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High-Energy Behaviour of Electromagnetic Scattering Cross-Sections.

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(ricevuto il 12 Dicembre 1960)

Summary. — The theory of the renormalization group is used in order to show that for large momentum transfer the perturbation expansion in α of the differential cross-section for the scattering of an electron by a potential in first order Born approximation is expanded in terms of the effective parameter $(\alpha/\pi) \ln(q^2/m^2)$. (q =invariant momentum transfer, m =electron rest mass; $q/m \gg 1$.)

1. - Introduction.

In a recent note it was pointed out by PETERMAN and the author ⁽¹⁾ that the effective perturbation expansion parameter for cross-sections for electromagnetic processes at high energies and momentum transfers is at most

$$(1) \quad \frac{\alpha}{\pi} \ln^{\frac{1}{2}} \left(\frac{q^2}{m^2} \right),$$

where q is the invariant momentum transfer in the process and m is the electron rest mass.

This result was obtained from considerations of Feynman diagrams and though we conjectured that the effective expansion parameter should in fact be

$$(2) \quad \frac{\alpha}{\pi} \ln \frac{q^2}{m^2},$$

⁽¹⁾ K. E. ERIKSSON and A. PETERMAN: *Phys. Rev. Lett.*, **5**, 444 (1960).

we could not prove this due to difficulties in handling diagrams containing closed electron loops.

By use of the properties of the renormalization group (*) it has now been possible to show that (2) is indeed the effective perturbation expansion parameter for electromagnetic scattering cross-sections at high energies and momentum transfers.

Here this theorem will be proved for the scattering of an electron by a potential in first order Born approximation. Since that process involves all features that are characteristic for electromagnetic processes this is not a severe limitation, and it will be quite straightforward to extend the proof to more complicated scattering processes.

2. - Preliminary discussion.

Consider an electron with initial momentum p that is scattered by an external electromagnetic field A_μ and thereby obtains the final momentum p' . The kinematics are given by

$$(3) \quad \begin{cases} p = (\sqrt{\frac{1}{4}q^2 + m^2}, \frac{1}{2}q, 0, 0), \\ p' = (\sqrt{\frac{1}{4}q^2 + m^2}, -\frac{1}{2}q, 0, 0), \end{cases}$$

where q is the invariant momentum transfer. We assume that

$$(4) \quad q^2 \gg m^2.$$

Let the energy resolution ΔE be isotropic in the Lorentz-frame determined by (3). Assume further that

$$(5) \quad \Delta E \ll q, \quad \text{but} \quad \left| \ln \frac{\Delta E}{q} \right| = O(1).$$

We write down the probability density $P(q^2/m^2, \Delta E/q)$ for the process

(*) The existence of this group was first noticed by STUECKELBERG and PETERMAN (2). Further developments were made by BOGOLIUBOV and SHIRKOV (3), who used the properties of the group to investigate the asymptotic behaviour of Green's functions.

(2) E. C. G. STUECKELBERG and A. PETERMAN: *Helv. Phys. Acta*, **24**, 317 (1951); *Helv. Phys. Acta*, **26**, 499 (1953).

(3) N. N. BOGOLIUBOV and D. V. SHIRKOV: *Introduction to the Theory of Quantized Fields* (New York, London, 1959), Sect. 42-44. This book contains further references.

in the form

$$\begin{aligned}
 (6) \quad P\left(\frac{q^2}{m^2}, \frac{\Delta E}{q}\right) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int \dots \int d^3\mathbf{k}_1 \dots d^3\mathbf{k}_m d^3\mathbf{p}_{11} d^3\mathbf{p}_{12} \dots d^3\mathbf{p}_{n1} d^3\mathbf{p}_{n2} \cdot \\
 &\quad \cdot \theta\left(\Delta E - \sum_{i=1}^m |\mathbf{k}_i| - \sum_{j=1}^n (E_{j1} + E_{j2})\right) \cdot \\
 &\quad \cdot \sum_{\substack{\text{spins} \\ \text{polarizations}}} \left| M_{mn} \left(\frac{q^2}{m^2}; \mathbf{k}_1, e_1; \dots; \mathbf{k}_m, e_m; \mathbf{p}_{11}, s_{11}; \dots; \mathbf{p}_{n2}, s_{n2} \right) \right|^2, \\
 E_{jk} &= \sqrt{\mathbf{p}_{jk}^2 + m^2}; \quad \theta(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0, \end{cases}
 \end{aligned}$$

