CM-P00057230

Physical one nucleon and two nucleon
wave functions in fixed source theory.

by Jacques Mandelbrojt ^{*)} ×)
CERN, Geneva, Switzerland

In this paper we bring further developments to a new intermediate coupling technique ¹⁾ we introduced as a modification of Tomonaga's intermediate coupling theory. The main advantage of our technique over the usual intermediate coupling technique lies in the simplicity of the calculations and in the fact that we get explicit workable analytic expressions for the physical one nucleon and two nucleon states, thus avoiding the use of machines and giving more transparent results. Furthermore the approximation with which we approach the exact fixed source wave function can be evaluated in a straightforward manner. It is to be noted that the method we use can, for a number of problems, replace variational calculations.

We intend eventually to use the two nucleon wave function that we obtain to carry out the program outlined by Lévy ³⁾, which comprises two main points:

- To obtain in a straightforward way a nucleon - nucleon potential which comprises terms of all orders of perturbation and should therefore improve in particular the part not due to $\bar{\pi} - \pi$ interaction, of the spin orbit potential which begins to appear in forth order perturbation.

- To calculate in detail the imaginary potential in the interaction nucleon-antinucleon, Lévy's calculation being so far the only theoretical justification for a potential having the range which is implied by experimental results, that is $1,5 \cdot 10^{-13}$ cm. ⁺⁾

*) Present address: Physique Théorique, Faculté des Sciences,
Place Victor Hugo, Marseille, France.

×) Submitted for publication in Nuovo Cimento.

+) Range given in the 1960 Rochester Report on nucleon - anti-nucleon forces by M. Lévy.

I. Fixed source nucleon surrounded by P wave mesons

The hamiltonian representing P wave mesons interacting with a nucleon without recoil can be written as

$$H = \frac{1}{(2\pi)^3} \int \left[\omega(k) a_{i\alpha}^*(k) a_{i\alpha}(k) - \int R(k) \sigma_i \tau_\alpha (a_{i\alpha}^*(k) + a_{i\alpha}(k)) \right] d^3k \quad (1)$$

where $a_{i\alpha}^*(k)$ and $a_{i\alpha}(k)$ are the creation and annihilation operators of mesons of spin and isotopic spin indices i and α and momentum vector k , $\omega(k)$ its total energy, $R(k)$ is the spherically symmetrical source function.

The intermediate coupling theory consists in looking for the best approximation of an eigenstate of H in the form

$$|\Psi_T\rangle = \sum_{p_1, p_2, \dots, p_g} \alpha_{p_1, p_2, \dots, p_g} (a_{i_1}^*)^{p_1} (a_{i_2}^*)^{p_2} \dots (a_{i_g}^*)^{p_g} |\Psi_0\rangle$$

where $|\Psi_0\rangle$ is the state vector of the bare nucleon and $a_{i\alpha}^*$ is an average over momentum of $a_{i\alpha}^*(k)$

$$a_{i\alpha}^* = \frac{1}{(2\pi)^3} \int a_{i\alpha}^*(k) \Phi(k) d^3k$$

similarly we define $a_{i\alpha} = \frac{1}{(2\pi)^3} \int a_{i\alpha}(k) \Phi(k) d^3k$

$\Phi(k)$ is a normalized function of k (we suppose it to be a real function)

It is then easy to see that $\langle \Psi_T^* | H | \Psi_T \rangle = \langle \Psi_T^* | H_T | \Psi_T \rangle$

with $H_T = \omega_0 a_{i\alpha}^* a_{i\alpha} - \int_0 \sigma_i \tau_\alpha (a_{i\alpha}^* + a_{i\alpha})$

where $\omega_0 = \frac{1}{(2\pi)^3} \int \omega(k) \Phi^2(k) d^3k$

$$\int_0 = \frac{1}{(2\pi)^3} \int R(k) \Phi(k) d^3k$$

We look for the function $\bar{\Phi}(\underline{k})$ and the coefficients $\alpha_1, \dots, \alpha_5$ to which correspond the vector $|\Psi_T\rangle$ nearest to the ground state of the nucleon. To do this we can avoid both the problem of finding the eigenstate of H_T and the variational calculation of $\bar{\Phi}(\underline{k})$ (which are usual in this method) by simply writing the condition under which $|\Psi_T\rangle$ which we suppose to be an eigenstate of H_T , shall be eigenstate of H .

