CM-P00057229

Analytic properties of deuteron photodisintegration
matrix element for fixed energy *)

A. Martin

CERN - Geneva

R E S U M E

On étudie les propriétés analytiques de l'élément de matrice non-relativiste de photodésintégration du deutéron pour une énergie fixée, en fonction de l'angle. Le proton et le neutron interagissent par un potentiel statique qui est une superposition de potentiels de Yukawa. Les complications dues aux spins sont négligées.

Le résultat obtenu coïncide avec les prédictions d'une représentation de Mandelstam proposée par De Alfaro et Rossetti à partir des premiers termes du développement de cet élément de matrice, calculé dans une théorie des champs où le deutéron est traité comme une particule élémentaire couplée au neutron et au proton.

L'intérêt de cette étude réside dans le fait que d'une part l'une des particules initiales est un état lié et que d'autre part dans la représentation de Mandelstam proposée, les points de départ des coupures sont donnés par des seuils anormaux du type de ceux étudiés par Karplus, Sommerfield et Wichmann.

Un résultat analogue, concernant l'analyticité des amplitudes correspondant à une onde partielle donnée en fonction de l'énergie a déjà été obtenu par De Alfaro et Rossetti.

*) Submitted for publication in Nuovo Cimento.

I. Introduction.

De Alfaro and Rossetti ¹⁾ have recently studied, in the non-relativistic case, the analytic properties of the partial wave amplitudes for Deuteron photodisintegration, under the assumption ~~that~~ the n-p static potential has the form

$$V(r) = \int_0^{\infty} g(\sigma) e^{-\sigma r} d\sigma \quad (1)$$

On the other hand they study a field theoretical model in which the Deuteron is treated as an elementary particle coupled to the neutron and the proton; from the five first perturbation graphs they obtain, using the results of Karplus, Sommerfield and Wichmann ²⁾ a Mandelstam representation with anomalous thresholds. It happens ~~that~~ the predictions of the static model coincide, in the non-relativistic limit, with those obtained by projecting on a partial wave the Mandelstam representation obtained from field theoretical arguments. We think it is worthwhile to study more thoroughly the non-relativistic matrix element for two reasons :

- i) one of the incoming particles is a bound state and no such case has been investigated up to now.
- ii) anomalous thresholds are present and the proof of the Mandelstam representation to all orders in perturbation theory by Eden, Polkinghorne et al. ³⁾ does not include this case.

The present paper deals with analyticity with respect to the angle for fixed energy ^{*)}. Our method makes use of two results :

- i) the existence of a spectral representation for the deuteron wave function, which was first introduced by the author ⁴⁾ and independently by Blanckenbecker ⁵⁾, under the assumption of a n-p potential of the type (1).
- ii) The analytic properties of the T matrix for neutron proton scattering with respect to the angle, off the energy shell, for fixed lengths of the initial and final momenta. These properties are very similar to those obtained on the energy shell by various authors ^{6),7),8)}, but though they are probably known by all the people working in this field, they have not yet been stated. They are derived in the Appendix.

Our result coincides with the predictions based on the Mandelstam representation obtained by De Alfaro and Rossetti from low order perturbation arguments.

II. The matrix element

We make the same simplifying assumptions as De Alfaro and Rossetti : the nucleons and the photon have no spin, the deuteron is in an S state and the n - p potential is a superposition of Yukawa potentials; such a superposition may be written in the form (1) (the reciprocal is not true). Then the photodisintegration matrix element reads :

$$M(k, q, \cos\theta) = \int \psi_{\underline{q}}^* (\underline{r}) e^{i \frac{\underline{k} \cdot \underline{r}}{2}} \psi_D(\underline{r}) d^3 \underline{r} \quad (2)$$

in coordinate space, or

$$M(k, q, \cos\theta) = \int \tilde{\psi}_{\underline{q}}^* (\underline{p}) \tilde{\psi}_D (\underline{p} - \frac{\underline{k}}{2}) d^3 \underline{p} \quad (3)$$

in momentum space (2π factors are omitted), where $\underline{k}$ is the momentum of the incident photon, $\underline{q}$ is the c.m. momentum of each of the final nucleons, ψ_D the deuteron wave function and $\psi_{\underline{q}}$ the wave function of the final state.