where

$$M_{mn} \left(\frac{q^2}{m^2}; \mathbf{k}_1, e_1; \dots; \mathbf{k}_m, e_m; \mathbf{p}_{11}, s_{11}; \dots; \mathbf{p}_{n2}, s_{n2} \right)$$

is the invariant amplitude when the scattering process is accompanied by emission of m low energy photons with momenta $\mathbf{k}_1, \dots, \mathbf{k}_m$ and polarizations $e_1, \dots, e_m$ and n electron pairs with momenta $\mathbf{p}_{11}, \dots, \mathbf{p}_{n2}$ and spins $s_{11}, \dots, s_{n2}$. In general M_{mn} is infra-red divergent. However, the infra-red divergences inside the sum cancel and $P(q^2/m^2, \Delta E/q)$ is not infra-red divergent.

We define $X(q^2/m^2, \Delta E/q)$ by the equation

$$(7) \quad P\left(\frac{q^2}{m^2}, \frac{\Delta E}{q}\right) = X\left(\frac{q^2}{m^2}, \frac{\Delta E}{q}\right) P^{(0)}\left(\frac{q^2}{m^2}\right),$$

where

$$(8) \quad P^{(0)}\left(\frac{q^2}{m^2}\right) = \frac{\alpha}{\pi} |\bar{u}(p') \gamma_\mu A^\mu u(p)|^2$$

is the lowest order (in α) invariant amplitude for the elastic process. The function $X(q^2/m^2, \Delta E/q)$ now contains all radiative corrections.

Since we are interested in the q^2/m^2 -dependence of X , we suppress the dependence on $\Delta E/q$ and write

$$(9) \quad X\left(\frac{q^2}{m^2}, \frac{\Delta E}{q}\right) = X\left(\frac{q^2}{m^2}\right).$$

In the next section the renormalization group will be used to get information about how X depends on q^2/m^2 . When dealing with the renormalization group it is most natural to work in a gauge where the photon

propagator $D_{\mu\nu}(k)$ contains only transverse components

$$(10) \quad D_{\mu\nu}(k) = \frac{d(k^2/m^2)}{k^2} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right).$$

In order to study the renormalization properties of the dimensionless functions $d(q^2/m^2)$ and $X(q^2/m^2)$, we follow the methods of ref. (3) and introduce the more general functions

$$(11) \quad d\left(\frac{q^2}{\lambda^2}, \frac{m^2}{\lambda^2}, e^2\right) \quad \text{and} \quad X\left(\frac{q^2}{\lambda^2}, \frac{m^2}{\lambda^2}, e^2\right), \quad \lambda^2 > 0 \text{ (space-like),}$$

with the normalization

$$(12) \quad \begin{cases} d\left(1, \frac{m^2}{\lambda^2}, e^2\right) = 1, \\ X\left(1, \frac{m^2}{\lambda^2}, e^2\right) = 1. \end{cases}$$

Here e is the electric charge and λ^2 is a squared normalization momentum.

With e_0 = the physical electron charge, we define also

$$(13) \quad \begin{cases} d_0\left(\frac{q^2}{m^2}, e_0^2\right) = d\left(\frac{q^2}{m^2}\right), \\ X_0\left(\frac{q^2}{m^2}, e_0^2\right) = X\left(\frac{q^2}{m^2}\right), \end{cases}$$

which have the following normalization

$$(14) \quad \begin{cases} d_0(0, e_0^2) = 1, \\ X_0(0, e_0^2) = 1, \end{cases}$$

and thus correspond to the case that $\lambda^2 = 0$ in (11) and (12).

