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***S*-Matrix, Left-Hand Cut Discontinuity and Potential.**

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Summary. — Assuming a two body potential which is a continuous superposition of exponential potentials we first show in a straightforward way how to get, from the left-hand cut discontinuity of the *S*-wave scattering amplitude, the scattering amplitude itself, without any ambiguity. On the other hand, we establish a one to one correspondence between the unphysical cut discontinuity and the inverse Laplace transform of the potential. When the discontinuity is known in a certain range of values the inverse Laplace transform of the potential is exactly known in a corresponding range of values. An extension of the treatment when a repulsive hard core is present at short distance is given. The usefulness of the inverse Laplace transform of the potential as an intermediate tool in the calculation of the scattering amplitude is discussed.

1. — Introduction.

In previous studies ⁽¹⁻³⁾ we have established the analytic properties of the scattering matrix in the *S*-state for two particles interacting through a potential of the family

$$(1) \quad V(r) = \int_{\mu}^{\infty} C(\alpha) \exp[-\alpha r] d\alpha, \quad |C(\alpha)| < M\alpha^{-\epsilon}.$$

It was shown that, in the complex energy plane, in addition to the physical

⁽¹⁾ A. MARTIN: *Nuovo Cimento*, **14**, 403 (1959).

⁽²⁾ A. MARTIN: *Lecture notes at the summer school of Cargèse (Corsica)* (1960), CERN preprint 9432/TH. 120.

⁽³⁾ R. G. NEWTON: *Journ. Math. Phys.*, **1**, 319 (1960).

cut, an unphysical cut is present on the real axis from $E \equiv k^2 = -\infty$ to $E = -\mu^2/4$. These properties coincide with those obtained in the field theoretical case by projecting the Mandelstam representation for the nucleon-nucleon scattering matrix on the S -state, if μ is the meson mass.

In the dispersion approach to the nucleon-nucleon problem the discontinuity of the S -matrix across the left hand cut is known at least in a certain region, and from it one can get, if certain assumptions are made, the scattering matrix itself, as it is shown by NOYES and WONG ⁽⁴⁾. On the other hand, CHARAP, FUBINI and TAUSNER ⁽⁵⁾ have studied the problem of defining a potential reproducing the S -matrix, at least when one and two meson intermediate states are considered ⁽⁵⁾.

We wish to look at these two problems under the assumption that two nucleons interact through a potential of family (1). In this case we prove that the knowledge of the discontinuity across the unphysical cut determines unambiguously both the scattering matrix and the inverse Laplace transform of the potential. If the discontinuity of the S -matrix is not completely known it is still possible to say something about the inverse Laplace transform of the potential. More precisely if the left-hand cut discontinuity is known from $E = k^2 = -\mu^2/4$ to $E = -n^2\mu^2/4$ the inverse Laplace transform of the potential is known from $\alpha = \mu$ to $\alpha = n\mu$ and *conversely*.

However, we could not solve completely the problem to find necessary and sufficient conditions on the left-hand cut discontinuity, assumed to be completely known from $-\mu^2/4$ to $-\infty$, to ensure that the inverse Laplace transform $C(\alpha)$ has the correct behaviour, given by (1) for large α . Only partial results have been obtained. This problem is not physically so important because the left-hand cut discontinuity is never known up to $E = -\infty$.

2. - Basic equations.

We write here the main equations and the main results of reference ⁽¹⁾. A solution of the Schrödinger equation was written as $f(k, r) \exp[-ikr]$ such that $f(k, \infty) = 1$. It has been shown that with a potential of family (1) f may be written

$$(2) \quad f(k, r) = 1 + \int_{\mu}^{\infty} \rho_k(\alpha) \exp[-\alpha r] d\alpha,$$

⁽⁴⁾ H. P. NOYES and D. Y. WONG: *Phys. Rev. Lett.*, **3**, 191 (1959).