We now wish to put in evidence the angular properties of the two quantities entering in the matrix element. It has been shown in ¹⁾ and ⁴⁾ that if the n - p potential has the form (1) the deuteron wave function may be written

$$\psi_D(\underline{r}) = \frac{1}{N} \frac{e^{-\alpha r}}{r} \left[1 + \int_{\mu}^{\infty} \rho(\sigma) e^{-\sigma r} dr \right] \equiv \int_0^{\infty} \rho'(\sigma) \frac{e^{-\sigma r}}{r} d\sigma \quad (4)$$

with

$$\rho'(\sigma) = \frac{1}{N} \delta(\alpha - \sigma) + \frac{1}{N} \rho(\sigma - \alpha) \theta(\sigma - \alpha - \mu)$$

and where $\alpha = M|\epsilon|$, ϵ , binding energy of the deuteron and M , mass of the nucleon, so that

$$\tilde{\Psi}_D(\underline{p}, \underline{k}) = \frac{1}{pk} \int_0^\infty \frac{\rho'(\sigma) d\sigma}{\frac{p^2 + k^2/4 + \sigma^2}{pk} - \cos(pk)} \quad (5)$$

On the other hand, the wave function of the final state in momentum space may be written.

$$\tilde{\Psi}_q(\underline{p}) = \delta(\underline{p}-\underline{q}) + \frac{1}{p^2 - q^2 - i\epsilon} T(\underline{p}, \underline{q}, \underline{q}) \quad (6)$$

$T(\underline{p}, \underline{q}, \underline{q})$ is the off energy shell scattering matrix given by

$$T(\underline{p}, \underline{q}, \underline{r}) \equiv T(\underline{p}, \underline{q}, \underline{r}, \cos\theta) = \langle \underline{p} | V | \underline{q} \rangle + \int d^3 p' \langle \underline{p} | V | \underline{p}' \rangle \frac{1}{p'^2 - r^2 - i\epsilon} T(\underline{p}', \underline{q}, \underline{r}) \quad (7)$$

The analytic properties of $T(\underline{p}, \underline{q}, \underline{q}, \cos\theta)$ with respect to $\cos\theta$, for fixed real p and q are derived in the Appendix under the assumption that V is a superposition of Yukawa potentials :

$$\langle \underline{p} | V | \underline{p}' \rangle = \int_0^\infty \frac{w(\sigma) d\sigma}{(\underline{p}-\underline{p}')^2 + \sigma^2} \quad (8)$$

Let us repeat that family (8) is contained in family (1).

The properties are the following :

- T is a holomorphic function of $\cos\theta$ except for a cut on the real axis, from

$$\cos\theta = \frac{p^2 + p^2 + q^2}{2pq} \quad \text{to} \quad \cos\theta = +\infty$$

- the same properties hold for the function which, for physical values of $\cos\theta$, is $\text{Im } T$.

- For strong potentials we were not able to establish the behaviour as

$|\cos\theta| \rightarrow \infty$. For extreme cases the behaviour is known :

i) $p = q$ Regge ⁷⁾ has proven in this case that for potentials $V(r)$ analytic in r for $\text{Re } r > 0$ such that $|V(ir)| < A/r^2$ The T matrix can be bounded as $|\cos\theta| \rightarrow \infty$ by some power of $|\cos\theta|$.

ii) $p = \infty$ then $T \equiv 0$

iii) $p = 0$ then $T = \text{constant}$.

