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An analytical formula for the band width of a strong nonstructure octupole resonance induced by a space charge

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In a high intensity proton synchrotron nonstructure octupole resonances are excited by a space charge force when its designed periodicity is broken by quadrupole errors. This paper gives an analytical formula to calculate a band width of a nonstructure octupole resonance induced by a space charge force. Two kinds of modulations, beam envelope modulation and betatron-phase modulation are taken into account as perturbations. An approximated treatment shows that the phase modulation takes on a more important role. The formula is applied to the KEK-PS, at which the band widths of nonstructure octupole resonances induced by the space charge force are much larger than that produced by magnetic octupole errors.

1 Introduction

In a proton synchrotron the repulsive force among particles, namely the space charge force, produces a tune spread of a bunched beam; then, a part of the beam comes to a betatron resonance. It is said that imperfections of lattice magnets excite betatron resonances, blow up the emittance of the beam and then produce a beam loss. Usually, a particle cannot survive for a long time at resonances of and lower than the fourth order. In conventional modeling of resonance [1], the order of the resonances corresponds to the polar multiplicity of the magnetic error field. However, up to the present, other kinds of error sources have been found to be important, such as a down feeding effect of the transverse resonances [2] and space charge induced resonances.

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B. Montague [3] has pointed out that the non-linear space charge force itself could be the source of a 0th harmonic integer coupling resonance and calculated its strength. In a real machine, this resonance is mainly induced by a space charge force, not by machine imperfections[4]. G. Parzen [5] and S. Machida [6] have proposed the idea of an intrinsic space charge limit, which is a limit in the absence of magnetic field errors. Since the space charge force is modulated according to the beam envelope, it has large harmonic components of the superperiod times an integer. Especially when the harmonic number is identical to the periodicity of a focusing function, the resonance becomes very strong and is called a *superstructure resonance* [7,8]. However, in some high intensity proton synchrotrons, which take some hundreds of milliseconds for a multi-step injection, much weaker resonances produced by the space charge and quadrupole imperfections become effective. This *space charge induced non-structure resonance* was experimentally observed at the KEK-PS for the first time [7], although it has never been considered in any other synchrotron. A strong beam current dependence of a nonstructure octupole resonance was observed, which limited the working area in its tune space. The effect of this resonance was not clear at the millisecond (10^3 turns) level but was fatefully serious for a time interval of some hundreds of milliseconds. This resonance was analyzed numerically by S. Machida[8] first and analytically by us.

In this paper we focus our discussion on a vertical octupole resonance, $4Q_y = N$, where Q_y is the vertical betatron tune and N is the harmonic number. It would be easy to extend the result to any other normal octupole resonances. To simplify the calculations, the following assumptions are used.

1. The betatron tunes are much larger than unity.
2. There is no magnetic multipole field of or higher than the sextupole.
3. The quadrupole errors are very small, and higher order effects can be ignored.
4. The momentum spread is very small, so that the dispersive beam spread is negligibly small compared to the spread by the horizontal emittance.
5. The beam distributions are Gaussian in the horizontal and vertical planes.
6. Fields due to images in the walls are ignorable.
7. The modulation of the beam envelope, which produces nonsystematic harmonic octupoles, can be expressed by the modulation of beta functions of a particle in a bunch.

Consequently, there is no octupole resonance produced by only magnetic-field errors and the superperiodicity is effectively identical to the periodicity. The last three assumptions are not accurate in any high intensity proton machine with a large tune spread. The last one is not correct especially near to an integer or a half-integer tune, where the modulation of beta functions diverges to infinity. However, the formulae given below have practically sufficient accuracy for designing synchrotrons.

2 Band width of the resonance

We consider the band width of a space charge induced resonance as follows. The Hamiltonian of a longitudinal coasting beam which includes quadrupole magnet imperfections and a space charge force is written as

