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Radiative Corrections to Muon-Electron Scattering

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S U M M A R Y

The differential cross section for μ -e scattering
is calculated to the order e^6 .

I. Introduction.

Scattering of muons by electrons at high energies has been planned ¹⁾ as an experiment for testing the validity of quantum electrodynamics at small distances. In order to interpret the results obtained in such an experiment one must know, of course, to a good accuracy, what is really predicted by quantum electrodynamics. In a recent note it was pointed out by Peterman and the author ²⁾ that for experiments performed or planned nowadays, it is quite sufficient to calculate first order radiative corrections to scattering cross sections.

The present paper reports on such a calculation on $\mu^\pm e^-$ scattering with an energy resolution which is assumed to be isotropic in the c.m. system and much smaller than the c.m. energies of the muon and the electron ^{*)}. From the result of this calculation we conclude that the formula derived by Schwinger for scattering of an electron by a potential ⁴⁾ holds with a good accuracy also for μe scattering at the energies one is planning with at present. Some numerical results for the radiative corrections at different energies and angles are given in a table.

The inelastic process of soft photon emission that accompanies the basic process is here taken into account by the application of a general formula derived by the present author in another publication ^{5) **)}.

The result obtained here is not yet complete enough to be directly compared to experiments. A certain amount of hard photon emission will be permitted by the experimental geometry and the cross section for that process must be calculated separately for every particular experimental design. Good examples of how this can be done are found in the work of Tsai ⁷⁾ on electron-electron scattering.

In Section II the differential cross section is expressed in terms of the matrix elements corresponding to different diagrams. These matrix elements are calculated in Section III. The final formulae are then collected in Section IV, where also the table with numerical results is presented

*) A similar calculation has recently been performed independently by Nikishov ³⁾.

**) In the following this paper will be referred to as I. As in I we shall follow the conventions used in the text-book of Jauch and Rohrlich ⁶⁾ as regards metric, normalization factors, etc.

II. The cross section.

The kinematics for elastic scattering of muons by electrons are given in the c.m. system by the following equations [see Fig.1]

$$\begin{cases} p_1 = (E_1, q, 0, 0) \\ p_2 = (E_2, -q, 0, 0) \end{cases} \quad \begin{cases} p'_1 = (E_1, q \cos \Theta, q \sin \Theta, 0) \\ p'_2 = (E_2, -q \cos \Theta, -q \sin \Theta, 0) \end{cases} \quad (1)$$

$$E_i^2 = m_i^2 + q^2 \quad (i=1,2) \quad (2)$$

where we can choose $q > 0$. The following definitions will be useful :

$$\begin{cases} W^2 = -(p_1 + p_2)^2 = (E_1 + E_2)^2 \\ p_3 = p'_1 - p_1 = p_2 - p'_2 \\ \Delta^2 = p_3^2 = 2q^2(1 - \cos \Theta) \end{cases} \quad (3)$$

$$\begin{cases} t_i = \frac{4m_i^2}{\Delta^2} \\ T = \frac{1}{4}(t_1 + t_2) \end{cases} \quad (4)$$

$$\Delta^2 u = -2p_1 \cdot p_2 \quad (5)$$

Thus W is the total c.m. energy and Δ is the invariant momentum transfer. The definitions (3), (4) and (5) imply the relations

$$\begin{cases} -2p_1 \cdot p_3 = 2p_2 \cdot p_3 = \Delta^2 \\ -2p_i \cdot p'_i = \Delta^2(1 + \frac{1}{2} t_i) \\ 2p_1 \cdot p'_2 = \Delta^2(1 - u) \end{cases} \quad (6)$$

If we include the emission of one soft photon with an energy not exceeding ΔE , where

$$\Delta E \ll q, \quad (7)$$

then the transition probability density can be written

$$(2\pi)^{-4} \delta(p'_1 + p'_2 - p_1 - p_2) \frac{1}{4} \sum_{\text{spins}} P(\Delta E). \quad (8)$$

Here we have averaged over initial spins and summed over final spins. The differential cross section is given by the following equation

$$\frac{d\sigma}{d\Omega} = (2\pi)^2 \frac{E_1 E_2}{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \int d^3 p'_1 d^3 p'_2 \delta(p'_1 + p'_2 - p_1 - p_2) \frac{1}{4} \sum_{\text{spins}} P(\Delta E), \quad (9)$$

which can be simplified to

$$\frac{d\sigma}{d\Omega} = (2\pi)^2 \frac{(E_1 E_2)^2}{W^2} \frac{1}{4} \sum_{\text{spins}} P(\Delta E). \quad (10)$$

We now take from I [Eq. (I-77)] the following formula for $P(\Delta E)$:

$$P(\Delta E) = |M^{(0)}|^2 \left[1 + C \ln\left(\frac{\Delta E}{E}\right) + B \right] + 2 \operatorname{Re} \left\{ M^{(0)*} \bar{M}^{(1)} \right\} \quad (11)$$

Here $M^{(0)}$ is the lowest order (e^2) matrix element, corresponding to the diagram in Fig.2; $\bar{M}^{(1)}$ is the non infra-red part of the next order (e^4) matrix element, $M^{(1)} = \sum_{i=1}^6 M_i^{(1)}$, corresponding to the six diagrams in Fig.3. The quantities C and B are connected with the emission of one soft photon [see Fig.4]. They are given by formulae (I-54) and (I-59). According to (I-29) $\bar{M}^{(1)}$ can be expressed in terms of $M^{(0)}$ and $M^{(1)}$ by the relation

$$\bar{M}^{(1)} = M^{(1)} + \frac{1}{2} A M^{(0)} \quad (12)$$

Here A contains the whole infra-red divergent contribution to $M^{(1)}$.