So we believe that nothing really new will happen between these three values of p and that only a finite number of subtractions, less than some fixed number independent of p , is needed in an integral representation of T . With this assumption we can write

$$T(p, q, \cos\theta) = \sum_{i=1}^{n-1} A_i(p, q) (\cos\theta)^i + (\cos\theta)^n \int \frac{R(u, p, q) du}{u - \cos\theta} \quad (9)$$

$\frac{p^2 + q^2 + \mu^2}{2pq}$

- For weak potentials, i.e. $\int_{\mu}^{\infty} |w(\sigma)| d\sigma$ small enough, $w(\sigma)$ being a sufficiently smooth function, satisfying, for instance, some kind of double Lipschitz condition, it is possible to show that at most one subtraction is needed, and that the integral representation of T may be obtained from the behaviour of each term of the Born expansion. Unfortunately, we know that the n - p potential is strong enough to bind a deuteron so that this result is not of much use here.

III. Analyticity of the matrix element with respect to the angle.

Let us first disregard the problem of subtractions and set $n=0$ in formula (9). We can write

$$\begin{aligned}
M(k, q, \cos \theta) = & \frac{1}{qk} \int_0^\infty \frac{p'(\sigma) d\sigma}{\frac{q^2 + k^2/4 + \sigma^2}{qk} - \cos \theta} + \\
& + \frac{1}{k} \int_0^\infty \frac{p dp}{p^2 - q^2 + i\epsilon} \int_0^\infty d\sigma \int_{\frac{p^2 + q^2 + \mu^2}{2pq}}^\infty dU \int d\Omega_p \frac{R^*(U, p, q) p'(\sigma)}{\left[\frac{p^2 + k^2/4 + \sigma^2}{pk} - \cos(kp) \right] \left[U - \cos(pq) \right]} \quad (10)
\end{aligned}$$

Let us now replace the angular integration appearing in the second term by an integral over an auxiliary variable. It can be shown that

$$\int \frac{d\Omega_2}{\left[\frac{l^2 + m^2 + \alpha^2}{2lm} - \cos \theta_{12} \right] \left[\frac{l^2 + n^2 + \beta^2}{2ln} - \cos \theta_{23} \right]} = \int_{x_{\min}}^\infty \frac{f(x) dx}{x - \cos \theta_{13}} \quad (11)$$

where $x_{\min}$ is a function of l, m, n, α, β such that

$$x_{\min} \gg \frac{m^2 + n^2 + (\alpha + \beta)^2}{2mn} \quad \text{for } 0 \leq l \leq +\infty$$

the explicit form of $f(x)$ is given in the appendix.

Remembering that $p'(\sigma)$ consists of a delta function plus a function starting at $\sigma = \alpha + \mu$, we perform the angular integration using formula (11). Assuming that no difficulty arises in the remaining integrations over the parameters of the integral representations and over p , we finally get :

$$\begin{aligned}
M(k, q, \cos \theta) = & \frac{1}{N} \frac{1}{q^2 + k^2/4 + \alpha^2 - qk \cos \theta} + \\
& + \int_{\frac{(\alpha + \mu)^2 + q^2 + k^2/4}{kq}}^\infty \frac{du \chi(k, q, u)}{u - \cos \theta} \quad (12)
\end{aligned}$$

Let us now compare this formula with the predictions based on field theoretical perturbative arguments. We reproduce here formula (44) of reference ¹⁾

$$M(q, k, \cos \theta) \equiv F(q, \cos \theta) = \frac{a/4}{q^2 + \alpha^2} - \frac{b/2}{\alpha^2 + q^2 + k^2/4 - kq \cos \theta} + 2 \int_0^\infty \frac{dq'^2}{q'^2 - q^2} \int_{\mu}^\infty \frac{d\sigma (\sigma + \alpha) M(\sigma, q'^2)}{(\sigma + \alpha)^2 + q^2 + k^2/4 - qk \cos \theta} \quad (13)$$

If we perform the integral over q'^2 we see that formula (12) and (13) coincide except that in formula (13) the deuteron pole term $\frac{a/4}{q^2 + \alpha^2}$ is missing. This is not surprising since we have disregarded the problem of subtractions. Now it is very likely that since, in general

$$\lim_{p \rightarrow 0} T(p, q, \cos \theta) = \text{const} \neq 0$$

at least one subtraction is needed in representation (9).