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2!}K(x^2 - y^2) + \frac{r_p\lambda}{\beta^2\gamma^3} \left[-\frac{1}{\sigma_x(\sigma_x + \sigma_y)}x^2 - \frac{1}{\sigma_y(\sigma_x + \sigma_y)}y^2 + \frac{2\sigma_x + \sigma_y}{12\sigma_y^3(\sigma_x + \sigma_y)^2}x^4 + \frac{1}{2\sigma_x\sigma_y(\sigma_x + \sigma_y)^2}x^2y^2 + \frac{2\sigma_y + \sigma_x}{12\sigma_x^3(\sigma_x + \sigma_y)^2}y^4 + \dots \right], \quad (1)$$

where x and y are the horizontal and vertical coordinates; p_x and p_y are the momenta canonically conjugate to x and y , respectively; σ_x and σ_y are the horizontal and vertical rms beam size, respectively, β and γ are the Lorentz factors, and λ and r_p are the line charge density and classical proton radius, respectively. K is a focusing force, $K = (1/B\rho)(\partial B_y/\partial x)$, which includes the imperfection quadrupoles. The square brackets indicate the space charge terms of both the linear and nonlinear contributions. Once we know the Hamiltonian, the band width of the fourth-order resonance: $4Q_y = N$ can be estimated [1] by

$$\Delta e = \left| \frac{J_y}{2\pi} \frac{r_p\lambda}{\beta^2\gamma^3} \int_0^{2\pi R} \beta_y^2 \frac{2\sigma_y + \sigma_x}{12\sigma_y^3(\sigma_x + \sigma_y)^2} e^{-iN\phi_y} ds \right|, \quad (2)$$

where J_y is the vertical Courant Snyder invariant, β_y is the vertical beta function and ϕ_y is a phase defined by

$$N\phi_y = (N - 4Q_y)(s/R) + 4\mu_y. \quad (3)$$

Here, the betatron oscillation phase (μ_y) is given by

$$\mu_y(s) = \int_0^s [ds^*/\beta_y(s^*)]. \quad (4)$$

The effect of the linear defocusing force due to the space charge (quadrupole component) is included in the beta function, which means that the beta func-

tion depends on the space charge force. Using

$$\sigma_x = \sqrt{\beta_x \varepsilon_x}, \\ \sigma_y = \sqrt{\beta_y \varepsilon_y},$$

the stop band width can be rewritten in a more simple form as

$$\Delta e = \left| \frac{J_y}{2\pi\varepsilon_y^2} \frac{r_p\lambda}{12\beta^2\gamma^3} \int_0^{2\pi R} \frac{(1+2c)c}{(1+c)^2} e^{-iN\phi_y} ds \right|. \quad (5)$$

Here ε_x and ε_y are the horizontal and vertical rms emittances, respectively, and c is the ratio of σ_y to σ_x ,

$$c(s) = \sigma_y(s)/\sigma_x(s).$$

In the absence of magnetic-field errors the band width of a nonstructure resonance is zero, even when a space charge force exists because of the periodicity of the system. However when both a space charge force and magnetic-quadrupole errors (δK) exist, the band width is not zero. Since we assumed that δK is very small, the band width of a nonstructure resonance (Δe_{ns}) in the lowest order approximations is given by

$$\Delta e_{ns} = \left| [d\Delta e/d\delta K]_{\delta K=0} \delta K \right|. \quad (6)$$

When there are many small quadrupole error sources the resonance width is a sum of the contributions from each error. That is,

$$\Delta e_{ns} = \left| \sum_i [\partial \Delta e / \partial \delta K_i]_{\delta K_i=0} \delta K_i \right|. \quad (7)$$

where the summed items are complex because they have a harmonic phase.

3 Band width produced by a small single quadrupole error

We first look at the most simple case, a band width produced by a single quadrupole error set at $s = 0$. We go back to eq. (6) and calculate the band width. Δe_{ns} is written as

$$\Delta e_{ns} = \left| \frac{J_y}{2\pi\varepsilon_y^2} \frac{r_p\lambda}{12\beta^2\gamma^3} \int_0^{2\pi R} \left[\frac{1+3c}{(1+c)^3} \delta c - i \frac{c(1+2c)}{(1+c)^2} \delta(N\phi_y) \right] e^{-iN\phi_y} ds \right|. \quad (8)$$