From (I-24) and (I-18) we conclude that

$$A = -\frac{\alpha}{\pi} \cdot \frac{1}{i\pi^2} \int \frac{d^4 l}{l^2 + \lambda^2} \left\{ \sum_{i=1}^2 \left[\frac{2p_i^2}{(l^2 + 2l \cdot p_i)(l^2 - 2l \cdot p_i)} + \frac{2p_i \cdot p_i'}{(l^2 + 2l \cdot p_i)(l^2 + 2l \cdot p_i')} \right] + \right. \\ \left. + Q \left[\frac{4p_1 \cdot p_2}{(l^2 - 2l \cdot p_1)(l^2 + 2l \cdot p_2)} + \frac{4p_1 \cdot p_2'}{(l^2 + 2l \cdot p_1)(l^2 + 2l \cdot p_2')} \right] \right\} \quad (13)$$

where λ is the fictitious photon mass and Q is the product of the two charges ($Q = +1$, if the muon and the electron have equal charges, and $Q = -1$ if they have different charges).

In the next section it will be shown that the terms of $M^{(1)}$ corresponding to the diagrams in Fig. 3 can be written in the form

$$M_1^{(1)} + M_2^{(1)} = -\frac{\alpha}{2\pi} 1 M^{(0)} \\ M_3^{(1)} + M_4^{(1)} = -\frac{\alpha}{2\pi} \left\{ -\frac{1}{i\pi^2} \int \frac{d^4 l}{(l^2 + \lambda^2)} \sum_{i=1}^2 \left[\frac{2p_i^2}{(l^2 + 2l \cdot p_i)(l^2 - 2l \cdot p_i)} + \right. \right. \\ \left. \left. + \frac{2p_i \cdot p_i'}{(l^2 + 2l \cdot p_i)(l^2 + 2l \cdot p_i')} \right] + 1 \right\} M^{(0)} + \bar{M}_3^{(1)} + \bar{M}_4^{(1)} \quad (14)$$

$$M_5^{(1)} = \frac{\alpha}{2\pi} Q \frac{1}{i\pi^2} \int \frac{d^4 l}{(l^2 + \lambda^2)(l^2 - 2l \cdot p_1)(l^2 + 2l \cdot p_2)} \cdot M^{(0)} + \bar{M}_5^{(1)}$$

$$M_6^{(1)} = \frac{\alpha}{2\pi} Q \frac{1}{i\pi^2} \int \frac{d^4 l}{(l^2 + \Lambda^2)(l^2 + 2l \cdot p_1)(l^2 + 2l \cdot p_2')} \cdot M^{(0)} + \bar{M}_6^{(1)}$$

If we use now equations (10) to (14), the infra-red divergences cancel, and we get the following expression for the cross section

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)^{(0)} \left[1 + C \ln\left(\frac{4E}{E}\right) + B - \frac{\alpha}{\pi} (1+l') \right] - \frac{\alpha^3}{\pi} \frac{1}{W^2} (R_3 + R_4 + R_5 + R_6) \quad (15)$$

where $\left(\frac{d\sigma}{d\Omega}\right)^{(0)}$ is the lowest order cross section,

$$\left(\frac{d\sigma}{d\Omega}\right)^{(0)} = (2\pi)^2 \frac{(E_1 E_2)^2}{W^2} \cdot \frac{1}{4} \sum_{\text{spins}} |M^{(0)}|^2 \quad (16)$$

and R_j is defined by

$$- \frac{\alpha^3}{\pi} \frac{R_j}{W^2} = (2\pi)^2 \frac{(E_1 E_2)^2}{W^2} \cdot \frac{1}{4} \sum_{\text{spins}} 2 \operatorname{Re} \left\{ M^{(0)*} \bar{M}_j^{(1)} \right\} \quad (j=3,4,5,6) \quad (17)$$

III. Calculations of the contributions from the diagrams $M^{(0)}, \dots, M_6^{(1)}$.

A. The lowest order diagram $M^{(0)}$

Following the usual techniques for evaluation of diagrams, we obtain the contribution from the lowest order diagram (Fig. 2)

$$\frac{1}{4} \sum_{\text{spins}} |M^{(0)}|^2 = \frac{\alpha^2}{(2\pi)^2} \cdot \frac{1}{(E_1 E_2)^2} \cdot \frac{1}{\Delta^4} F_{1\mu\nu} F_2^{\mu\nu} \quad (18)$$

with

$$F_{i\mu\nu} = m_i^2 \sum_{\text{spins}} [\bar{u}_i(p_i') \gamma_\mu u_i(p_i)]^* [\bar{u}_i(p_i') \gamma_\nu u_i(p_i)] \quad (19)$$

A simple trace calculation gives

$$F_{i\mu\nu} = p_i'{}_\mu p_{i\nu} + p_i'{}_\nu p_{i\mu} + \frac{1}{2} \Delta^2 g_{\mu\nu} \quad (20)$$

Inserting this into (18) and using the definitions (4) and (5) we get

$$\frac{1}{4} \sum_{\text{spins}} |M^{(0)}|^2 = \frac{\alpha^2}{(2\pi)^2} \frac{1}{(E_1 E_2)^2} (u^2 - u - T + \frac{1}{2}) \quad (21)$$

Thus, the lowest order cross section given by eq. (16) becomes

$$\left(\frac{d\sigma}{d\Omega}\right)^{(0)} = \frac{\alpha^2}{W^2} (u^2 - u - T + \frac{1}{2}) \quad (22)$$