Let us now show that formula (14) is not seriously altered when n subtractions are present in representation (9). In this case we have to perform angular integrals of two types :

$$\text{i)} \quad \int d\Omega_2 \frac{(\cos \theta_{23})^n}{(A - \cos \theta_{12})(B - \cos \theta_{23})} \quad A > 1 \quad B > 1 \quad (14)$$

$$\text{ii)} \quad \int d\Omega_2 \frac{\sum_{i=0}^{n-1} a_i (\cos \theta_{23})^i}{A - \cos \theta_{12}} = \int d\Omega_2 \frac{\sum_{i=0}^{n-1} b_i P_i(\cos \theta_{23})}{A - \cos \theta_{12}} \quad (15)$$

Noticing that :

$$\frac{x^n}{B-x} = B^n \left[\frac{1}{B-x} - \frac{1}{B} - \frac{x}{B^2} \cdots - \frac{x^{n-1}}{B^n} \right] \quad (16)$$

we can reduce integral i) to the angular integral without subtraction (for fixed B and A there is no problem of convergence) and to an integral of the type ii).

The integral of the type ii) may be computed; it is just

$$4\pi \sum_{i=0}^{n-1} b_i Q_i(A) P_i(\cos\theta_{13})$$

So, after the angular integration we get an expression

$$\sum_{i=1}^{n-1} C_i(A,B)(\cos\theta_{13})^i + \int_{x_{\min}(A,B)}^{\infty} \frac{f(x,A,B)dx}{x - \cos\theta_{13}} \quad (17)$$

Now it is clear that when one integrates over B , with the convenient weight function, the terms appearing in (17), one gets for each term a divergent result because a factor B^n appears in the right hand side of (16). It turns out that n subtractions have to be made in the integral appearing in (17). Finally one gets the result (12) with n subtractions.

IV. Concluding remarks.

Though the results of the present paper on the analyticity for fixed energy, combined with the results of De Alfaro and Rossetti on partial waves are not sufficient to establish that the non-relativistic photodisintegration matrix element satisfies the Mandelstam representation (13), they are nevertheless very encouraging. In the scattering problem, all the proofs of Mandelstam representation use a combination of the analytic properties for fixed energy and fixed momentum transfer. One may hope to apply the same method here. So, what remains to be done is to derive dispersion relations for fixed momentum transfer. (This momentum transfer has to be defined conveniently). This is being currently investigated now at CERN.

Acknowledgements

The author wishes to thank Professors S. Fubini, M. Gourdin and J. Mandelbrojt for very stimulating discussions.

APPENDIX

1) Angular integrals.

By means of Feynman parametrisation technique one may prove

$$\int_{A > 1} \frac{d\Omega_2}{(A - \cos\theta_{12})(B - \cos\theta_{23})} = 4\pi \int_1^\infty \frac{dz}{\sqrt{z^2 - 1} [z \sqrt{A^2 - 1} \sqrt{B^2 - 1} + AB - \cos\theta_{13}]} \quad (A1)$$

So

$$\int_{A_0}^\infty \int_{B_0}^\infty \int \frac{dA dB d\Omega_2 f(A)g(B)}{(A - \cos\theta_{12})(B - \cos\theta_{23})} = \int_{C_0}^\infty \frac{dc h(c)}{C - \cos\theta_{13}} \quad (A2)$$

$$\text{with } C_0 = \sqrt{A_0^2 - 1} \sqrt{B_0^2 - 1} + A_0 B_0.$$

Further one can prove the following inequality :

$$\int_{C_0}^\infty \frac{|h(c)| dc}{C - 1} \leq 4\pi \int_{A_0}^\infty \frac{|f(A)| dA}{A - 1} \int_{B_0}^\infty \frac{|g(B)| dB}{B - 1} \quad (A3)$$

Now if

$$A_0 = \frac{1^2 + m^2 + \alpha^2}{2lm} \quad \text{and} \quad B_0 = \frac{1^2 + n^2 + \beta^2}{2ln}$$

The minimum of C_0 with respect to l for $0 < l < \infty$ is

$$C_{0 \min} = \frac{m^2 + n^2 + (\alpha + \beta)^2}{2mn}$$

This relation was derived in another way in reference ⁶⁾.

