; \vec{I}) \quad (42)$$

where the 'a priori' probability is given by

$$P(p_1, \dots, p_n; r_1, \dots, r_n; \vec{P}=0; \vec{I}) \equiv \frac{1}{(4\pi)^{2n}} \int \dots \int d\vec{e}_1 \dots d\vec{e}_n d\vec{n}_1 \dots d\vec{n}_n$$

$$\delta^{(3)}(\sum_i p_i \vec{n}_i) \delta^{(3)}(\sum_i p_i r_i \vec{e}_i \times \vec{n}_i - \vec{I}) \quad (43)$$

with $\vec{e}_i$ and $\vec{n}_i$ unit vectors.

To obtain a simple approximate expression for this 'a priori' probability we apply the random coupling approximation to Eq. (43) [cf. Eq. (3) and following]. The vectors $\vec{n}_i$ and $\vec{e}_i \times \vec{n}_i$ which occur in the delta functions do not have independent directions; their projections on any axis are, however, uncorrelated in sign. Hence we can regard $\vec{n}_i$ and $\vec{e}_i \times \vec{n}_i$ as effectively uncorrelated for the purpose of random coupling and we can regard the delta functions in Eq. (43) as referring to independent variables. The error introduced in this way is only of the order of that of the RCA itself. Application of Eqs. (4)-(10) to $\sum_i p_i \vec{n}_i$ and $\sum_i p_i r_i \vec{e}_i \times \vec{n}_i$ yields immediately

$$P(p_1, \dots, p_n; r_1, \dots, r_n; \vec{P}=0; \vec{I}) \approx \frac{1}{(2\pi\sigma_1^2)^{3/2}} \frac{e^{-\vec{I}^2/2\sigma_2^2}}{(2\pi\sigma_2^2)^{3/2}} \quad (44)$$

where

$$\sigma_1^2 = \sum_i p_i^2 \langle n_{ix}^2 \rangle = \frac{1}{3} \sum_i p_i^2$$

and

$$\sigma_2^2 = \sum_i p_i^2 r_i^2 \langle (\vec{e}_i \times \vec{n}_i)_x^2 \rangle = \frac{1}{3} \sum_i p_i^2 r_i^2 \langle \sin^2(\vec{r}_i, \vec{e}_i) \rangle = \frac{2}{9} \sum_i p_i^2 r_i^2$$

The relations for σ_1^2 and σ_2^2 follow immediately from the fact that no direction in space is preferred by $\vec{e}_i$ and $\vec{n}_i$. In order to simplify Eq. (44) even further we observe that the expression for σ_2^2 contains the sum over all particles of $p_i^2 r_i^2$. Obviously such a sum will depend only weakly on any of the individual r_i ; we therefore replace r_i^2 by its mean value $\langle r^2 \rangle$. This gives the approximate value of σ_2^2

$$\sigma_2^2 \approx \frac{2}{9} \langle r^2 \rangle \sum_i p_i^2 \quad (45)$$

The same statement applies equally well to the momentum variables of σ_1^2 and σ_2^2 . We therefore replace $\sum_i p_i^2$ by its mean value as obtained from momentum phase space without momentum and angular momentum conservation in Eq. (14). [This argument is the same as that leading from Eq. (11) to Eq. (13); the error introduced is discussed in section 2.] The expression for the available phase space becomes approximately

$$R_n(E, \vec{P}=0, \vec{I}) \approx \frac{1}{\left(\frac{2\pi}{3} \langle \sum_i p_i^2 \rangle\right)^{3/2}} \frac{\exp\left\{-\vec{I}/\frac{4}{9} \langle r^2 \rangle \langle \sum_i p_i^2 \rangle\right\}}{\left(\frac{4\pi}{9} \langle r^2 \rangle \langle \sum_i p_i^2 \rangle\right)^{3/2}} \Omega_{on}^{nN^*(E)} \quad (46)$$

where $N_n^*(E)$ is the momentum phase space as defined by Eq. (12), i.e. restricted by energy conservation only, and Ω_o is the interaction volume.