B. The vacuum polarization diagrams, $M_1^{(1)}$ and $M_2^{(1)}$

From ref. ⁶⁾ equations (9-66) and (9-71) it is immediately seen that the contribution from the diagrams $M_1^{(1)}$ and $M_2^{(1)}$ in Fig. 3 is of the form indicated in (14) and that

$$1 = \frac{2}{3} \sum_{i=1}^2 \left[\frac{5}{3} - t_i - (1 - \frac{1}{2}t_i) \sqrt{1+t_i} \ln \frac{\sqrt{1+t_i}+1}{\sqrt{1+t_i}-1} \right] \quad (23)$$

C. The vertex diagrams, $M_3^{(1)}$ and $M_4^{(1)}$

We start with a general vertex diagram of order e^3 with free external fermion lines (Fig. 5). The renormalized vertex function $\Lambda_{f\mu}(p', p)$ is given by

$$\Lambda_{f\mu}(p', p) = \Lambda_{\mu}(p', p) - \Lambda_{\mu}(p, p) \quad (24)$$

$$\Lambda_{\mu}(p', p) = \frac{\alpha}{2\pi} \frac{1}{i\pi^2} \int \frac{d^4 l}{(l^2 + \lambda^2)(l^2 - 2l \cdot p')(l^2 - 2l \cdot p)} N_{\mu}(p', p)$$

where

$$\begin{aligned} N_{\mu} &= \frac{1}{2} \gamma_{\nu} (i\not{p}' - i\not{l} - m) \gamma_{\mu} (i\not{p} - i\not{l} - m) \gamma^{\nu} = \\ &= -2p' \cdot p \gamma_{\mu} + 1^{\nu} (\gamma_{\mu} \gamma_{\nu} \not{p}' + \not{p}' \gamma_{\nu} \gamma_{\mu}) - 1^{\nu} 1^{\rho} (\frac{1}{2} g_{\nu\rho} \gamma_{\mu} - 2g_{\mu\nu} \gamma_{\rho}) - \frac{1}{2} 1^2 \gamma_{\mu} \end{aligned} \quad (25)$$

Separating out the infra-red divergent terms we get

$$\Lambda_{f\mu}(p', p) = \frac{\alpha}{2\pi} \cdot \frac{1}{i\pi^2} \int \frac{d^4 l}{(l^2 + \lambda^2)} \int \frac{2p^2}{(l^2 + 2l \cdot p)(l^2 - 2l \cdot p)} +$$

$$+ \frac{2p \cdot p'}{(l^2 + 2l \cdot p)(l^2 + 2l \cdot p')} \int \gamma_\mu + \bar{\Lambda}_{f\mu}(p', p) \quad (26)$$

where $\bar{\Lambda}_{f\mu}(p', p)$ is the non-infra-red part of the vertex function.

$$\bar{\Lambda}_{f\mu}(p', p) = -\frac{\alpha}{2\pi} \frac{1}{i\pi^2} \int d^4 l \left\{ \frac{l^\nu (\gamma_\mu \gamma_\nu \not{p}' + \not{p} \gamma_\nu \gamma_\mu) - l^\nu l^\rho (\frac{1}{2} g_{\nu\rho} \gamma_\mu - 2g_{\mu\nu} \gamma_\rho)}{l^2(l^2 - 2l \cdot p)(l^2 - 2l \cdot p')} - \right.$$

$$- \frac{2ilml_\mu - l^\nu l^\rho (\frac{1}{2} g_{\nu\rho} \gamma_\mu - 2g_{\mu\nu} \gamma_\rho)}{l^2(l^2 - 2l \cdot p)^2} - \gamma_\mu \int \frac{l \cdot (p' - p)}{(l^2 - 2l \cdot p)^2(l^2 - 2l \cdot p')} -$$

$$\left. - \frac{4p^2}{(l^2 - 2l \cdot p)^2(l^2 + 2l \cdot p)} \int \right\} \quad (27)$$

It can be evaluated with Feynman's integral technics ^{*)} and the result is

$$\bar{\Lambda}_{f\mu}(p', p) = -\frac{\alpha}{2\pi} \int \gamma_\mu (4 - 3/2 \sqrt{1+t} \ln \frac{\sqrt{1+t}+1}{\sqrt{1+t}-1}) - \frac{im(p'_\mu + p_\mu)}{(p' - p)^2} \cdot$$

$$\cdot \frac{1}{\sqrt{1+t}} \ln \frac{\sqrt{1+t}+1}{\sqrt{1+t}-1} \int \quad t = \frac{4m^2}{(p' - p)^2} \quad (28)$$

^{*)} See ref. 6) Section A 5-1

From (26) and (28) it follows that $M_3^{(1)} + M_4^{(1)}$ (Fig. 3) is of the form indicated in (14) and that

$$1' = \sum_{i=1}^2 \left(4 - \frac{3}{2} \sqrt{1+t_i} \ln \frac{\sqrt{1+t_i} + 1}{\sqrt{1+t_i} - 1} \right) \quad (29)$$

For the magnetic contribution $R_3 + R_4$, defined by (17), we get

$$R_3 + R_4 = \frac{2m_1^2 m_2^2}{\Delta^6} \operatorname{Re} \left\{ F_{1\mu} F_2^\mu \right\} \sum_{i=1}^2 \frac{1}{\sqrt{1+t_i}} \ln \frac{\sqrt{1+t_i} + 1}{\sqrt{1+t_i} - 1} \quad (30)$$

with

$$F_{i\mu} = m_i \sum_{\text{spins}} \bar{u}_i(p'_i) \gamma_\mu u_i(p_i) \left[\bar{u}_i(p'_i) u_i(p_i) \right]^* \quad (31)$$