To do this we write that

$$\sigma^2(H, \Psi_T) = \langle \Psi_T^* H^2 \Psi_T \rangle - \langle \Psi_T^* H \Psi_T \rangle^2 = 0$$

this can be identically written in the form

$$\begin{aligned} \sigma^2(H, \Psi_T) &= \frac{1}{(2\pi)^3} \int \langle \Psi_T^* (\omega(\underline{k}) a_{i\alpha}^* \bar{\Phi}(\underline{k}) - \sigma_i \tau_\alpha \int R(\underline{k})) (\omega(\underline{k}) a_{i\alpha} \bar{\Phi}(\underline{k}) - \sigma_i \tau_\alpha \int R(\underline{k}) \Psi_T) d\underline{k} \\ &\quad - \frac{1}{(2\pi)^6} \langle \Psi_T^* \int (\omega(\underline{k}) a_{i\alpha}^* \bar{\Phi}^2(\underline{k}) - \sigma_i \tau_\alpha \int R(\underline{k}) \bar{\Phi}(\underline{k})) d\underline{k} \int (\omega(\underline{k}) a_{i\alpha} \bar{\Phi}^2(\underline{k}) - \sigma_i \tau_\alpha \int R(\underline{k}) \bar{\Phi}(\underline{k})) d\underline{k} \Psi_T \rangle \end{aligned}$$

It is then easily seen that $\sigma^2(H, \Psi_T) = 0$ implies

$$(\omega(\underline{k}) a_{i\alpha} \bar{\Phi}(\underline{k}) - \sigma_i \tau_\alpha \int R(\underline{k})) \Psi_T = A \bar{\Phi}(\underline{k})$$

where A is a vector independent of $\underline{k}$. Taking the scalar product of both sides of this equation by $\langle \Psi_T^* a_{i\alpha}^*$ then by $\langle \Psi_T^* \sigma_i \tau_\alpha$ we get for $\bar{\Phi}(\underline{k})$ two expressions

$$\bar{\Phi}(\underline{k}) = \frac{\int M R(\underline{k})}{N \omega(\underline{k}) - \lambda} \quad \text{and} \quad \bar{\Phi}(\underline{k}) = \frac{g \int R(\underline{k})}{M \omega(\underline{k}) - \lambda'}$$

where $M = \langle \Psi_T^* \sigma_i \tau_\alpha a_{i\alpha} \Psi_T \rangle$, $N = \langle \Psi_T^* a_{i\alpha}^* a_{i\alpha} \Psi_T \rangle$, λ and λ' are two constants; equating these two expressions of $\bar{\Phi}(\underline{k})$ we get

$$gN = M^2$$

that is

$$\langle \Psi_T^* \sigma_i \tau_\alpha \sigma_i \tau_\alpha \Psi_T \rangle \langle \Psi_T^* a_{i\alpha}^* a_{i\alpha} \Psi_T \rangle = \langle \Psi_T^* \sigma_i \tau_\alpha a_{i\alpha} \Psi_T \rangle^2$$

which implies

$$a_{i\alpha} \Psi_T \rangle = K \sigma_i \sigma_{i\alpha} \Psi_T \rangle \quad (2)$$

where K is a constant. It is easily seen that the value of K which gives the minimum energy for $\langle \Psi_T^* H \Psi_T \rangle$ is $K = \frac{\hbar \omega_0}{2}$.

From equation (2) we can obtain the two equations

$$a_{i\alpha}^* a_{i\alpha} \Psi_T \rangle = \frac{\hbar \omega_0}{2} \sigma_i \sigma_{i\alpha} a_{i\alpha}^* \Psi_T \rangle \quad (3)$$

$$\sigma_i \sigma_{i\alpha} a_{i\alpha} \Psi_T \rangle = g \Psi_T \rangle \quad (g \text{ being a constant}) \quad (4)$$

It is easily checked that, if these two equations are satisfied simultaneously, $\Psi_T \rangle$ will be an eigenstate of H with the energy $E = -\frac{\hbar \omega_0}{2} g$. These two equations are in fact satisfied together in the weak and in the strong coupling limit,¹⁾ in the weak coupling limit $g = g \frac{\hbar \omega_0}{2}$ as would be expected from equation (2), in the strong coupling limit $g = \frac{\hbar \omega_0}{2}$.

We can now proceed in either of two ways to get from these equations an approximation for the ground state of H. Either solve the first equation and see with what approximation the second equation is satisfied, or look for the linear combination of solutions of the first and second equation which minimises H.