When the interaction volume has cylindrical symmetry around the beam direction (taken as the z-axis), we divide $\vec{r}_i$ and $\vec{p}_i$ into parts parallel to the beam direction, $\vec{r}_{iz}$ and $\vec{p}_{iz}$, and transverse to the beam direction, $\vec{r}_{iT}$ and $\vec{p}_{iT}$. We also divide the total angular momentum into corresponding parts, $\vec{I}_z$ and $\vec{I}_T$.

$$\begin{aligned} \vec{I}_z &= \sum_i \vec{r}_{iT} \times \vec{p}_{iT} \\ \vec{I}_T &= \sum_i (\vec{r}_{iT} \times \vec{p}_{iz} + \vec{r}_{iz} \times \vec{p}_{iT}) \end{aligned} \quad (47)$$

We treat the effects of momentum conservation identically as before, while we let the angular momentum components couple randomly in the z-direction and in the transverse plane according to Eq. (47). We have

$$P(p_1, \dots, p_n; r_{1T}, \dots, r_{nT}; r_{1z}, \dots, r_{nz}; \vec{P}=0; \vec{I}) \simeq \frac{1}{\left(\frac{2\pi}{3} \sum_i p_i^2\right)^{3/2}} \quad (48)$$

$$\frac{1}{(2\pi\sigma_{2z}^2)^{1/2}} \cdot \frac{1}{(2\pi\sigma_{2T}^2)} e^{-I_T^2/2\sigma_{2T}^2 - I_z^2/2\sigma_{2z}^2}$$

with

$$\sigma_{2z}^2 = \sum_i \langle (\vec{r}_{iT} \times \vec{p}_{iT})^2 \rangle = \sum_i r_{iT}^2 p_{iT}^2 \langle \sin^2(\vec{r}_{iT}, \vec{p}_{iT}) \rangle = \frac{1}{2} \sum_i r_{iT}^2 p_{iT}^2$$

$$\sigma_{2T}^2 = \sum_i \left\{ \langle (\vec{r}_{iT} \times \vec{p}_{iz})^2 \rangle + \langle (\vec{r}_{iz} \times \vec{p}_{iT})^2 \rangle \right\} = \frac{1}{2} \sum_i (r_{iT}^2 p_{iz}^2 + r_{iz}^2 p_{iT}^2)$$

We express Eq. (48) more conveniently by observing that only the linear dimensions of the volume in the z-direction should be changed. Denote by $\langle r_o^2 \rangle$ the mean value of the uncontracted radius squared and let

$$\langle r_{iz}^2 \rangle = \alpha^2 \langle r_{oz}^2 \rangle = \frac{\alpha^2}{3} \langle r_o^2 \rangle$$

where α is the linear contraction factor. As we have imposed no condition on the p_i in the coupling, the momenta are isotropic in space. Thus

$$\begin{aligned} \sigma_{2z}^2 &= \frac{2}{9} \langle r_o^2 \rangle \sum_i p_i^2 \\ \sigma_{2T}^2 &= \frac{1}{9} (1+\alpha^2) \langle r_o^2 \rangle \sum_i p_i^2 \end{aligned} \quad (49)$$

We have used the same arguments as previously in replacing the square of positions vectors by average values. In the same approximation as Eq. (46) we have

$$R_n(E, \vec{P}=0, \vec{I}) \simeq \frac{1}{\left(\frac{2\pi}{3} \langle \sum_i p_i^2 \rangle\right)^{3/2}} \frac{\exp\left\{-\frac{1}{2}\left(\frac{I_T^2}{1+\alpha^2} + \frac{I_Z^2}{2}\right)\left(\frac{1}{9} \langle r_o^2 \rangle \langle \sum_i p_i^2 \rangle\right)^{-1}\right\}}{\left(\frac{4\pi}{9} \langle r_o^2 \rangle \langle \sum_i p_i^2 \rangle\right)^{\frac{1}{2}} \left(\frac{2\pi}{9}(1+\alpha^2) \langle r_o^2 \rangle \langle \sum_i p_i^2 \rangle\right)} \quad (50)$$

$$\propto \Omega_0^n N_n^*(E)$$

where Ω_0 is the uncontracted volume.

Eqs. (46) and (50) give the branching ratios between different processes for given total angular momentum $\vec{I}$. The total angular momentum is entirely dominated by the orbital angular momentum of the initially colliding particles and is therefore perpendicular to the beam-direction; $I_Z=0$. The cross-section for different processes is obviously obtained by analysing the cross-section of the initially colliding particles into partial waves, which again can be made classically.

$$\sigma_n(E) \simeq \pi \lambda_i^2 \int_0^{I_m} 2I \frac{R_n(E, \vec{P}=0, I, I_Z=0)}{\sum_{all\ n} R_n(E, \vec{P}=0, I, I_Z=0)} dI \quad (51)$$

The maximum value of the total angular momentum is close to $I_m \simeq k_i R$, where R is the radius of the interaction volume.