A simple calculation gives

$$F_{i\mu} = -i (p_{i\mu} + p'_{i\mu}) \quad (32)$$

which inserted into (30) yields

$$R_3 + R_4 = 1/8 t_1 t_2 (2u-1) \sum_{i=1}^2 \frac{1}{\sqrt{1+t_i}} \ln \frac{\sqrt{1+t_i} + 1}{\sqrt{1+t_i} - 1} \quad (33)$$

D. The two photon exchange diagrams, $M_5^{(1)}$ and $M_6^{(1)}$

The contributions from the diagrams $M_5^{(1)}$ and $M_6^{(1)}$ in Fig. 3 can be expressed in terms of each other by changing particle 2 into its antiparticle and reversing the signs of its momenta (q_2 is the charge)

$$\begin{cases} (2) \rightarrow (\bar{2}) \\ q_2 \rightarrow -q_2 \end{cases} \quad \begin{cases} p_2 \rightarrow -p_2 \\ p'_2 \rightarrow -p'_2 \end{cases} \quad (34)$$

The cross section terms are invariant under the transformation (34). This fact is used in Fig. 6 to show that

$$\text{Re} \left\{ M^{(0)*} M_6^{(1)} \right\} = -\text{Re} \left\{ M^{(0)*} M_5^{(1)} \right\} \quad (p_2 \leftrightarrow -p'_2) \quad (35)$$

Therefore, it is sufficient to consider only the diagram $M_5^{(1)}$. The corresponding matrix element is given by

$$\begin{aligned} QM_5^{(1)} &= -\frac{\alpha}{2\pi} \left(\frac{\alpha}{i\pi} \frac{m_1 m_2}{E_1 E_2} \right)^{\frac{1}{2}} \frac{1}{i\pi^2} \int d^4k \bar{u}_1(p'_1) \gamma_\nu \frac{2ip_{1\mu} - ik \gamma^\mu}{k^2 - 2k \cdot p_1} u_1(p_1) \\ &\cdot \frac{1}{k^2 + \lambda^2} \frac{1}{(p_3 + k)^2 + \lambda^2} \bar{u}_2(p'_2) \gamma^\nu \frac{2ip_2^\mu + ik \gamma^\mu}{k^2 + 2k \cdot p_2} u_2(p_2) = \\ &= -\frac{\alpha}{2\pi} \frac{1}{i\pi^2} \int \frac{d^4k}{(k^2 - 2k \cdot p_1)(k^2 + 2k \cdot p_2)} \left\{ -2p_1 \cdot p_2 \left[\frac{1}{k^2 + \lambda^2} + \frac{1}{(p_3 + k)^2 + \lambda^2} - \frac{2k(k + p_3)}{k^2(p_3 + k)^2} \right] M^{(0)} + \right. \\ &+ \left(\frac{\alpha}{i\pi} \frac{m_1 m_2}{E_1 E_2} \right) \frac{1}{k^2(p_3 + k)^2} \left[k^\sigma (\bar{u}_1(p'_1) \gamma_\nu \gamma_\sigma \not{p}_2 u_1(p_1) \bar{u}_2(p'_2) \gamma^\nu u_2(p_2) - \right. \\ &- \bar{u}_1(p'_1) \gamma^\nu u_1(p_1) \bar{u}_2(p'_2) \gamma_\nu \gamma_\sigma \not{p}_1 u_2(p_2)) + \frac{1}{2} k^\sigma k_\tau \bar{u}_1(p'_1) \gamma_\nu \gamma_\sigma \gamma_\tau \gamma^\mu u_1(p_1) \\ &\cdot \left. \left. \bar{u}_2(p'_2) \gamma^\nu \gamma^\tau \gamma^\mu u_2(p_2) \right] \right\} \quad (36) \end{aligned}$$

The infra-red contribution here is easily transformed into that of eq. (14).

For R_5 , defined by (17), we obtain

$$QR_5 = -2u(u^2 - u - T + \frac{1}{2}) \text{I} + \text{Re} \left\{ -\frac{1}{i\pi^2} \int d^4k \frac{k^\sigma N_\sigma + \frac{1}{2} k^\sigma k^\tau N_\sigma^\tau}{(k^2 - 2k \cdot p_1)(k^2 + 2k \cdot p_2)k^2(k+p_3)^2} \right\} \quad (37)$$

where

$$\text{I} = \text{Re} \left\{ \Delta^2 \frac{1}{i\pi^2} \int \frac{d^4k k \cdot (k+p_3)}{(k^2 - 2k \cdot p_1)(k^2 + 2k \cdot p_2)k^2(k+p_3)^2} \right\} \quad (38)$$

and

$$N_\sigma = \frac{1}{\Delta^2} \left[p_2^\mu F_1 g_{\nu\sigma\mu} F_2 g^\nu - p_1^\mu F_1 g^\nu F_2 g_{\nu\sigma\mu} \right] \quad (39)$$

$$N_\sigma^\tau = \frac{1}{\Delta^2} F_1 g_{\nu\sigma\mu} F_2 g^{\nu\tau\mu}$$

The second rank tensor $F_{i\mu\nu}$ appearing in (39) is given by (19) and (20) and the fourth rank tensor is defined by

$$F_{i\mu\nu\sigma\mu} = m_i^2 \sum_{\text{spins}} \left[\bar{u}_i(p'_i) \gamma_\mu u_i(p_i) \right]^* \left[u_i(p'_i) \gamma_\nu \gamma_\sigma \gamma_\mu u_i(p_i) \right] \quad (40)$$