First solution : Solving the first equation gives

$$\Psi_T \rangle = B e^{\frac{\hbar \omega_0}{2} \sum_i \sigma_i \alpha_i a_{i\alpha}^*} \Psi_0 \rangle \quad (5)$$

where B is a normalisation constant defined by $\langle \Psi_T^* \Psi_T \rangle = 1$. For practical calculations of mean values of operators this form is

1) For a detailed discussion and for the approximation with which $\Psi_T \rangle$ is an eigenvector of H see J. Mandelbrojt, Nuovo Cimento Serie X Vol. 14, page 625 (1959)

not convenient because of the complications of the commutation relations of high powers of $\sigma_i \tau_\alpha$. We therefore give a reduced form to Ψ_T following a method given by Friedman, Lee and Christian.²⁾

The ground state of the nucleon being like the bare nucleon of spin and isotopic spin $\frac{1}{2}$, it is obtained from the bare nucleon by application of an operator constructed with the operators σ_i , τ_α and $a_{i\alpha}^*$ which must be a scalar in space and in isospace. F.L.C. have proved that the most general form of such an operator is

$$f_0(s_1, s_2, s_3) + f_1(s_1, s_2, s_3) \sigma_i \tau_\alpha a_{i\alpha}^* + f_2(s_1, s_2, s_3) \sigma_i \tau_\alpha \epsilon_{ijk} \epsilon_{\alpha\beta\gamma} a_{j\beta}^* a_{k\gamma}^* + \\ + f_3(s_1, s_2, s_3) \sigma_i \tau_\alpha a_{j\beta}^* a_{i\beta}^* a_{j\alpha}^*$$

where

$$s_1 = a_{i\alpha}^* a_{i\alpha}^* \\ s_2 = \epsilon_{ijk} \epsilon_{\alpha\beta\gamma} a_{i\alpha}^* a_{j\beta}^* a_{k\gamma}^* \\ s_3 = a_{i\alpha}^* a_{j\beta}^* a_{i\beta}^* a_{j\alpha}^*$$

f_0, f_1, f_2, f_3 being scalar functions.

Furthermore it has been noticed²⁾ that variables in the $a_{i\alpha}^*$ space can be introduced for which the wave function Ψ_T has cubic symmetry, and approximating this cubic symmetry by a spherical symmetry is equivalent to considering that f_0, f_1, f_2, f_3 are functions of the only variables.

To represent $\Psi_T = B e^{\frac{i}{\hbar} \sigma_i \tau_\alpha a_{i\alpha}^*} \Psi_0$ in the reduced form (6)

$$\Psi_{TR} = B \left\{ f_0(s_1) + f_1(s_1) \sigma_i \tau_\alpha a_{i\alpha}^* + \right. \\ \left. + f_2(s_1) \sigma_i \tau_\alpha \epsilon_{ijk} \epsilon_{\alpha\beta\gamma} a_{j\beta}^* a_{k\gamma}^* + f_3(s_1) \sigma_i \tau_\alpha a_{j\beta}^* a_{i\beta}^* a_{j\alpha}^* \right\} \Psi_0 \quad (6)$$

2) M.H. Friedman, T.D. Lee and R. Christian, Phys. Rev. 100, 1494 (1955)

3) M. Lévy, Nuovo Cimento, Vol. VIII no. 1, p. 92 (1958)

where we will write:

$$f_0(\phi_i) = \sum u_n \phi_i^n, \quad f_1(\phi_i) = \sum v_n \phi_i^n, \quad f_2(\phi_i) = \sum w_n \phi_i^n, \quad f_3(\phi_i) = \sum c_n \phi_i^n$$

or, what is equivalent, to look for the best approximation of the form (6) to a solution of $a_{i\alpha}^* a_{i\alpha} \Psi_T \rangle = \frac{f_0}{\omega_0} \sigma_i \epsilon_{\alpha} a_{i\alpha}^* \Psi_T \rangle$, we determine the coefficients u_n, v_n, w_n, c_n by the condition that

$$\left| \left(a_{i\alpha}^* a_{i\alpha} - \frac{f_0}{\omega_0} \sigma_i \epsilon_{\alpha} a_{i\alpha}^* \right) \left(f_0(\phi_i) + f_1(\phi_i) \sigma_i \epsilon_{\alpha} a_{i\alpha}^* + \right. \right. \\ \left. \left. + f_2(\phi_i) \sigma_i \epsilon_{\alpha} \sum_{j\beta} \epsilon_{j\beta} \epsilon_{\alpha\beta} + f_3(\phi_i) \sigma_i \epsilon_{\alpha} a_{j\beta}^* a_{i\beta}^* a_{j\alpha}^* \right) \Psi_0 \right|$$