3. - Proof of the theorem.

Under multiplicative renormalization each $|M_{mn}|^2$ occurring in (6), and hence also $P(q^2/m^2, \Delta E/q)$, transforms like $|\bar{\psi}e\Gamma\psi|^2$, where ψ is the electron field, and Γ is the vertex function. From (8) is seen that $P^{(0)}(q^2/m^2)$ transforms like $|\bar{\psi}e\psi|^2$. Because of (7) this implies that X transforms like

$$(15) \quad |\Gamma|^2.$$

We consider the following renormalization

$$(16) \quad \begin{cases} d\left(\frac{q^2}{\lambda_2^2}, \frac{m^2}{\lambda_2^2}, e_2^2\right) = z_3 d\left(\frac{q^2}{\lambda_1^2}, \frac{m^2}{\lambda_1^2}, e_1^2\right), \\ X\left(\frac{q^2}{\lambda_2^2}, \frac{m^2}{\lambda_2^2}, e_2^2\right) = z^2 X\left(\frac{q^2}{\lambda_1^2}, \frac{m^2}{\lambda_1^2}, e_1^2\right), \\ e_2^2 = z_3^{-1} e_1^2. \end{cases}$$

The renormalization constants z and z_3 can be eliminated, and using (12) we are led to the following functional equations (see ref. (3), Section 42'3)

$$(17) \quad \begin{cases} d\left(\frac{q^2}{\lambda_2^2}, \frac{m^2}{\lambda_2^2}, e_2^2\right) = \frac{d(q^2/\lambda_1^2, m^2/\lambda_1^2, e_1^2)}{d(\lambda_2^2/\lambda_1^2, m^2/\lambda_1^2, e_1^2)}, \\ X\left(\frac{q^2}{\lambda_2^2}, \frac{m^2}{\lambda_2^2}, e_2^2\right) = \frac{X(q^2/\lambda_1^2, m^2/\lambda_1^2, e_1^2)}{X(\lambda_2^2/\lambda_1^2, m^2/\lambda_1^2, e_1^2)}, \\ e_2^2 = e_1^2 d\left(\frac{\lambda_2^2}{\lambda_1^2}, \frac{m^2}{\lambda_1^2}, e_1^2\right). \end{cases}$$

Taking $\lambda_1^2 = 0$, $\lambda_2^2 = \lambda^2$ and using the functions defined by (13), we then have as a special case of (17)

$$(18) \quad \begin{cases} d\left(\frac{q^2}{\lambda^2}, \frac{m^2}{\lambda^2}, e^2\right) = \frac{d_0(q^2/m^2, e_0^2)}{d_0(\lambda^2/m^2, e_0^2)}, \\ X\left(\frac{q^2}{\lambda^2}, \frac{m^2}{\lambda^2}, e^2\right) = \frac{X_0(q^2/m^2, e_0^2)}{X_0(\lambda^2/m^2, e_0^2)}, \\ e^2 = e_0^2 d_0\left(\frac{\lambda^2}{m^2}, e_0^2\right). \end{cases}$$

With

$$(19) \quad x = \frac{q^2}{\lambda^2}; \quad y = \frac{m^2}{\lambda^2}; \quad \lambda^2 = \lambda_2^2,$$

the following Lie differential equations can be derived from the eq. (17) (see ref. (3), Section 42'4)

$$(20) \quad \begin{cases} \frac{\partial e^2 d(x, y, e^2)}{\partial x} = \frac{e^2 d(x, y, e^2)}{x} \varphi\left(\frac{y}{x}, e^2 d(x, y, e^2)\right), \\ \frac{\partial \ln X(x, y, e^2)}{\partial x} = \frac{1}{x} \psi\left(\frac{y}{x}, e^2 d(x, y, e^2)\right), \end{cases}$$

where

$$(21) \quad \begin{cases} \varphi(y, e^2) = \left[\frac{\partial}{\partial \xi} d(\xi, y, e^2) \right]_{\xi=1}, \\ \psi(y, e^2) = \left[\frac{\partial}{\partial \xi} \ln X(\xi, y, e^2) \right]_{\xi=1}. \end{cases}$$