Trace calculations give

$$\begin{aligned} F_{i\mu\nu\sigma\mu} = & F_{i\mu\nu} g_{\sigma\mu} + p_{i\mu} (p_{i\nu} g_{\mu\sigma} - p_{i\sigma} g_{\nu\mu}) + p_{i\nu} (p_{i\mu} g_{\mu\sigma} - p_{i\sigma} g_{\mu\nu}) + \\ & + (p_{i\sigma} p_{i\mu} - p_{i\mu} p_{i\sigma}) g_{\nu\mu} + \frac{1}{2} \Delta^2 (g_{\mu\sigma} g_{\nu\mu} - g_{\mu\nu} g_{\sigma\mu}) + \\ & + p_{i\mu} (p_{i\nu} g_{\mu\sigma} - p_{i\sigma} g_{\nu\mu}) - p_{i\nu} (p_{i\mu} g_{\mu\sigma} - p_{i\sigma} g_{\mu\nu}) \end{aligned} \quad (41)$$

After calculating the tensor products in (39) we get

$$\begin{aligned}
 \Delta^2(k^\sigma N_\sigma + \frac{1}{2}k^\sigma k_\tau N_\sigma^\tau) &= \int 4(p_1 p_2)^2 - \frac{1}{2} \Delta^2_{W^2} \int [(k^2 - 2k \cdot p_1) + (k^2 + 2k \cdot p_2)] - \\
 &- 2p_1 \cdot p_2 \int m_2^2 (k^2 - 2k \cdot p_1) + m_1^2 (k^2 + 2k \cdot p_2) + (\Delta^2_{-W^2}) k \cdot (k + p_3) \int + \\
 &+ k^2 \Delta^2(p_1 \cdot p_2 + \frac{1}{2} \Delta^2) + k_\sigma k_\tau \int \Delta^2(p_1^\sigma p_1^\tau + p_2^\sigma p_2^\tau) + 4 p_1 \cdot p_2 p_1^\sigma p_2^\tau + \\
 &+ 2(p_1 p_2 - \frac{1}{2} \Delta^2)(p_2^\sigma - p_1^\sigma) p_3^\tau - 2p_1 \cdot p_2 p_3^\sigma p_3^\tau \int
 \end{aligned} \tag{42}$$

This inserted into (37) gives us

$$QR_5 = -u(2u^2 - u - T) I + \sum_{i=1}^2 (u^2 - \frac{1}{2}u - \frac{1}{2}T + \frac{1}{4}ut_i) I_i + \left\{ \frac{1}{2}(1-u)g^{\mu\nu} + \right. \tag{43}$$

$$\left. + \frac{1}{\Delta^2} \int p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + u(p_3^\mu p_3^\nu - 2p_1^\mu p_2^\nu) + (1+u)(p_1^\mu - p_2^\mu) p_3^\nu \int \right\} I_{\mu\nu}$$

where

$$I_i = \text{Re} \left\{ \Delta^2 \frac{1}{i\pi^2} \int \frac{d^4 k}{(0)(i)(\bar{3})} \right\} \quad (i=1,2) \tag{44}$$

and

$$I_{\mu\nu} = \text{Re} \left\{ \Delta^2 \frac{1}{i\pi^2} \int \frac{d^4 k k_\mu k_\nu}{(0)(1)(2)(\bar{3})} \right\} \tag{45}$$

Here we have used the abbreviations

$$\begin{cases} (0) = k^2 \\ (3) = (p_3 + k)^2 \end{cases} \quad \begin{cases} (1) = k^2 - 2k \cdot p_1 \\ (2) = k^2 + 2k \cdot p_2 \end{cases} \tag{46}$$

The quantities

$$T_{ij} = \frac{1}{\Delta^2} p_i^\mu p_j^\nu I_{\mu\nu} \quad (i, j = 1, 2, 3) \quad (47)$$

can be expressed in terms of I , I_i and the integrals

$$X = g^{\mu\nu} I_{\mu\nu} = \text{Re} \left\{ \Delta^2 \frac{1}{i\pi^2} \int \frac{d^4 k}{(1)(2)(3)} \right\} \quad (48)$$

$$\begin{cases} a = \text{Re} \left\{ \frac{1}{i\pi^2} \int d^4 k \frac{(3) - (0) + \Delta^2}{(1)(2)(3)} \right\} \\ a' = \text{Re} \left\{ \frac{1}{i\pi^2} \int d^4 k \frac{(0)(3) - (1)(2)}{(0)(1)(2)(3)} \right\} \\ a_i = \text{Re} \left\{ \frac{1}{i\pi^2} \int d^4 k \frac{(3) - (i)}{(0)(i)(3)} \right\} \end{cases} \quad (49)$$

in the following way

$$\begin{cases} T_{11} = -\frac{1}{4} (a_2 + a - a' - X) + \frac{1}{4(t_2+1)} \int (2u-1)a_2 - (u+\frac{1}{2}t_2)I_2 \\ T_{33} = \frac{1}{4}(4X-2I-a) + \frac{1}{4u^2-t_1t_2} \int (u+\frac{1}{2}t_1)a_1 + (u+\frac{1}{2}t_2)a_2 - 2(u+T)a' \\ T_{12} = \frac{1}{4} (a_1 + a_2 + a - a' - X) \\ T_{13} = \frac{1}{4} (I_2 + a - 2X) \quad (\text{similarly for } T_{22} \text{ and } T_{23}) \end{cases} \quad (50)$$