is

minimum. Using the value given in appendix I of $\sigma_i \epsilon_{\alpha} a_{i\alpha}^*$ applied to each of the terms of $\Psi_{TR} \rangle$ and the obvious value of $a_{i\alpha}^* a_{i\alpha}$ applied to each of the terms of $\Psi_{TR} \rangle$ we get

$$u_n = \frac{3^n \left(\frac{f_0}{\omega_0} \right)^{2n-1}}{(2n)!}$$

$$v_n = \frac{3^{n+1} \left(\frac{f_0}{\omega_0} \right)^{2n}}{(2n+1)!}$$

$$w_n = \frac{-3^{n+1} \left(\frac{f_0}{\omega_0} \right)^{2n+1}}{(2n+2)!}$$

$$c_n = \frac{-3^{n+1} \left(\frac{f_0}{\omega_0} \right)^{2n+2}}{(2n+3)!}$$

The ground state expectation value of an operator can now be obtained explicitly using the relations given in appendix III and the values of u_n, v_n, w_n, c_n . It is essential to note that

$\langle \Psi_{TR}^* \Psi_{TR} \rangle$, $\langle \Psi_{TR}^* a_{i\alpha}^* a_{i\alpha} \Psi_{TR} \rangle$ and $\langle \Psi_{TR}^* \sigma_i \epsilon_{\alpha} a_{i\alpha} \Psi_{TR} \rangle$ can by this calculation be given in analytically closed form for instance

$$\begin{aligned}
 \langle \Psi_{TR}^* \Psi_{TR} \rangle &= \frac{2B^2}{5.6.7} \left[\text{ch } z (z^4 + 2.6z^3 + 14.1z^2 + 56.1z + 105) + \right. \\
 &\quad \left. + \text{sh } z (z^4 + 2.2z^3 + 20.7z^2 + 27.9z + 384) \right] \\
 \langle \Psi_{TR}^* \sigma_1 \sigma_2 a_{12} \Psi_{TR} \rangle &= \frac{2B^2}{5.6.7} \frac{k_0}{\omega_0} \left[\text{sh } z (6.7z^3 + 9.75z) + \right. \\
 &\quad \left. + \text{ch } z (z^4 + 2.85z^2 + 9.45) \right] + \frac{2B}{5.6.7} \frac{1}{\omega_0} \left[\frac{\text{sh } z (z^4 + 20.7z^2 + 384) + \text{ch } z (2.6z^3 + 56.1z)}{z^3} \right] \\
 &\quad \text{with } z = \left(\frac{k_0}{\omega_0} \right)^2
 \end{aligned}$$

It is important to note that, as we mentioned in our first paper, $\sigma(H, \Psi_{TR})$ gives from Weinstein's inequality an upper bound for the error made in replacing the ground state energy by $\langle \Psi_{TR}^* H \Psi_{TR} \rangle^{(1)}$.

Second solution : We can improve the preceding wave function by

calculating the solution of the form Ψ'_{TR} of equation (4) in a similar way to the way we calculated Ψ_{TR} , using this time the expressions given in appendix II, and looking for the linear combination $\Psi_{TR} + C \Psi'_{TR}$ which minimizes $\frac{\langle \Psi_{TR}^* + C \Psi_{TR}'^* | H | \Psi_{TR} + C \Psi_{TR}' \rangle}{\langle \Psi_{TR}^* + C \Psi_{TR}'^* | \Psi_{TR} + C \Psi_{TR}' \rangle}$

the constant C can be calculated explicitly. Let us note that this is a way of calculating the constant λ which appears in the expression of $\Phi(\underline{k})$, the value of λ is related in a simple way to C, in particular C = 0 gives $\lambda = 0$.

II. Two nucleon wave function.

We shall relate the calculation of the two nucleon wave function to that of the clothed nucleon wave function we calculated in the previous paragraph.