⁽⁵⁾ J. CHARAP and S. FUBINI: *Nuovo Cimento*, **14**, 540 (1959); **15**, 73 (1960); J. CHARAP and M. TAUSNER: *E.F.I.N.S.* 60-26, to be published in *Nuovo Cimento*.

with

$$(3) \quad \alpha(\alpha + 2ik) \varrho_k(\alpha) = C(\alpha) + \int_{\mu}^{\alpha-\mu} C(\alpha - \beta) \varrho_k(\beta) d\beta. \quad (C(\alpha) = 0 \text{ for } \alpha < \mu).$$

Eq. (2) is an especially simple integral equation from which one can get $\varrho_k(\alpha)$ for any finite value of α by a finite number of iterations. When k is outside the cut $i(\mu/2) \rightarrow i\infty$, $\varrho_k(\alpha)$ is well defined for all the values of α . The S -matrix is just

$$(4) \quad S(k) = \exp [2i\delta(k)] = \frac{f(k, 0)}{f(-k, 0)},$$

where $f(k, 0)$ (*) has a cut along the imaginary k axis from $k = i(\mu/2)$ to $k = i\infty$ and no other singularities. Consequently $f(-k, 0)$ has a cut from $k = -i(\mu/2)$ to $k = -i\infty$, and one can further prove that $f(-k, 0)$ has no zeros in the upper half plane except on the imaginary axis. We should add the symmetry property $f(k, 0) = f^*(-k^*, 0)$. The behaviour at infinity is given by

$$(5) \quad \lim_{|k| \rightarrow \infty} |k|^\varepsilon |f(k, 0) - 1| \leq \text{const.} \quad 0 < \varepsilon < 1.$$

3. - S -matrix and unphysical cut.

The above mentioned analytic properties and, especially, inequality (5) enable us to write spectral representations for $f(k, 0) - 1$ and $f(-k, 0) - 1$ without any subtraction. Hence, S may be written

$$(6) \quad S(k) = \frac{1 - i \int_{\mu/2}^{\infty} w(x)/(k - ix) dx}{1 + i \int_{\mu/2}^{\infty} w(x)/(k + ix) dx},$$

where $-2\pi i w(x)$ is the purely imaginary discontinuity of $f(k, 0)$ for $k = ix$.

Let us call $2\pi i D(x)$ the discontinuity of $S(k)$ across the upper cut, which corresponds to the left-hand cut in the variable k^2 (the lower cut corresponds to a cut in the second Riemann sheet of the k^2 plane).

(*) This function was first introduced by R. JOST⁽⁶⁾.

(6) R. JOST: *Helv. Phys. Acta*, **19**, 256 (1946).

Then one readily gets from eq. (6)

$$\frac{S(i\kappa_0 + \varepsilon) - S(i\kappa_0 - \varepsilon)}{2i\pi} \equiv D(\kappa_0) = \frac{w(\kappa_0)}{1 + \int_{\mu/2}^{\infty} w(\kappa) d\kappa / (\kappa + \kappa_0)},$$

or

$$(7) \quad w(\kappa_0) = D(\kappa_0) \left[1 + \int_{\mu/2}^{\infty} \frac{w(\kappa) d\kappa}{\kappa + \kappa_0} \right].$$

With some changes in the notations this is exactly the Noyes and Wong final Fredholm equation in the non-relativistic case, established without any ambiguity in the case of potential scattering. The Fredholm equation determines the weight function w which, inserted in eq. (6), gives $S(k)$. It is not clear, however, that when $w(\kappa)$ is inserted in expression (6), $S(k)$ has the right analytic properties. The cuts are in the right place, but nothing prevents $f(-k, 0)$ from vanishing in isolated points of the upper half k plane, outside the imaginary axis. In an actual calculation, starting from a given $D(\kappa)$, one has to check *a posteriori* that this does not happen. The same difficulty appears in the field theoretical treatment.

If we try to compare our method of solution with that of NOYES and WONG we see that the main difference comes from the fact that they use the variable k^2 . Then the cuts are the left-hand cut and the physical cut; S , in their case is separated as $\mathcal{N}/\mathcal{D}$, where $\mathcal{N}$ has a discontinuity across the left-hand cut and $\mathcal{D}$ across the right hand cut. $\mathcal{N}$ and $\mathcal{D}$ are related by two coupled dispersion relations taking into account unitarity and the knowledge of $D(\kappa)$. In our case, however, where the physical cut is replaced by the lower cut in the k plane, unitarity is automatically fulfilled from the beginning. This is why we directly get a unique integral equation. On the other hand, in our treatment, the subtraction problem does not appear, and, further, the possible bound states should come out of the solution, without being *a priori* postulated.