While we can obtain the influence of angular momentum conservation on branching ratios from Eqs. (46), (48) and (51), it is even simpler to see the structure of the angular distribution of the emitted particles ^{*)}. If the particle angular momenta were free to take any direction in space the angular distribution of the emitted particles would be isotropic. Because of angular momentum conservation the particle angular momenta $\vec{l}_i = \vec{r}_i \times \vec{p}_i$ are forced to align partially with the total angular momentum $\vec{I}$. We will therefore obtain the maximum anisotropy possible in the SM, if we consider the case when the particle angular momenta are completely aligned with $\vec{I}$; any decoupling of $\vec{l}_i$ from $\vec{I}$ will cause a smearing of the angular distribution. In case of complete alignment, the particle will be emitted entirely in the plane perpendicular to $\vec{I}$, so that the angular distribution may be described by the delta function $\delta(\vec{n}_i \cdot \vec{I})$. Here $\vec{n}_i$ is the direction of the particle. The total angular momentum is perpendicular to the beam direction. The actually measured angular distribution $W_a(\vec{n}_i)$ is the average over the directions of $\vec{I}$ in the plane perpendicular to the beam. In terms of the azimuthal angle φ of $\vec{I}$ and with θ_i the angle between $\vec{n}_i$ and the beam

$$W_a(\vec{n}_i) = \frac{1}{2\pi} \int_0^{2\pi} \delta(\vec{n}_i \cdot \vec{I}) d\varphi = \frac{1}{2\pi} \int_0^{2\pi} (I \sin\theta_i \cos\varphi) d\varphi = \frac{1}{\pi I \sin\theta_i} \quad (52)$$

The aligned angular distribution $W_a(\vec{n}_i)$ is thus strongly peaked forward and backward with a minimum at $\theta_i = \frac{\pi}{2}$. The typical forward-backward peaking of the angular distribution is due to the fact that the beam direction is the only direction which is perpendicular to all the $\vec{I}$ over which we average. The contributions of particles emitted perpendicular to $\vec{I}$ will therefore pile up for $\sin\theta_i \simeq 0$. Any decoupling of $\vec{l}_i$ from the direction $\vec{I}$ will tend to smear the angular distribution in the neighbourhood of the beam direction inside the

^{*)} We will use the technique of Ericson and Strutinski ⁸⁾ for treating angular distributions in compound nucleus reactions. We refer also to the review article by Ericson ⁹⁾.

decoupling angle, while the angular distribution for other angles remains essentially that of the aligned one. Using Eq. (44) we can easily estimate the extent to which $\vec{l}_i$ is decoupled from the direction of $\vec{I}$ in the case of a spherically symmetric interaction volume. If the length of l_i is given as well as I , the other particles must couple to an angular momentum $\vec{I}-\vec{l}_i$. The 'a priori' probability of this is proportional to $\exp\{-(\vec{I}-\vec{l}_i)^2/2\sigma_2^2\}$, where σ_2^2 refers to the (n-1) remaining particles. We study this as function of the angle ψ_i between $\vec{I}$ and $\vec{l}_i$ and obtain

$$e^{-(\vec{I}-\vec{l}_i)^2/2\sigma_2^2} = e^{-(I-l_i)^2/2\sigma_2^2} e^{-2Il_i(1-\cos\psi_i)/2\sigma_2^2} \approx e^{-(I-l_i)^2/2\sigma_2^2} e^{-Il_i\psi_i^2/2\sigma_2^2} \quad (53)$$

We thus see that $\vec{l}_i$ will be forced to align with $\vec{I}$ to within an angle $\psi \approx (2\sigma_2^2/l_i I)^{1/2}$. The angular distribution will therefore deviate from the aligned distribution $1/\sin\theta_i$ within an angle ψ_0 around the beam direction, where it will be strongly smeared (see Fig.3). It will be practically unchanged elsewhere. The cross-section will thus retain its typical feature of forward-backward peaking with a minimum at 90° to the beam. The angular distributions from a contracted volume will be very closely similar to those considered for a spherical volume.