The fixed source hamiltonian of the nucleon nucleon interaction is

$$H = \frac{1}{(2\pi)^3} \int \left\{ \omega(k) c_\alpha^*(k) c_\alpha(k) + \frac{k_j}{P} \sum_{j=1}^2 \frac{1}{(2\pi)^3} \int R(k) c_\alpha^{(j)} \left[i(\sigma^{(j)} \cdot \underline{k}) c_\alpha(k) e^{i \underline{k} \cdot \underline{x}_j} - i(\sigma^{(j)} \cdot \underline{k}) c_\alpha^*(k) e^{-i \underline{k} \cdot \underline{x}_j} \right] d\underline{k} \right\}$$

where $c_\alpha^*(k)$ and $c_\alpha(k)$ are the creation and annihilation operators of the meson field, α is the charge indice.⁺ We now separate the P-wave part of H around each nucleon and derive the intermediate coupling hamiltonian by defining the two sets of commuting creation operators independent of $\underline{k}$ but dependent on the coordinates x_1 and x_2 of the two nucleons; (We take $\underline{x}_1 = -\underline{x}_2 = \frac{\underline{z}}{2}$)

$$\bar{b}_{\alpha i}^+(z) = \frac{i\sqrt{6}}{(2\pi)^3} \int \Phi_1^+(k, z) Q_{ij}^+(k, z) \frac{k_j}{k} \cos \frac{\underline{k} \cdot \underline{z}}{2} c_\alpha^*(k) d\underline{k}$$

$$\bar{b}_{\alpha i}^-(z) = \frac{i\sqrt{6}}{(2\pi)^3} \int \Phi_2^-(k, z) Q_{ij}^-(k, z) \frac{k_j}{k} \sin \frac{\underline{k} \cdot \underline{z}}{2} c_\alpha^*(k) d\underline{k}$$

and similar definitions for the annihilation operators $\bar{b}_{\alpha i}^+(z)$ and $\bar{b}_{\alpha i}^-(z)$. The weight functions $\Phi_1^\pm(k, z)$ are supposed to depend only on $|\underline{z}|$. The operators $Q_{ij}^\pm$ are introduced to ensure that the operators $\bar{b}_{\alpha i}^\pm$ and $\bar{b}_{\alpha j}^\pm$ commute if $i \neq j$, they satisfy the property

$$\frac{3}{4\pi} Q_{ij}^\pm Q_{kl}^\pm \int \frac{k_j k_l}{k^2} (1 \pm \cos \frac{\underline{k} \cdot \underline{z}}{2}) d\Omega_k = \frac{\delta_{ij} \delta_{kl}}{4\pi} \int (1 \pm \cos \frac{\underline{k} \cdot \underline{z}}{2}) d\Omega_k$$

where $\int d\Omega_k$ is an integration over the angles of $\underline{k}$ only.

+) For notations and details of calculations of the intermediate coupling hamiltonien see Lévy's article ³⁾ We have changed Lévy's notation for the wave function which we call $|\Psi\rangle$ instead of $|\Phi\rangle$.

If we then want to approximate an eigenstate of H by a state containing only the application of a function of the operators $\bar{b}_{\alpha i}^{\pm*}$, on the product of the two body wave function by the meson vacuum state vector, we have

$$\langle \Psi_T^* | H | \Psi_T \rangle = \langle \Psi_T^* | H_T | \Psi_T \rangle \quad \text{with}$$

$$H_T = H_T^+ + H_T^-$$

with

$$H_T^+ = \omega_+(z) \bar{b}_{\alpha i}^{\pm*} b_{\alpha i}^+ + \sqrt{2} f_{ij}^+ (\sigma_i' \tau_{\alpha}^1 + \sigma_i^2 \tau_{\alpha}^2) (b_{\alpha j}^{\pm*} + \bar{b}_{\alpha j}^+)$$

$$H_T^- = \omega_-(z) \bar{b}_{\alpha i}^{\pm*} \bar{b}_{\alpha i}^- + i\sqrt{2} f_{ij}^- (\sigma_i' \tau_{\alpha}^1 - \sigma_i^2 \tau_{\alpha}^2) (\bar{b}_{\alpha j}^- - b_{\alpha j}^{\pm*})$$

$$\text{where } \omega_{\pm}(z) = \frac{1}{(2\pi)^3} \int \omega(k) \Phi_{\pm}^{\pm}(k, z) (1 \pm \cos k z) dk$$

$$f_{ij}^{\pm} = \frac{1}{\sqrt{3}} \frac{f_-}{(4\pi)^3} \int \Phi_{\pm}^{\pm}(k, z) k (1 + \cos k z) R(k) \bar{Q}_{ij}^{\pm} dk$$

$$\text{where } \bar{Q}_{ij}^{\pm} \text{ are the transposed matrices of } Q_{ij}^{\pm}.$$