Now the expression for QR_5 (43) can be written in terms of I , I_i , a , a' , a_i and X . It turns out that the coefficient for X vanishes (as it should do, because X is infra-red divergent) and we arrive at

$$\begin{aligned}
QR_5 = & -(2u^2 - u - T + \frac{1}{2}) \int u I^{-\frac{1}{2}} (I_1 + I_2) \int - \frac{1}{4} u a + \frac{1}{2} (1 + u - \frac{u(u+T)}{u^2 - \frac{1}{4} t_1 t_2}) a' + \\
& + \frac{1}{4} \sum_{i=1}^2 \int t_i I_i (u - \frac{\frac{1}{2} - u}{1 + t_i}) - a_i (1 + 2u + \frac{1 - 2u}{1 + t_i} - \frac{u(u + \frac{1}{2} t_i)}{u^2 - \frac{1}{4} t_1 t_2}) \int
\end{aligned} \tag{51}$$

Evaluation of the remaining integrals gives the result

$$\begin{aligned}
I = & \frac{1}{4 \sqrt{u^2 - \frac{1}{4} t_1 t_2}} \sum_{i=1}^2 \int \ln^2 \frac{t_i}{4} - \ln \frac{(u + \frac{1}{2} t_i + \sqrt{u^2 - \frac{1}{4} t_1 t_2})^2}{4(u^2 - \frac{1}{4} t_1 t_2)} \ln \frac{t_i}{4} + \\
& + \ln \frac{u+T}{u^2 - \frac{1}{4} t_1 t_2} \ln \frac{(u + \frac{1}{2} t_i + \sqrt{u^2 - \frac{1}{4} t_1 t_2})^2}{4(u+T)} + 2 \phi \left(\frac{1}{2} (1 - \frac{u + \frac{1}{2} t_i}{\sqrt{u^2 - \frac{1}{4} t_1 t_2}}) \right) - \\
& - 2 \phi \left(\frac{1}{2} (1 + \frac{u + \frac{1}{2} t_i}{\sqrt{u^2 - \frac{1}{4} t_1 t_2}}) \right) - \pi^2 \theta(u + \frac{t_1}{2}) \theta(u + \frac{t_2}{2}) \int; \quad \theta(x) = \begin{cases} 0, x \leq 0 \\ 1, x > 0 \end{cases}
\end{aligned} \tag{52}$$

$$I_i = - \frac{1}{\sqrt{1+t_i}} \int \phi(z_i^+) - \phi(z_i^-) + 2\phi(\frac{1}{z_i^+}) - 2\phi(\frac{1}{z_i^-}) \int; \quad z_i^\pm = \frac{2}{t_i} (-1 \pm \sqrt{1+t_i})$$

$$a = - \frac{1}{\sqrt{u^2 - \frac{1}{4} t_1 t_2}} \sum_{i=1}^2 \int \ln \frac{(u + \frac{1}{2} t_i + \sqrt{u^2 - \frac{1}{4} t_1 t_2})^2}{4(u+T)} - \ln \frac{t_i}{4} \int$$

$$a' = - \frac{\sqrt{u^2 - \frac{1}{4} t_1 t_2}}{2(u+T)} \sum_{i=1}^2 \int \ln \frac{(u + \frac{1}{2} t_i + \sqrt{u^2 - \frac{1}{4} t_1 t_2})^2}{4(u+T)} - \left(1 - \frac{u + \frac{1}{2} t_i}{\sqrt{u^2 - \frac{1}{4} t_1 t_2}} \right) \ln \frac{t_i}{4} \int$$

$$a_i = - \ln \frac{t_i}{4}$$

Here $\phi(x)$ is a so-called Spence function defined by

$$\phi(x) = \int_0^x \frac{dy}{y} \ln |1-y| \quad (53)$$

Because of (35) we may write

$$R_5 + R_6 = (u^2 - u - T + \frac{1}{2}) \sum_{i=1}^2 \int R_i(u) - R_i(1-u) \quad (54)$$

since the interchange $p_2 \leftrightarrow -p_2'$ is equivalent to $u \leftrightarrow (1-u)$ as follows from (5) and (6). We combine (51) and (52) and approximate $\sqrt{u^2 - \frac{1}{4}t_1 t_2}$ with u , which is correct when $(E_{\mu \text{lab}})^2 \gg m_{\mu}^2$. ($E_{\mu \text{lab}}$ = muon energy in the rest frame of the electron.) Then we obtain the following expression for $R_i(u)$

$$\begin{aligned} R_i(u) = & \frac{1}{4}Q(u^2 - u - T + \frac{1}{2})^{-1} \left\{ -(2u^2 - u - T + \frac{1}{2}) \int \ln^2 \frac{t_i}{4} - 2I_i - 2 \ln \frac{t_i}{4} \ln(1 + \frac{t_i}{4u}) - \right. \\ & - \ln \frac{u^2}{u+T} \ln \frac{(u + \frac{t_i}{4})^2}{u+T} + 2\phi(-\frac{t_i}{4u}) - 2\phi(1 + \frac{t_i}{4u}) - \pi^2 \theta(u + \frac{t_1}{2}) \theta(u + \frac{t_2}{2}) \int + \\ & \left. + t_i \int -\frac{1}{2}(\ln \frac{t_i}{4})(\frac{1+u}{u+T} + \frac{2-4u}{1+t_i}) + I_i(u - \frac{1-u}{1+t_i}) \int + \int 2 - \frac{u(1+u)}{u+T} \int \ln \frac{(u + \frac{t_i}{4})^2}{u+T} \right\} \quad (55) \end{aligned}$$

$$I_i = - \frac{1}{\sqrt{1+t_i}} \int \phi(z_i^+) - \phi(z_i^-) + 2\phi\left(\frac{1}{z_i^+}\right) - 2\phi\left(\frac{1}{z_i^-}\right) \int; \quad z_i^{\pm} = \frac{2}{t_i} (-1 \pm \sqrt{1+t_i})$$

IV. Results

We can now add the contributions from the different diagrams.