We are interested in the case when $q^2 \gg m^2$, i.e. when $x \gg y$. If we choose $\lambda^2 = O(q^2)$, we have $y \ll 1$. Putting $y = 0$ in (20) and (21) we then get

$$(22) \quad \begin{cases} \frac{\partial e^2 d(x, e^2)}{\partial x} = \frac{e^2 d(x, e^2)}{x} \varphi(e^2 d(x, e^2)), \\ \frac{\partial \ln X(x, e^2)}{\partial x} = \frac{1}{x} \psi(e^2 d(x, e^2)), \end{cases}$$

and

$$(23) \quad \begin{cases} \varphi(e^2) = \left[\frac{\partial}{\partial \xi} d(\xi, e^2) \right]_{\xi=1}, \\ \psi(e^2) = \left[\frac{\partial}{\partial \xi} \ln X(\xi, e^2) \right]_{\xi=1}, \end{cases}$$

with the definitions

$$(24) \quad \begin{cases} d(x, e^2) = d(x, 0, e^2), \\ X(x, e^2) = X(x, 0, e^2). \end{cases}$$

Integration of the eq. (22) gives with use of (12)

$$(25) \quad \begin{cases} \ln x = \int_{e^2}^{e^2 d(x, e^2)} \frac{dz}{z \varphi(z)}, \\ \ln X(x, e^2) = \int_{e^2}^{e^2 d(x, e^2)} \frac{dz}{z \varphi(z)} \psi(z). \end{cases}$$

In perturbation theory $d(x, e^2)$ and $X(x, e^2)$ can be expanded in the following way

$$(26) \quad \begin{cases} d(x, e^2) = 1 + \sum_{i,j=1}^{\infty} a_{ij} e^{2i} (\ln x)^j, \\ X(x, e^2) = 1 + \sum_{i,j=1}^{\infty} b_{ij} e^{2i} (\ln x)^j, \end{cases}$$

where b_{ij} are functions of $\Delta E/q$. Because of the condition on $\ln \Delta E/q$ in (5) the effective expansion parameter does not depend appreciably on $\Delta E/q$ (see ref. (4) Section 5).

Insertion of (26) into (23) gives

$$(27) \quad \begin{cases} \varphi(e^2) = \sum_{i=1}^{\infty} a_i e^{2i}, \\ \psi(e^2) = \sum_{i=1}^{\infty} b_i e^{2i}, \end{cases}$$

where

$$(28) \quad \begin{cases} a_i = a_{i1}, \\ b_i = b_{i1}. \end{cases}$$

If

$$(29) \quad e^2 d(x, e^2) \ll 1,$$

then $z \ll 1$ in (25) and the following expansions obtainable from (27) are useful

$$(30) \quad \begin{cases} \frac{1}{z\varphi(z)} = \frac{1}{z^2} \sum_{i=0}^{\infty} \alpha_i z^i, \\ \frac{\psi(z)}{z\varphi(z)} = \frac{1}{z} \sum_{i=0}^{\infty} \beta_i z^i. \end{cases}$$

The coefficients α_i and β_i can be determined from a_i, b_i by the recurrence formulae

$$(31) \quad \begin{cases} \alpha_0 a_1 = 1, \\ \sum_{i=0}^{n-1} \alpha_i a_{n-i} = 0, \quad n > 1, \\ \sum_{i=0}^{n-1} \beta_i a_{n-i} = b_n. \end{cases}$$