2) Analytic properties of T off the energy shell for weak potentials.

Let us first write the n^{th} term of the Born expansion

$$T_n(p, q, r) = \int \langle p | V | k_1 \rangle \frac{1}{k_1^2 - r^2 - i\epsilon} \langle k_1 | V | k_2 \rangle \cdots \frac{1}{k_{n-1}^2 - r^2 - i\epsilon} \langle k_{n-1} | V | q \rangle \quad (\text{A5})$$

$$d^3 k_1 \cdots d^3 k_{n-1}$$

if V is a superposition of Yukawa potential, a typical term appearing in $\langle k_i | V | k_{i+1} \rangle$ will be

$$\frac{1}{2k_i k_{i+1}} \frac{1}{\frac{\alpha^2 + k_i^2 + k_{i+1}^2}{2k_i k_{i+1}} - \cos(k_i k_{i+1})}$$

So, applying formulae (A1) and (A4) we can write :

$$T_n(p, q, r) = \int \prod_{i=1}^{n-1} \left(\frac{dk_i}{k_i^2 - r^2 - i\epsilon} \right) \int_{C_0(k_1 \cdots k_{n-1}, p, q)}^{\infty} \frac{W(C, k_1, k_2 \cdots k_{n-1}, p, q) dC}{C - \cos \theta}$$

where

$$C_0 \geq \frac{n^2 \mu^2 + p^2 + q^2}{2pq}$$

which, if no trouble arises in the radial integrals gives

$$T_n(p, q, r) = \int_{\frac{n^2 \mu^2 + p^2 + q^2}{2pq}}^{\infty} \frac{R_n(p, q, r, u) du}{u - \cos \theta} \quad (\text{A6})$$

If the last radial integrals did not contain principal value integrals, it would not be difficult to get, by using inequality (A3), an upper bound for

$$\int \frac{|R_n(u)| du}{u-1}$$

and to show that for sufficiently weak potentials $T = \sum T_n$ and $R = \sum R_n$. Unfortunately we have to study more carefully the integration region from $k_i = r - \Delta$ to $k_i = r + \Delta$. A convenient procedure is to write

$$\begin{aligned} & \frac{1}{k_i^2 - r^2} V(\underline{k}_i, \underline{k}_{i+1}) k_i k_{i+1} \frac{1}{k_{i+1}^2 - r^2} = \\ & = \frac{1}{k_i^2 - r^2} r^2 V(\hat{\underline{k}}_i, \hat{\underline{k}}_{i+1}) \frac{1}{k_{i+1}^2 - r^2} + \\ & + \frac{1}{k_i^2 - r^2} W(\hat{\underline{k}}_i, \underline{k}_{i+1}) + \\ & + \frac{1}{k_{i+1}^2 - r^2} W(\underline{k}_i, \hat{\underline{k}}_{i+1}) + \\ & + Z(\underline{k}_i, \underline{k}_{i+1}) \end{aligned} \quad (A7)$$

where

$$\hat{\underline{k}}_i = \frac{k_i}{k_i} r$$

then one gets rid of all the singularities. If W , and Z have integral representations in $\cos\theta$ of the same type as V itself, with weight functions

such that $\int \frac{\rho(x) dx}{x-1}$ is bounded, one can still get an upper bound for $\int \frac{|R_n(u)| du}{u-1}$.