Since $\bar{b}_{\alpha i}^{\pm}$ and $b_{\alpha i}^{\pm}$ commute, the equation

$$H_T(z) |\Psi_T(z)\rangle = \mathcal{E}(z) |\Psi_T(z)\rangle \text{ splits into two equations:}$$

$$H_T^{\pm}(z) |\Psi_T^{\pm}(z)\rangle = \mathcal{E}_{\pm}(z) |\Psi_T^{\pm}(z)\rangle \quad \text{with} \quad \mathcal{E} = \mathcal{E}_+ + \mathcal{E}_-$$

Solution of $H_T^+(z) |\Psi_T^+(z)\rangle = \mathcal{E}_+ |\Psi_T^+(z)\rangle$:

By the same calculation as in the first part we can replace this eigenvector equation by the equation

$$\bar{b}_{\alpha i}^+ |\Psi_T^+\rangle = \sqrt{2} \frac{f_{ij}^+}{\omega_+(z)} (\sigma_i' \tau_{\alpha}^1 + \sigma_i^2 \tau_{\alpha}^2) |\Psi_T^+\rangle$$

Using our first method of calculation we see that we can write a solution of this in the form (given by Lévy)

$$|\Psi_T^+\rangle = B^+ e^{i\sqrt{2} \frac{f_{ij}^+}{\omega_+(z)} (\sigma_i' \tau_{\alpha}^1 + \sigma_i^2 \tau_{\alpha}^2) \bar{b}_{j\alpha}^{\pm*}} |\Psi_0\rangle$$

the solution Ψ_T^- will be

$$\Psi_T^- = B^- e^{\frac{i\sqrt{2}}{\omega-(2)} \frac{f_0^-}{\omega-(2)} (\sigma_i^2 \tau_{\alpha}^2 - \sigma_i^2 \tau_{\alpha}^2) \bar{b}_{j\alpha}^{+*}} \Psi_0$$

We can write both these functions in the reduced form. Writing the part of Ψ_{TR}^+ which contributes to the central potential we have

$$\begin{aligned} \Psi_{TR}^+ = B^+ & \left(\sum u_n (\bar{b}_{i\alpha}^{+*})^n + \sum v_n (\bar{b}_{i\alpha}^{+*})^n \sigma_i^2 \tau_{\alpha}^2 \bar{b}_{i\alpha}^{+*} + \sum w_n (\bar{b}_{i\alpha}^{+*})^n \sigma_i^2 \tau_{\alpha}^2 \varepsilon_{ijk} \varepsilon_{lps} \bar{b}_{j\beta}^{+*} \bar{b}_{l\gamma}^{+*} \right. \\ & + \sum c_n (\bar{b}_{i\alpha}^{+*})^n \sigma_i^2 \tau_{\alpha}^2 \bar{b}_{j\beta}^{+*} \bar{b}_{i\beta}^{+*} \bar{b}_{j\alpha}^{+*} \left(\sum u_n (\bar{b}_{i\alpha}^{+*})^n + \sum v_n (\bar{b}_{i\alpha}^{+*})^n \sigma_i^2 \tau_{\alpha}^2 \bar{b}_{i\alpha}^{+*} + \right. \\ & \left. \left. + \sum w_n (\bar{b}_{i\alpha}^{+*})^n \sigma_i^2 \tau_{\alpha}^2 \varepsilon_{ijk} \varepsilon_{lps} \bar{b}_{j\beta}^{+*} \bar{b}_{l\gamma}^{+*} + \sum c_n (\bar{b}_{i\alpha}^{+*})^n \sigma_i^2 \tau_{\alpha}^2 \bar{b}_{j\beta}^{+*} \bar{b}_{i\beta}^{+*} \bar{b}_{j\alpha}^{+*} \right) \Psi_0 \right) \end{aligned}$$

where u_n, v_n, w_n, c_n have the values previously calculated values

where $\frac{f_0}{\omega_0}$ is replaced by $\frac{\sqrt{2} f^+}{\omega_+(2)}$

Ψ_{TR}^- will have a similar expression with $\frac{f_0}{\omega_0}$ replaced $-\frac{i\sqrt{2} f^-}{\omega_-(2)}$

Calculations using these expressions to carry out the program outlined in the introduction are under way.

Acknowledgements

We wish to thank Professor Lévy for his interest in this work and Dr. André Martin for enjoyable discussions. We are also grateful to Professor Fubini for having read this paper and for useful criticism.