In the case where the discontinuity across the unphysical cut is not completely known (in practice one can obtain $D(\kappa)$ from $\kappa = \mu/2$ to $\kappa = 3\mu/2$, corresponding to the knowledge of the $N\bar{N} \rightarrow \pi$ and $N\bar{N} \rightarrow 2\pi$ amplitudes) eq. (7) cannot be solved without making phenomenological assumptions on $D(\kappa)$ for large values of κ . However, as we shall see it in the next section, it is nevertheless possible to obtain a partial but accurate information on the potential itself.

4. - Unphysical cut and inverse Laplace transform of the potential.

We shall try first to evaluate the discontinuity $D(\kappa)$ in terms of the inverse Laplace transform of the potential. For this purpose we must look more carefully at what happens for $k \rightarrow i\kappa$.

For $k = i\kappa$, the function $\varrho_k(\alpha)$, introduced in eq. (3) is well defined provided $\alpha < 2\kappa$. In the neighbourhood of $\alpha = 2\kappa$ this is no longer true because a factor in the left-hand side of eq. (3) vanishes. However, from $\alpha = 2\kappa + \varepsilon$ to $\alpha = 2\kappa + \mu - \varepsilon$ $\varrho_{i\kappa}(\alpha)$ is again well defined because the right-hand side of eq. (3) makes use of values of α smaller than 2κ . When $\varrho_{i\kappa}(\alpha)$ is not well defined, it is still possible to define $\varrho_{i\kappa}^+(\alpha)$ and $\varrho_{i\kappa}^-(\alpha)$ such that

$$\varrho_{i\kappa}^+(\alpha) = \lim_{\varepsilon \rightarrow 0} \varrho_{i\kappa+\varepsilon}(\alpha), \quad \varrho_{i\kappa}^-(\alpha) = \lim_{\varepsilon \rightarrow 0} \varrho_{i\kappa-\varepsilon}(\alpha).$$

From eq. (3) one gets, in the neighbourhood of $\alpha = 2\kappa$

$$(8) \quad \varrho_{i\kappa}^+(\alpha) - \varrho_{i\kappa}^-(\alpha) = -i\pi \delta(\alpha - 2\kappa) \frac{C(2\kappa) + \int_{\mu}^{2\kappa-\mu} C(2\kappa - \beta) \varrho_{i\kappa}(\beta) d\beta}{\kappa}.$$

f^+ and f^- can be defined from the corresponding ϱ 's. So for $k = i\kappa$ there are three solutions of the Schrödinger equation:

$$f^+(i\kappa, r) \exp[\kappa r], \quad f^-(i\kappa, r) \exp[\kappa r], \quad f(-i\kappa, r) \exp[-\kappa r];$$

these solutions are not independent, and since

$$\lim_{r \rightarrow \infty} f^+(i\kappa, r), \quad f^-(i\kappa, r), \quad f(-i\kappa, r) = 1$$

the only possible relation which can be identically satisfied is

$$(9) \quad [f^+(i\kappa, r) - f^-(i\kappa, r)] \exp[2\kappa r] = A(\kappa) f(-i\kappa, r).$$

If we evaluate eq. (9) at $r = 0$ we get

$$\frac{f^+(i\kappa, 0)}{f(-i\kappa, 0)} - \frac{f^-(i\kappa, 0)}{f(-i\kappa, 0)} = A(\kappa).$$