This is not the case for a pure Yukawa potential corresponding to $w(\sigma) = \int (\sigma - \mu)$ in

$$V(r) = \int w(\sigma) \frac{e^{-\sigma r}}{r} d\sigma,$$

but it is the case for potentials with a weight function $w(\sigma)$ sufficiently

smooth. For instance one can require for $w(\sigma)$ a double Lipschitz condition:

$$\left| \frac{w(\alpha) - w(\beta)}{\alpha - \beta} - \frac{w(\gamma) - w(\delta)}{\gamma - \delta} \right| < A \frac{[|\alpha - \gamma| + |\beta - \delta|][|\alpha - \delta| + |\beta - \gamma|]}{(\alpha + \beta)^2 (\gamma + \delta)^2}$$

which, with $w(\infty) = 0$ implies

$$\left| \frac{w(\alpha) - w(\beta)}{\alpha - \beta} \right| < \frac{A}{(\alpha + \beta)^2} \quad \text{and} \quad |w(\alpha)| < \frac{A}{\alpha}$$

Then one can show that V, W, Z have the desired properties and that

$$T_n = \int \frac{R_n(p, q, r, u) du}{u - \cos \theta} \quad \text{with} \quad \int \frac{|R_n| du}{u - 1} < 2^n A^n$$

$$\frac{h^2 p^2 + p^2 + q^2}{2pq}$$

where λ is some fixed number. Then, for A small enough, the series can be summed and

$$T(p, q, r) = \int_{\frac{\mu^2 + p^2 + q^2}{2pq}}^{\infty} \frac{R(p, q, r, u) du}{u - \cos \theta},$$

with

$$\int_{\frac{\mu^2 + p^2 + q^2}{2pq}}^{\infty} \frac{|R(p, q, r, u)| du}{u - 1} \quad (A8)$$

finite.

One can understand why one subtraction is necessary for $p=0$: as $p \rightarrow 0$ the lower limit for u goes to infinity. So it is quite possible that $R(p, q, r, u) \rightarrow \infty$ as $p \rightarrow 0$, though the integral remains finite. On the other hand, property (A8) ensures that $\int \frac{R(u) du}{u - \cos \theta}$ exists everywhere outside the cut and goes to zero as $|\cos \theta| \rightarrow \infty$ in any direction except the direction of the cut.

It is interesting to note that the case of pure Yukawa potential cannot be treated in this way and that in Regge's proof for $p=q=r$ ⁷⁾ it is also excluded.

3) Analytic properties for strong potentials.

There we combine the properties of the terms of the perturbation expansion with general properties of T for potentials decreasing faster than an exponential:

$$T(\underline{p}, \underline{q}, r) = \int e^{-i \underline{p} \cdot \underline{x}} V(\underline{x}) G(\underline{x}, \underline{x}', r) V(\underline{x}') e^{i \underline{q} \cdot \underline{x}'} d^3 \underline{x} d^3 \underline{x}' \quad (\text{A9})$$

where G is the total Green's function which is known to be bounded⁹⁾. A sufficient condition for the existence of T is that the integrals at both ends be absolutely convergent. Setting

$$\begin{aligned} \underline{p} &= n_1 \underline{p} \cos \alpha \theta + n_2 \underline{p} \sin \alpha \theta \\ \underline{q} &= n_1 \underline{q} \cos(1-\alpha)\theta - n_2 \underline{q} \sin(1-\alpha)\theta \end{aligned}$$

with

$$n_1 \perp n_2 \quad \text{and} \quad 0 < \alpha < 1$$

we get the simultaneous conditions

$$\begin{aligned} \text{sh}^2[\alpha \text{Im}(\theta)] &< \frac{\mu^2}{p^2} \\ \text{sh}^2[(1-\alpha) \text{Im}(\theta)] &< \frac{\mu^2}{q^2} \end{aligned} \quad (\text{A10})$$

which define, in the $\cos\theta$ plane two ellipses with foci -1 $+1$. α must be chosen to make the domain as large as possible. This is the case when the two ellipses coincide. Then the semi-major axis is

$$\sqrt{1 + \frac{\mu^2}{p^2}} \sqrt{1 + \frac{\mu^2}{q^2}} + \frac{\mu^2}{pq} > \sqrt{1 + \frac{\mu^2}{q^2}} \quad (\text{A11})$$