Neglecting the magnetic contribution (33) which involves the small factor $1/8t_1t_2$, we obtain from (15), (I-54), (I-59), (23), (29) and (54), after some calculations

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega} \right)^{(0)} \left[1 - \frac{\alpha}{\pi} (\delta_1 + \delta_2) \right] \quad (56)$$

$$\delta_i = L_i + K_i(u) \ln \frac{q}{4E} + H_i(u) + \frac{1}{2}H'(u) + R_i(u) - R_i(1-u) \quad (57)$$

$$\begin{cases} L_i = \frac{28}{9} - \frac{2}{3}t_i - \left(\frac{13}{6} - \frac{1}{3}t_i \right) \sqrt{1+t_i} \ln \frac{\sqrt{1+t_i} + 1}{\sqrt{1+t_i} - 1} \\ K_i(u) = 2 \left[\frac{1+\frac{1}{2}t_i}{\sqrt{1+t_i}} \ln \frac{\sqrt{1+t_i} + 1}{\sqrt{1+t_i} - 1} - 1 + \ln \left| \frac{u+T}{1-u+T} \right| - 2 \ln \left| \frac{u+\frac{1}{4}t_i}{1-u+\frac{1}{4}t_i} \right| \right] \end{cases} \quad (58)$$

$$H_1(u) + H_2(u) + H'(u) = - \left(\frac{\alpha}{\pi} \right)^{-1} B+4 = 2 \sum_{i=1}^2 \int \frac{p_i \cdot p'_i}{q^2} \operatorname{Re} \left\{ \int_0^1 dx \, h \left(\frac{p_i x + p'_i (1-x)}{q} \right) \right\} - \quad (59)$$

$$- \frac{p_i^2}{q^2} h \left(\frac{p_i}{q} \right) + 4Q \int \frac{p_1 \cdot p_2}{q^2} \operatorname{Re} \left\{ \int_0^1 dx \, h \left(\frac{p_1 x - p_2 (1-x)}{q} \right) \right\} + \frac{p_1 \cdot p'_2}{q^2} \operatorname{Re} \left\{ \int_0^1 dx \, h \left(\frac{p_1 x + p'_2 (1-x)}{q} \right) \right\} -$$

$$h(p) = - \frac{1}{-p} \int_{-\frac{1}{2}}^{\frac{1}{2}} \ln \frac{-p^2}{4} + \int_0^1 dy \frac{p_0^2}{p_0^2 - p^2 y^2}; \quad (\text{Poles are avoided by detours into the upper half-plane}).$$

The zero order cross section is given above in (22). The function $R_i(u)$ is expressed in (55). From (59) we separate out the more explicit formulae

$$\begin{aligned}
 H_i(u) = & \frac{E_i - q}{q} \ln \frac{E_i - q}{2q} - \frac{E_i + q}{q} \ln \frac{E_i + q}{2q} + \\
 & + 2 \sin^2 \frac{\theta}{2} (1 + \frac{1}{2} t_i) \int_0^1 \frac{dx}{\varphi(x)} \left[\frac{q}{E_i - q \varphi(x)} \ln \frac{E_i + q \varphi(x)}{2q} - \right. \\
 & \left. - \frac{q}{E_i + q \varphi(x)} \ln \frac{E_i - q \varphi(x)}{2q} \right];
 \end{aligned} \tag{60}$$

$$\begin{aligned}
 H'(u) = & 2q \left\{ \sum_{i=1}^2 \left[\phi\left(\frac{E_i - q}{2q}\right) - \phi\left(\frac{E_i + q}{2q}\right) \right] + 2 \sin^2 \frac{\theta}{2} (1 - u) \int_0^1 \frac{dx}{\psi(x)} \left[\frac{q}{E_1 + (E_2 - E_1)x + q \psi(x)} \right. \right. \\
 & \left. \left. \cdot \ln \frac{E_1 + (E_2 - E_1)x - q \psi(x)}{2q} - \frac{q}{E_1 + (E_2 - E_1)x - q \psi(x)} \ln \frac{E_1 + (E_2 - E_1)x + q \psi(x)}{2q} \right] \right\};
 \end{aligned} \tag{61}$$

$$\begin{aligned}
 \varphi(x) &= \left[1 - 4 \sin^2 \frac{\theta}{2} x(1-x) \right]^{\frac{1}{2}} \\
 \psi(x) &= \left[1 - 4 \cos^2 \frac{\theta}{2} x(1-x) \right]^{\frac{1}{2}}
 \end{aligned} \tag{62}$$

By examining the result stated in equations (55) to (62) one can see that when $\Delta \gg m_e$ - as is the case in high energy experiments - the pre-dominant terms of the radiative corrections are those proportional to $\ln(\frac{\Delta^2}{m_e^2})$. (m_e (μ) = electron (muon) rest mass). This is true until the momentum transfer becomes so large that the $\ln(\frac{\Delta^2}{m_\mu^2})$ - terms are also important. It appears that the $\ln(\frac{\Delta^2}{m_e^2})$ - terms derive extensively from the "electron diagrams" $M_1^{(1)}$ and $M_3^{(1)}$ in m_e Fig. 3 and a and b in Fig. 4. Similarly, the $\ln(\frac{\Delta^2}{m_\mu^2})$ - terms derive extensively from the "muon diagrams", $M_2^{(1)}$ and $M_4^{(1)}$ in Fig. 3 and c and d in Fig. 4. The two-photon exchange diagrams $M_5^{(1)}$ and $M_6^{(1)}$, on the other hand, do not contribute any terms that become large with large Δ .