It is also interesting to consider the features of the angular distributions in single events, even though interference effects will be important, since no averaging can be performed. For a purely classical individual event the total angular momentum $\vec{I}$ has a well defined direction perpendicular to the beam. Since the angular momenta of the emitted particles add up to the total angular momentum, there must be a tendency for the particles to come out in a plane perpendicular to $\vec{I}$; the angular distribution must therefore show azimuthal asymmetry in high multiplicity events. The angular distribution in the plane perpendicular to $\vec{I}$ is on the average isotropic for a statistical process, but will show strong fluctuations, since interference has not been averaged out. The statistical azimuthal distribution will be nearly independent of interference effects, however, and will have a half width of the order of ψ_0 , since the uncertainty in azimuthal angle is due to the decoupling of the particle angular momentum $\vec{l}_i$ from $\vec{I}$, which smears the distribution by the decoupling angle (cf. preceding paragraph).

We will finally estimate the influence of angular momentum conservation on the shape of the energy spectra of the emitted particles and on the branching ratios between different processes.

We immediately notice that both momentum and angular momentum conservation must leave the shape of the energy spectra nearly unchanged, as soon as the multiplicity is not very low. The reason is that these conservation laws enter by the 'a priori' probability $P(p_1, \dots, p_n; r_1, \dots, r_n; \vec{P}=0; \vec{I})$, which depends on the particle momenta only through the factor $\sum_i p_i^2$ in the random coupling approximation [Eqs. (44) and (45)]. Since this factor is very insensitive to the momentum of an individual particle, it will have little effect on the energy distribution, which thus is easily obtained from the momentum phase space restricted only by energy conservation.

The multiplicities will be dramatically changed by inclusion of angular momentum. We have to distinguish between the limits of very low angular momentum (annihilation) with relatively small changes and that of high angular momentum with strong changes in multiplicity. We assume in the following for the sake of argument that the particles are relativistic in order to easily expose the qualitative changes. Since the particles are relativistic $\langle \sum_i p_i^2 \rangle \simeq \frac{4}{3} E^2/n$ [see Eq. (23)]. When $I \simeq 0$ we immediately conclude from Eq. (46) that the effect of angular momentum conservation is to multiply the ordinary phase space without angular momentum conservation by a factor $(\langle \sum_i p_i^2 \rangle)^{-3/2} \propto n^{3/2}$. This factor favours higher multiplicities. It is well known that the calculated average multiplicity in annihilation events comes out too small, if the natural interaction volume, the pion volume $\frac{4\pi}{3}(m_\pi)^{-3}$ is used; the interaction volume which reproduces the observed multiplicity is 10 times larger ^{14) *)}.

*) The large increase in the interaction needed to reproduce the experimental average multiplicity reflects the very slow variation of this quantity with Ω_0 . For relativistic particles it can easily be shown that the average multiplicity is proportional to $\Omega_0^{1/4}$.

Since the inclusion of angular momentum favours higher multiplicity, it is no longer necessary to use quite as large an interaction volume as before. An estimate, which allows for the low multiplicity in annihilation, indicates that the observed multiplicities are reproduced by an interaction volume 6-7 times larger than the natural volume. Thus while the inclusion of angular momentum decreases the size of the interaction volume needed, it still leaves us with a quite large volume to explain.

For large values of I the factor in Eq. (46) due to angular momentum conservation is proportional to $n^{3/2} \exp\{-na I^2 / \langle r^2 \rangle E^2\}$, where a is a constant of order unity. This factor, which thus multiplies the result without angular momentum restrictions, is dominated by its exponential part which rapidly decreases with n . In order to maintain the same multiplicities as have been previously calculated without angular momentum and which reproduce experiments with Ω_0 equal the natural volume, it is thus necessary to increase this volume. The increase is essentially by a factor $\exp\{aI^2 / \langle r^2 \rangle E^2\}$, which typically is of the order of 3 and possibly larger. The correct interaction volume is thus at least 3 times larger than the natural volume. A comparison with the interaction volume for annihilation shows that the difference is about a factor 2 or less. We therefore tentatively conclude that there is no significant difference between the interaction volumes needed to reproduce observed multiplicities in low and high angular momentum events. While the discrepancy in interaction volumes between these different types of reactions thus seems to disappear, we are still left with the unexplained problem that the interaction volume is several times the natural volume ^{*)}.

^{*)} We intend to look more quantitatively into the dependence of multiplicities on angular momentum.