We now insert the expansions (30) obtained from perturbation theory into the integral equations (25). This gives

$$(32) \quad \begin{cases} (\alpha_0 - e^2 \ln x) d(x, e^2) - \alpha_0 + \alpha_1 e^2 d(x, e^2) \ln d(x, e^2) + \\ \quad + e^2 d(x, e^2) \sum_{n=1}^{\infty} \frac{\alpha_{n+1}}{n} e^{2n} [(d(x, e^2))^n - 1] = 0, \\ X(x, e^2) - [d(x, e^2)]^{\beta_0} \exp \left[\sum_{n=1}^{\infty} \frac{\beta_n}{n} e^{2n} [(d(x, e^2))^n - 1] \right] = 0. \end{cases}$$

(4) K. E. ERIKSSON: *Nuovo Cimento*, **19**, 1010 (1961).

To solve the equations (32) we make the following ansatz

$$(33) \quad \begin{cases} d(x, e^2) = \sum_{n=0}^{\infty} e^{2n} \bar{d}^{(n)}(\eta), \\ X(x, e^2) = \sum_{n=0}^{\infty} e^{2n} X^{(n)}(\eta), \end{cases}$$

with

$$(34) \quad \eta = e^2 \ln x$$

which we introduce into (32). The condition that all coefficients for different powers of e^2 vanish gives then the solutions for $\bar{d}^{(0)}(\eta), \bar{d}^{(1)}(\eta), \dots, X^{(0)}(\eta), X^{(1)}(\eta), \dots$. So the ansatz is correct, and if we expand also in powers of η we see that the expansions (26) are unnecessarily general. In fact we have that

$$(35) \quad a_{ij} = b_{ij} = 0, \quad \text{if } i < j.$$

Recalling the definitions (19) and (24) and using (18) we then conclude that in the expansions of

$$(13) \quad \begin{cases} d\left(\frac{q^2}{m^2}\right) = d_0\left(\frac{q^2}{m^2}, e_0^2\right), \\ X\left(\frac{q^2}{m^2}\right) = X_0\left(\frac{q^2}{m^2}, e_0^2\right), \end{cases}$$

in powers of $\alpha = e_0^2/4\pi$ the terms with the highest power of $\ln(q^2/m^2)$ are of the form

$$(36) \quad \left(\frac{\alpha}{\pi} \ln \frac{q^2}{m^2}\right)^n.$$

Hence (2) is the effective expansion parameter of $X(q^2/m^2)$ and our theorem is proved.

4. - Concluding remark.

The above treatment indicates that the use of the renormalization group is a good method to improve the results calculated by ordinary perturbation methods. If the $2n$ coefficients $a_1, \dots, a_n, b_1, \dots, b_n$ in the perturbation expansions of $d(x, e^2)$ and $X(x, e^2)$ (see (26) and (28)) are known, then by (31) they determine $\alpha_0, \dots, \alpha_{n-1}, \beta_0, \dots, \beta_{n-1}$, and by (32) and

(33) these latter coefficients determine the $2n$ functions $d^{(0)}(\eta), \dots, d^{(n-1)}(\eta), X^{(0)}(\eta), \dots, X^{(n-1)}(\eta)$. In ref. (3) it is shown how this works in calculations of Green's functions.

* * *

I am indebted to Dr. A. PETERMAN for many stimulating discussions from which the basic idea of the proof originates. His helpful criticism of the manuscript is also gratefully acknowledged.

RIASSUNTO (*)

Si fa uso della teoria del gruppo di rinormalizzazione allo scopo di mostrare come per grandi trasferimenti di impulso lo sviluppo in α della perturbazione della sezione d'urto differenziale per lo scattering di un elettrone da parte di un potenziale nella approssimazione di Born di primo ordine si effettui in termini del parametro effettivo $(\alpha/\pi) \ln(q^2/m^2)$ (q =trasferimento invariabile dell'impulso, m =massa di riposo dell'elettrone; $q/m \approx 1$).

(*) Traduzione a cura della Redazione.