APPENDIX I

$$\sigma_i \tau_\alpha a_{i\alpha}^* (a_{i\alpha}^{*2})^n = \sigma_i \tau_\alpha a_{i\alpha}^* (a_{i\alpha}^{*2})^n$$

$$\sigma_j \tau_\beta a_{j\beta}^* \sigma_i \tau_\alpha a_{i\alpha}^* (a_{i\alpha}^{*2})^n = (a_{i\alpha}^{*2})^{n+1} - \sigma_k \tau_\gamma \varepsilon_{ijk} \varepsilon_{\alpha\beta\gamma} a_{i\alpha}^* a_{j\beta}^* (a_{i\alpha}^{*2})^n$$

$$\begin{aligned} \sigma_i \tau_\alpha a_{i\alpha}^* \sigma_i \tau_\alpha \varepsilon_{ijk} \varepsilon_{\alpha\beta\gamma} a_{j\beta}^* a_{k\gamma}^* (a_{i\alpha}^{*2})^n &= a_{i\alpha}^* a_{j\beta}^* a_{k\gamma}^* \varepsilon_{ijk} \varepsilon_{\alpha\beta\gamma} (a_{i\alpha}^{*2})^n - \\ &- 2 \sigma_k \tau_\gamma a_{k\gamma}^* (a_{i\alpha}^{*2})^{n+1} + 2 \sigma_k \tau_\beta a_{j\beta}^* a_{k\gamma}^* a_{j\gamma}^* (a_{i\alpha}^{*2})^n \end{aligned}$$

$$\begin{aligned} \sigma_i \tau_\alpha a_{i\alpha}^* \sigma_i \tau_\alpha a_{j\beta}^* a_{i\beta}^* a_{j\alpha}^* (a_{i\alpha}^{*2})^n &= a_{i\alpha}^* a_{j\beta}^* a_{i\beta}^* a_{j\alpha}^* (a_{i\alpha}^{*2})^n - \\ &- \sigma_k \tau_\gamma \varepsilon_{ijk} \varepsilon_{\alpha\beta\gamma} a_{i\alpha}^* a_{j\beta}^* a_{i\beta}^* a_{j\alpha}^* (a_{i\alpha}^{*2})^n \end{aligned}$$

APPENDIX II

$$\sigma_i \tau_\alpha a_{i\alpha} (a_{i\alpha}^{*2})^n \Psi_0 = 2n \sigma_i \tau_\alpha (a_{i\alpha}^{*2})^{n-1} \Psi_0$$

$$\begin{aligned} \sigma_i \tau_\alpha a_{i\alpha} \sigma_i \tau_\alpha a_{i\alpha}^* (a_{i\alpha}^{*2})^n \Psi_0 &= \left\{ (g+2n) (a_{i\alpha}^{*2})^n - \right. \\ &- 2n \sigma_k \tau_\beta \varepsilon_{\alpha\beta\gamma} \varepsilon_{ijk} a_{i\alpha}^* a_{j\beta}^* (a_{i\alpha}^{*2})^n \left. \right\} \Psi_0 \end{aligned}$$

$$\begin{aligned} \sigma_i \tau_\alpha a_{i\alpha} \sigma_i \tau_\alpha \varepsilon_{ijk} \varepsilon_{\alpha\beta\gamma} a_{j\beta}^* a_{k\gamma}^* (a_{i\alpha}^{*2})^n \Psi_0 &= \left\{ 2n \varepsilon_{ijk} \varepsilon_{\alpha\beta\gamma} a_{j\beta}^* a_{k\gamma}^* a_{i\alpha}^* (a_{i\alpha}^{*2})^{n-1} + \right. \\ &+ 4n \sigma_k \tau_\beta a_{j\beta}^* a_{k\gamma}^* a_{j\gamma}^* (a_{i\alpha}^{*2})^{n-1} - (4n+8) \sigma_k \tau_\gamma a_{k\gamma}^* (a_{i\alpha}^{*2})^n \left. \right\} \Psi_0 \end{aligned}$$

$$\begin{aligned} \sigma_i \tau_\alpha a_{i\alpha} \sigma_i \tau_\alpha a_{j\beta}^* a_{i\beta}^* a_{j\alpha}^* (a_{i\alpha}^{*2})^n \Psi_0 &= \left\{ 2n a_{j\beta}^* a_{i\beta}^* a_{j\alpha}^* a_{i\alpha}^* (a_{i\alpha}^{*2})^{n-1} - \right. \\ &- 2n \sigma_k \tau_\beta \varepsilon_{ijk} \varepsilon_{\alpha\beta\gamma} a_{j\beta}^* a_{i\beta}^* a_{j\alpha}^* a_{i\alpha}^* (a_{i\alpha}^{*2})^{n-1} + 7 (a_{i\alpha}^{*2})^{n+1} - \\ &- \sigma_k \tau_\beta \varepsilon_{ijk} \varepsilon_{\alpha\beta\gamma} a_{i\alpha}^* a_{j\beta}^* (a_{i\alpha}^{*2})^n \left. \right\} \Psi_0 \end{aligned}$$