The first term in the square brackets of the integrand represents the contribution of the beam envelope modulation, and the second represents the vertical betatron phase modulation. They are calculated using the modulations of the beta functions ($\delta\beta_x$ and $\delta\beta_y$), the betatron phase advance ($\delta\mu_y$) and the tune shifts (δQ_y and δQ_x) as

$$\delta c = \frac{c}{2} \left(\frac{\delta\beta_y}{\beta_y} - \frac{\delta\beta_x}{\beta_x} \right) \quad (9)$$

and

$$\delta(N\phi_y) = 4[\delta\mu_y - \delta Q_y(s/R)]. \quad (10)$$

Since the error is set at $s = 0$, the modulations ($\delta\beta_x$, $\delta\beta_y$ and $\delta\mu_y$) are given by

$$\frac{\delta\beta_x(s)}{\beta_x(s)} = -\frac{2\pi\delta Q_x}{\sin(2\pi Q_x)} \cos[2\mu_x(s) - 2\pi Q_x], \quad (11)$$

$$\frac{\delta\beta_y(s)}{\beta_y(s)} = -\frac{2\pi\delta Q_y}{\sin(2\pi Q_y)} \cos[2\mu_y(s) - 2\pi Q_y] \quad (12)$$

and

$$\delta\mu_y(s) = \frac{2\pi\delta Q_y}{2\sin(2\pi Q_y)} \sin[2\mu_y(s) - 2\pi Q_y] + \pi\delta Q_y \quad (13)$$

for $0 \leq s < 2\pi R$. Using eq. (13), eq. (10) is rewritten as

$$\delta(N\phi_y) = 4 \left[\frac{\sin(2\mu_y - 2\pi Q_y)}{2\sin(2\pi Q_y)} + \left(\frac{1}{2} - \frac{\mu_y}{2\pi Q_y} \right) + \left(\frac{\mu_y}{2\pi Q_y} - \frac{s}{2\pi R} \right) \right] 2\pi\delta Q_y. \quad (14)$$

Since the last round brackets within the square brackets contain only the structure harmonics, they disappear after integration. Using the above equations we obtain Δe_{ns} as

$$\Delta e_{ns} = \left| \frac{J_y}{2\pi\epsilon_y^2} \frac{r_p\lambda}{12\beta^2\gamma^3} \int_0^{2\pi R} \left\{ \frac{c(1+3c)}{2(1+c)^3} \left[\frac{\cos(2\mu_x - 2\pi Q_x)}{\sin(2\pi Q_x)} 2\pi\delta Q_x - \frac{\cos(2\mu_y - 2\pi Q_y)}{\sin(2\pi Q_y)} 2\pi\delta Q_y \right] - i \frac{4c(1+2c)}{(1+c)^2} \left[\frac{\sin(2\mu_x - 2\pi Q_x)}{2\sin(2\pi Q_x)} + \left(\frac{1}{2} - \frac{\mu_y}{2\pi Q_y} \right) \right] 2\pi Q_y \right\} \right|$$

$$\} e^{-iN\phi_y} ds|. \quad (15)$$

4 Harmonic components

In order to simplify eq. (15), we separate the terms in the integrand into a Fourier series. The terms which contain nonstructure harmonics are

$$\begin{aligned} \frac{\cos(2\mu_x - 2\pi Q_x)}{\sin(2\pi Q_x)} &= \frac{2}{2\pi Q_x} \sum_{n=0}^{\infty} S_{x,n} \cos(n \frac{\mu_x}{Q_x}) \\ &= \frac{1}{2\pi Q_x} \sum_{n=0}^{\infty} \sum_{j=1,-1} S_{x,n} \exp(ijn \frac{\mu_x}{Q_x}), \end{aligned} \quad (16)$$

$$\begin{aligned} \frac{\sin(2\mu_y - 2\pi Q_y)}{\sin(2\pi Q_y)} &= \frac{2}{2\pi Q_y} \sum_{n=0}^{\infty} S_{y,n} \cos(n \frac{\mu_y}{Q_y}) \\ &= \frac{1}{2\pi Q_y} \sum_{n=0}^{\infty} \sum_{j=1,-1} S_{y,n} \exp(ijn \frac{\mu_y}{Q_y}), \end{aligned} \quad (17)$$

and

$$\begin{aligned} \frac{\sin(2\mu_y - 2\pi Q_y)}{2\sin(2\pi Q_y)} + \frac{1}{2} - \frac{\mu_y}{2\pi Q_y} &= \frac{2}{2\pi Q_y} \sum_{n=1}^{\infty} T_{y,n} \sin(n \frac{\mu_y}{Q_y}) \\ &= \frac{1}{2\pi Q_y} \sum_{n=1}^{\infty} \sum_{j=1,-1} T_{y,n} \exp(ijn \frac{\mu_y}{Q_y}). \end{aligned} \quad (18)$$