Let us now write

$$T(\underline{p}, \underline{q}, q) = T_1 + \dots + T_n + \int T_n(\underline{p}, \underline{p}', q) \frac{1}{p'^2 - q^2 - i\epsilon} T(\underline{p}', q, q) d^3 \underline{p}' \quad (\text{A12})$$

and expand T and T_n under the integral in Legendre polynomials

$$T_n(\underline{p}, \underline{p}', q) = \sum B_n P_n(\cos(\underline{p}, \underline{p}')) \quad (A13)$$

$$T(\underline{p}', q, q) = \sum C_n P_n(\cos(\underline{p}', q)) \quad (A14)$$

and

$$\int d\Omega_p T_n(\underline{p}, \underline{p}', q) T(\underline{p}', q, q) = \sum \frac{4\pi}{2n+1} B_n C_n P_n(\cos(\underline{p}, q)) \quad (A15)$$

Now if (A13) converges in an ellipse with semi-major axis b and (A14) in an ellipse with semi-major axis c

$$\limsup_{n \rightarrow \infty} (B_n)^{\frac{1}{n}} < \frac{1}{b + \sqrt{b^2 - 1}}$$

$$\limsup_{n \rightarrow \infty} (C_n)^{\frac{1}{n}} < \frac{1}{c + \sqrt{c^2 - 1}}$$

and expansion (A15) converges in an ellipse with semi-major axis a where

$$a + \sqrt{a^2 - 1} = (b + \sqrt{b^2 - 1})(c + \sqrt{c^2 - 1}),$$

i.e.

$$a = bc + \sqrt{b^2 - 1} \sqrt{c^2 - 1}$$

where

$$b = \frac{n^2 \mu^2 + p'^2 + p^2}{2pp'}$$

$$c = \sqrt{1 + \frac{\mu^2}{p'^2}} \sqrt{1 + \frac{\mu^2}{q^2}} + \frac{\mu^2}{p'q}$$

the exact minimum of a with respect to p' is difficult to get but it is clear that

$$a > \sqrt{\left(1 + \frac{n^2 \mu^2}{p^2}\right) \left(1 + \frac{\mu^2}{q^2}\right)}$$

so that $T - T_1 - T_2 \dots T_n$ is holomorphic inside a larger and larger ellipse as $n \rightarrow \infty$. This method establishes the analyticity in $\cos\theta$ outside the cut but cannot give any information on the behaviour at infinity.

We add that all the properties we have obtained are valid for $\text{Im } T$ where

$$\text{Im } T(\underline{p}, \underline{q}, r) = \int e^{-i\underline{p}\underline{x}} V(\underline{x}) \text{Im } G(\underline{x}, \underline{x}', r) V(\underline{x}') e^{i\underline{q}\underline{x}'} d^3\underline{x} d^3\underline{x}' .$$

REFERENCES

- 1) V. De Alfaro and C. Rossetti, Preprint (Torino), to be published in Nuovo Cimento.
- 2) R. Karplus, C. Sommerfield and E. Wichmann, Phys.Rev. 114, 378 (1959).
- 3) J. Eden, Preprint - and
J. Eden and J.C. Polkinghorne, to be published in the Proceedings of the
Ninth Annual International Conference on
High Energy Physics, Rochester (1960).
- 4) A. Martin, Nuovo Cimento 14, 403 (1959).
- 5) R. Blankenbeckler, Preprint, to be published in Phys.Rev,
- 6) J. Bowcock and A. Martin, Nuovo Cimento 14, 516 (1959).
- 7) T. Regge, Nuovo Cimento 14, 951 (1959).
- 8) R. Blankenbeckler, M.L. Goldberger, N.N. Khuri and S.B. Treiman,
Annals of Physics 10, 62 (1960).
A. Klein, J.Math.Phys. 1, 41 (1960).
- 9) A. Klein and C. Zemach, Annals of Physics 7, 440 (1959).