5. Conclusion

The great drawback of the SM in recent years is its ever increasing complexity, which is particularly regrettable, since the basic idea of the model is most simple : cross-sections proportional to the accessible phase space. The reasons for the increased complexity are two : 1) accelerator energies have increased and with them the average multiplicity per event and the number of energetically possible events; 2) various conservation laws, which were initially neglected in the theory, are now included and must be so unless shown to be unimportant. The methods developed up to now for treating problems of the SM assume that the number of emitted particles remains moderate. They consequently use techniques which are most suited for systems of a small number of particles. This is particularly evident when the isospin and angular momentum conservation are accounted for by Clebsch-Gordan techniques. It is therefore no surprise that these methods rapidly lead to excessive labour with increasing multiplicities. On the other hand it is obvious that systems of high multiplicities should asymptotically become simple and be describable by thermodynamical methods. The accuracy of thermodynamics is, however, not sufficient, since even high multiplicities still represent a small number of particles thermodynamically. The random coupling method developed in the preceding sections represents in a sense a compromise between these extremes. It has been chosen in such a way that the phase space is obtained exactly in the limit of high multiplicities and for a finite multiplicity n is correct to order $1/n$. It is therefore a deliberate high multiplicity method and our philosophy in applying it is thus the opposite of that in the previous low multiplicity methods. There is, however, a large region of moderate multiplicities in which both approaches overlap, and they therefore complete each other.

One of the greatest advantages of the random coupling approximation is that it gives an excellent insight into the effects of the conservation laws on the phase space. Both for momentum and isospin conservation we have found that their net effect is to multiply the phase space without these conservation laws by the 'a priori' probability that momentum and isospin vectors of the particles couple to correct total quantum numbers. We have derived simple analytical approximation

formulas for these 'a priori' probabilities. They show very clearly that energy spectra are nearly unaffected by momentum conservation, as might have been expected, since momentum conservation is a restriction on the system as a whole, rather than on any individual particle. We have furthermore shown by this approximation technique that the average angular correlation of two particles due to momentum conservation is proportional to $1/n$. The goodness of the asymptotic semi-classical expressions for statistical weight factors demonstrate that the corresponding quantum mechanical results rapidly turn to the classical limit as the number of particles increases.

In sections 2 and 3 we derived the results of the random coupling approximation for the usual versions of the SM, i.e. neglecting angular momentum conservation. In a complete description this conservation law must be included; this will not only influence the predicted multiplicities but leads to uniquely predicted angular distributions. The angular distributions are, however, considerably more sensitive than multiplicities and energy spectra to the validity of the basic statistical assumption: independence of formation and decay modes (but for exact quantum numbers). If the SM is applied outside its range of full validity, it is often useful as a rough guide for multiplicities and energy spectra, but angular distributions are then hardly expected to be well represented. On the other hand it is of considerable interest for the understanding of strong interaction mechanisms to know the extent of validity of the SM in different cases; it is therefore necessary to work out its exact predictions so that it can be subjected to stringent experimental tests. Consequently we have analysed the basic hypothesis of the SM in section 4. We found that the statistical description implicitly assumes that the matrix elements have random phases; there will thus be no interference effects on the average in the SM. This has the important consequence that the angular distribution of a particle emitted in the CM must have forward-backward symmetry on the average. Since this is a statistical prediction independent on any parameters of the model, it should be well suited as a simple test for separating statistical from non-statistical processes. It is of course to be expected that the high

multiplicity events show more statistical features than others, since they imply a more complicated interaction. We have furthermore examined the qualitative features of the angular distributions in the CM system on the basis of a classical approximation. The particles are emitted preferentially in the forward-backward direction with a minimum at 90° to the beam direction. The anisotropy of the particles is limited by a distribution $[\sin\theta_i]^{-1}$, where θ_i is the angle between the beam and the particle direction.

The influence of angular momentum conservation on the energy spectra of the emitted particles is weak, but the multiplicities are strongly affected. The qualitative effect is an increase of multiplicity for low total angular momentum and a decrease in multiplicity for large total angular momentum. effect, reinterpreted in terms of interaction volumes, leads probably to the same interaction volume in annihilation events as for high energy nucleon-nucleon collisions. The large discrepancy between the interaction volumes needed for reproduction of observed multiplicities seems therefore to have been removed.

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Fig.1

Fig. 2

Fig. 3