Here $S_{x,n}$, $S_{y,n}$ and $T_{y,n}$ are scaling factors, given by

$$S_{x,n} = \frac{(2Q_x)^2}{(2Q_x)^2 - n^2}, \quad (19)$$

$$S_{y,n} = \frac{(2Q_y)^2}{(2Q_y)^2 - n^2}, \quad (20)$$

$$(21)$$

and

$$T_{y,n} = \frac{2Q_y}{n} S_{y,n} \quad (n \neq 0). \quad (22)$$

The terms which contain only structure harmonics are separated as

$$\frac{c(1+3c)}{(1+c)^3} \beta_y Q_y = \sum_{k=-\infty}^{\infty} A_{ye,kM} \exp(ikM \frac{\mu_y}{Q_y}), \quad (23)$$

$$\frac{c(1+2c)}{(1+c)^2} \beta_y Q_y = \sum_{k=-\infty}^{\infty} A_{yp,kM} \exp(ikM \frac{\mu_y}{Q_y}), \quad (24)$$

$$(25)$$

and

$$\frac{c(1+3c)}{(1+c)^3} \beta_x Q_x \exp(i \frac{\mu_x}{Q_x} - i \frac{\mu_y}{Q_y}) = \sum_{k=-\infty}^{\infty} A_{xe,kM} \exp(ikM \frac{\mu_x}{Q_x}). \quad (26)$$

Here, M is the periodicity and $A_{xe,n}$, $A_{ye,n}$ and $A_{yp,n}$ are the Fourier amplitudes.

Because our interest is on the band width for a particle on the resonance, we set $Q_y = N/4$. Then,

$$N\phi_y = N \frac{\mu_y}{Q_y}. \quad (27)$$

Using the above equation, Δe_{ns} becomes

$$\Delta e_{ns} = \left| \frac{J_y}{2\pi\epsilon_y^2} \frac{r_p \lambda}{12\beta^2 \gamma^3} \sum_{k=-\infty}^{\infty} \sum_{j=1,-1} [A_{xe,kM} S_{x,j(N-kM)} \frac{\delta Q_x}{2Q_x} + A_{ye,kM} S_{y,j(N-kM)} \frac{\delta Q_y}{2Q_y} - 8ij A_{yp,kM} T_{y,j(N-kM)} \frac{\delta Q_y}{2Q_y}] \right|. \quad (28)$$

5 Approximated formula

We now perform calculations concerning the simple FODO lattice, which is used in most high intensity proton synchrotrons. Since we intend to obtain a simple formula that is accurate enough for an order estimation, we can accept rather rough approximations.

We use the approximation that

$$A_{ye,kM} \approx A_{yp,kM}, \quad (29)$$

because two functions, $c(1+2c)/(1+c)^2$ and $c(1+3c)/(1+c)^3$, do not differ much in the region $0 < c < 3$. In addition, $A_{xe,kM}$ cannot be much larger than

$A_{ye,kM}$. Therefore, the magnitudes of the scaling factors determine the main contributions to the resonance width.

Although the scaling factors become very large near $2Q_x = kM$ and $2Q_y = kM$, the operating point $2Q_x \approx kM$ or $2Q_y \approx kM$ is not practically used in an actual machine. $T_{yp,n}$ is large when $n \approx 0$ ($Q_y \approx M/4$). In this region since the contribution from phase modulation is much larger than that from envelope modulation, the resonance width is approximated by

$$\Delta e_{ns} \approx \left| \frac{J_y}{2\pi\epsilon_y^2} \frac{r_p \lambda}{12\beta^2 \gamma^3} 8\pi A_{yp,M} T_{y,\pm(N-M)} \frac{\delta Q_y}{2Q_y} \right| \approx |\Delta e_{ss} \frac{8\delta Q_y}{N-M}| \quad \text{where } N \text{ is close, but not equal to, } M. \quad (30)$$

Here, Δe_{ss} is the band width of the nearby superstructure resonance ($4Q_y = M$) induced by the space charge in the absence of a quadrupole error. We have used eq. (29) and the smallness of the dependence of $A_{ye,n}$ on Q_y . This equation says that the quadrupole errors produce side bands of a strong resonance. When there are many quadrupole errors and they are random, eq. (30) can be understood as being a statistical equation. Δe_{ns} is the expected band width, where δQ_y is half of the expected band width of a half-integer resonance.