APPENDIX III

The following expectation values obtained by comparison of the expressions given in appendix I and II

$$A_n = \langle (a_{i\alpha}^\dagger)^n (a_{i\alpha}^\dagger)^n \rangle = \frac{(2n+7)!}{4 \cdot 5 \cdot 6 \cdot 7 (n+1)(n+2)(n+3)}$$

$$B_n = \langle ()^n \sigma_i \tau_\alpha a_{i\alpha} \sigma_i \tau_\alpha a_{i\alpha}^\dagger ()^n \rangle = (2n+3) A_n$$

where $()^n$ stands for $(a_{i\alpha}^\dagger)^n$ and $\langle ()^n$ stands for $\langle (a_{i\alpha}^\dagger)^n$

$$C_n = \langle ()^{n+1} \sigma_i \tau_\alpha a_{i\alpha} \sigma_i \tau_\alpha a_{j\beta}^\dagger a_{i\beta}^\dagger a_{j\alpha}^\dagger ()^n \rangle = 4n(n+3) C_{n-1} + 7 A_{n+1}$$

$$D_n = \langle ()^n \sigma_i \tau_\alpha a_{i\alpha} \sigma_i \tau_\alpha a_{i\alpha} \sigma_i \tau_\alpha \sum_{j\beta} \sum_{k\gamma} a_{j\beta}^\dagger a_{k\gamma}^\dagger ()^n \rangle = 4n C_{n-1} - (4n+8) B_n$$

$$E_n = \langle ()^n \sigma_i \tau_\alpha \sum_{j\beta} \sum_{k\gamma} \sum_{\alpha'\beta'\gamma'} a_{j\beta'} a_{k'\gamma'} \sigma_i \tau_\alpha a_{i\alpha} \sigma_i \tau_\alpha a_{j\beta}^\dagger a_{i\beta}^\dagger a_{j\alpha}^\dagger ()^n \rangle =$$

$$= 4n^2 E_{n-1} - 16 C_{n-1} + D_n$$

$$F_n = \langle ()^{n+1} \sigma_i \tau_\alpha a_{j\alpha} a_{i\beta} a_{j\beta} \sigma_i \tau_\alpha a_{i\alpha} \sigma_i \tau_\alpha \sum_{j\beta} \sum_{k\gamma} a_{j\beta}^\dagger a_{k\gamma}^\dagger ()^n \rangle =$$

$$= \frac{1}{2n} E_n - \frac{1}{2n} D_n$$

$$H_n = \langle ()^n \sigma_i \tau_\alpha \sum_{j\beta} \sum_{k\gamma} \sum_{\alpha'\beta'\gamma'} a_{j\beta'} a_{k'\gamma'} \sigma_i \tau_\alpha a_{i\alpha} \sigma_i \tau_\alpha a_{i\alpha}^\dagger ()^{n+1} \rangle = 2(n+1) D_n$$

$$L_n = \langle ()^n \sigma_i \tau_\alpha a_{j\beta} a_{i\beta} a_{j\alpha} \sigma_i \tau_\alpha a_{i\alpha} ()^{n+2} \rangle = \frac{1}{2(n+1)} C_{n+1} - \frac{7}{2(n+1)} A_{n+2}$$

$$M_n = \langle ()^n \sigma_i \tau_\alpha \sum_{j\beta} \sum_{k\gamma} \sum_{\alpha'\beta'\gamma'} a_{j\beta'} a_{k'\gamma'} \sigma_i \tau_\alpha \sum_{j\beta} \sum_{k\gamma} a_{j\beta}^\dagger a_{k\gamma}^\dagger ()^n \rangle = -D_n$$

$$N_n = \langle ()^n \sigma_i \tau_\alpha a_{j\beta} a_{i\beta} a_{j\alpha} \sigma_i \tau_\alpha a_{j\beta}^\dagger a_{i\beta}^\dagger a_{j\alpha}^\dagger ()^n \rangle = \frac{1}{2} E_n + C_n$$