The momentum transfers that occur in experiments planned at present are such that the $\ln(\frac{\Delta^2}{m_e^2})$ - terms will dominate the radiative corrections. If we take into account only the "electron diagrams", $M_1^{(1)}$, $M_3^{(1)}$ (Fig. 3) and a, b (Fig. 4) we get (56) and (57) replaced by the simpler formulae

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)^{(0)} \left[1 - \frac{\alpha}{\pi} \delta \right] \quad (63)$$

$$\delta = L_1 + K_1(u) \ln \frac{q}{4E} + H_1(u) \quad (64)$$

where L_1 and $K_1(u)$ are given by (58) and $H_1(u)$ by (60) and (62). Neglecting terms that are linear in $\frac{m_e}{\Delta}$ we then obtain the following simple expression for δ :

$$\begin{aligned} \delta = & \frac{28}{9} - \frac{13}{6} \ln \left(\frac{\Delta^2}{m_e^2} \right) + 2 \left[\ln \left(\frac{\Delta^2}{m_e^2} \right) - 1 \right] \ln \left(\frac{q}{4E} \right) + \\ & + 2 \sin \frac{\theta}{2} \int_{\cos \frac{\theta}{2}}^1 \frac{dx}{\sqrt{x^2 - \cos^2 \frac{\theta}{2}}} \left[\frac{1}{1-x} \ln \left(\frac{1+x}{2} \right) - \frac{1}{1+x} \ln \left(\frac{1-x}{2} \right) \right] \end{aligned} \quad (65)$$

As one could indeed expect this is precisely the result obtained by Schwinger (ref. ⁴⁾, Eq. (2.105) and (2.97)) for the radiative corrections to the scattering of an electron by a potential ^{*}). A similar result was obtained by Tsai ⁷⁾ for electron-electron scattering.

The muon diagrams and the two-photon exchange diagrams - which are neglected when δ is approximated by the formula (65) - give corrections to δ of the order

$$\frac{\Delta}{m_{\mu}} \quad (56)$$

So (65) is obtained in the "static limit" when $m_{\mu} \rightarrow \infty$.

When $\Delta \gg m_{\mu}$, the expression (65) has to be extended by a muon term similar to the electron term.

Numerical results based on (65) are shown in Table 1 for some energies and angles at which this formula gives a reasonable approximation. The errors due to the neglect of "muon diagrams" and two-photon-exchange diagrams have been estimated to be of an order of magnitude certainly less than 0.01.

Table 1

First order radiative corrections to μ -e scattering in per cent ($100 \frac{\delta}{\pi}$)

$E_{\mu \text{ lab}}$	q/m_e	c.m. scattering angle, θ		
		45°	90°	180°
2 GeV	17.7	$1.9 \ln(\frac{q}{\Delta E}) - 1.8$	$2.5 \ln(\frac{q}{\Delta E}) - 2.2$	$2.8 \ln(\frac{q}{\Delta E}) - 2.5$
10 GeV	68.3	$3.2 \ln(\frac{q}{\Delta E}) - 3.2$	$3.8 \ln(\frac{q}{\Delta E}) - 3.6$	$4.1 \ln(\frac{q}{\Delta E}) - 3.8$
50 GeV	202	$4.2 \ln(\frac{q}{\Delta E}) - 4.3$	$4.8 \ln(\frac{q}{\Delta E}) - 4.7$	$5.1 \ln(\frac{q}{\Delta E}) - 4.9$

^{*}) The integral appearing in (65) is in general not analytically evaluable. An approximative formula for this integral is given in Schwinger's paper (Eq. (2.99)).

As mentioned in the introduction, the cross section for the hard photon emission, permitted by the experimental geometry has also to be calculated before the radiative corrections can be compared to experimental results.

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Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

$$\begin{aligned}
 & \left(\begin{array}{cc} p'_1 & p'_2 \\ \uparrow & \uparrow \\ e & e \\ \uparrow & \uparrow \\ p_1 & p_2 \end{array} \right)^* \left(\begin{array}{cc} p'_1 & p'_2 \\ \uparrow & \uparrow \\ e & e \\ \uparrow & \uparrow \\ p_1 & p_2 \end{array} \right) = \left(\begin{array}{cc} p'_1 & p_2 \\ \uparrow & \downarrow \\ e & e \\ \uparrow & \downarrow \\ p_1 & p'_2 \end{array} \right)^* \left(\begin{array}{cc} p'_1 & p_2 \\ \uparrow & \downarrow \\ e & e \\ \uparrow & \downarrow \\ p_1 & p'_2 \end{array} \right) = \\
 & = \left(\begin{array}{cc} p'_1 & -p_2 \\ \uparrow & \downarrow \\ e & -e \\ \uparrow & \downarrow \\ p_1 & -p'_2 \end{array} \right)^* \left(\begin{array}{cc} p'_1 & -p_2 \\ \uparrow & \downarrow \\ e & -e \\ \uparrow & \downarrow \\ p_1 & -p'_2 \end{array} \right) = - \left(\begin{array}{cc} p'_1 & -p_2 \\ \uparrow & \downarrow \\ e & e \\ \uparrow & \downarrow \\ p_1 & -p'_2 \end{array} \right)^* \left(\begin{array}{cc} p'_1 & -p_2 \\ \uparrow & \downarrow \\ e & e \\ \uparrow & \downarrow \\ p_1 & -p'_2 \end{array} \right)
 \end{aligned}$$

Fig. 6