6 Example

Since the ring lattice of the KEK-PS consists of 28 FODO cells, 28th harmonic resonances are superstructure resonances. The operating point (bare tunes) had been $(Q_x, Q_y) = (7.12, 7.25)$ but was changed to $(7.12, 5.25)$ in 1996 [9]. Before the change the working area in the tune space was limited by three resonances: $Q_x = 7$ (or $4Q_y = 28$), $Q_x - Q_y = 0$ (or $2Q_x - 2Q_y = 0$) and $4Q_y = 29$. One of them, $4Q_y = 29$, was a nonstructure octupole resonance.

The gross of the harmonic quadrupole imperfections and that of the harmonic octupole imperfections were estimated from the experimental data of some quadrupole and octupole resonance corrections. Since the half-integer band width was about 0.04, the expected resonance width of $4Q_y = 29$ ($N - M = 1$) is roughly 0.16 times that of $4Q_y = 28$. On the other hand, a resonance width of $4Q_y = 21$ ($N - M = -7$) is roughly 0.02 times that of $4Q_y = 28$. We thought that this difference was the reason why the beam survival at the KEK-PS was improved by changing the operating point.

The band width of $4Q_y = 29$ was calculated using the following parameters:

$$\begin{aligned}
\text{proton energy (injection)} &= 500 \text{ MeV}, \\
\text{line density } \lambda &= 10^{11} \text{ protons/m}, \\
\text{horizontal emittance } \varepsilon_x \pi &= 12 \pi \text{ mm} \cdot \text{mrad}, \\
\text{vertical emittance } \varepsilon_y \pi = J_y \pi &= 6 \pi \text{ mm} \cdot \text{mrad}.
\end{aligned}$$

The assumed vertical emittance is one fifth of the physical aperture. Under these conditions the number of protons in the ring is 9×10^{12} ppp with a bunching factor of 0.3. Also the Laslett tune shift was calculated to be -0.7 (a rearrangement of the beam shape was not considered). The band width of $4Q_y = 29$ was calculated to be $\Delta e_{ns} = 4 \times 10^{-4}$. On the other hand the band width of the same $4Q_y = 29$ at the lowest beam current, which was thought to be produced by octupole field imperfections, was only $\Delta e_{ns} = 1 \times 10^{-5}$. The nonstructure resonances were produced mainly by the space charge and quadrupole errors, not by octupole errors.

7 Discussions

A space charge induced nonstructure resonance can be stronger than that induced by magnetic octupole imperfections, even when far from a superstructure resonance. The best cure is to reduce the quadrupole imperfections of the ring. A commonly used method to reduce the quadrupole error is to energize the correction quadrupoles appropriately in order to reduce the amplitude modulation of β_x and β_y , however the measurement of the amplitudes is less sensitive to the lower harmonic phase modulations. Although measurements of the transfer matrix gave phase information, they were not analyzed from the view point of the resonance. The other way, applying a magnetic octupole field to cancel it, would work, but is not practical. The field required for a cancellation depends on the line density and beam shape, which are not uniform along the azimuthal direction, and is not constant with time. Another way is to observe the emittance blow up or the beam loss due to this resonance, and to optimize the harmonic correction quadrupoles. At least two correction knobs are required for one resonance, which are diagonal in the N th harmonic phase. The complex number between the absolute signs of eq. (28) gives the driving term of the correction knob. When the phase modulation is dominant, an independent correction of the horizontal and vertical resonances is possible because the driving power to $4Q_y = N$ is roughly proportional to δQ_y , while those for the $4Q_x = N$ is roughly proportional to δQ_x .

The non-periodicity of an octupole mirror-charge field, which we ignored in this report, could be another source of an intensity dependent resonance. The non-periodic quadrupole would introduce a nonlinear dependence of the band width to λ . The mirror-charge tune shift is normally in the range of 10^{-1} .

If 1/10 of the vacuum chamber is special and is not symmetric, the non-periodic mirror-charge field produces a half-integer band width of 10^{-2} , which is considerably large.

In a real machine, especially with a high intensity beam, the behavior of the beam is not simple; even so our formula gives a practical guideline in designing a machine. It is important to reduce the quadrupole imperfections of any harmonics for the good performance of a machine.

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