Therefore $A(\kappa)$ is just the discontinuity of the S -matrix across the cut. So we can rewrite eq. (9) as:

$$2i\pi D(\kappa) f(-i\kappa, r) = [f^+(i\kappa, r) - f^-(i\kappa, r)] \exp[2\kappa r]$$

and hence

$$2i\pi D(\kappa) = \lim_{r \rightarrow \infty} [f^+(i\kappa, r) - f^-(i\kappa, r)] \exp[2\kappa r]$$

or, expressing f^+ and f^- in terms of ϱ^+ and ϱ^-

$$2i\pi D(\kappa) = \lim_{r \rightarrow \infty} \exp[2\kappa r] \int_{2\kappa-\varepsilon}^{2\kappa+\varepsilon} [\varrho_{i\kappa}^+(\alpha) - \varrho_{i\kappa}^-(\alpha)] \exp[-\alpha r] d\alpha + \\ + \lim_{r \rightarrow \infty} \exp[2\kappa r] \int_{2\kappa+\mu-\varepsilon}^{\infty} \exp[-\alpha r] [\varrho_{i\kappa}^+(\alpha) - \varrho_{i\kappa}^-(\alpha)] d\alpha.$$

The second term vanishes because it behaves like $\exp[-(\mu-\varepsilon)r]$. The first term is the δ -function contribution. Finally

$$(10) \quad -2\kappa D(\kappa) = C(2\kappa) + \int_{\mu}^{2\kappa-\mu} C(2\kappa-\beta) \varrho_{i\kappa}(\beta) d\beta.$$

Eq. (10) is the key for our problem. First it gives $D(\kappa)$ in terms of the inverse Laplace transform of the potential. $\varrho_{i\kappa}(\beta)$ appears in the equation, but the argument β is less than $2\kappa-\mu$, and ϱ can therefore be obtained from the inverse Laplace transform of the potential by using eq. (3). The important point is that the *reverse* is *true*. Assume we know

$$D(\kappa) \quad \text{from} \quad \kappa = \mu/2 \quad \text{to} \quad \kappa = \kappa_0 + (\mu/2)$$

and

$$C(\alpha) \quad \text{from} \quad \alpha = \mu \quad \text{to} \quad \alpha = 2\kappa_0$$

then it is clear that we also know $\varrho_{i\kappa}(\beta)$ from $\beta = \mu$ to $\beta = 2\kappa_0$ provided $\kappa > \kappa_0$.

Let us now apply eq. (10) for a value

$$\kappa_0 < \kappa < \kappa_0 + (\mu/2).$$

Then $D(\kappa)$ is known, the quantities appearing in the integral are known and one gets *unambiguously* $C(2\kappa)$ or $C(\alpha)$ for $2\kappa_0 < \alpha < 2\kappa_0 + \mu$. So if $D(\kappa)$ is known up to $\kappa = \kappa_{\max}$, $C(\alpha)$ is known up to $\alpha = 2\kappa_{\max}$. We can illustrate this for low values of κ :

$$\text{— for } \kappa < \mu/2 \quad C(2\kappa) = D(\kappa) = 0,$$

$$\text{— for } \mu/2 < \kappa < \mu \quad C(2\kappa) = -2\kappa D(\kappa),$$

$$\text{— for } \mu < \kappa < \frac{3\mu}{2}, \quad C(2\kappa) = -2\kappa D(\kappa) - \int_{\mu}^{2\kappa-\mu} D\left(\kappa - \frac{\beta}{2}\right) D\left(\frac{\beta}{2}\right) d\beta,$$

and so on....

If we look at the expression for $\mu < x < 3\mu/2$ we see that its structure reminds us of the result of CHARAP and FUBINI, namely that to get the fourth order potential one must subtract from the discontinuity of the scattering amplitude for $-\mu^2 > k^2 > -(3\mu/2)^2$ the iteration of the second order result.

In our opinion this approach, though it does not furnish directly the scattering amplitude, is more satisfactory than the one considered in the preceding section because one does not mix known and unknown things concerning the left-hand cut discontinuity. The last step, which unavoidably involves some phenomenology, is to go from the inverse Laplace transform of the potential to the scattering amplitude. Let us first notice that one can work directly with the inverse Laplace transform of the potential as was shown by the author in ref. (1); the method of computation we proposed is rapidly convergent and an upper limit of the error made in the calculation of phase shifts can be obtained. However, one has to assume something about the high components of the inverse Laplace transform, but the assumptions made are easily interpreted because these high components mainly affect the short distance part of the potential. For instance to introduce a short range repulsion one can take $C(\alpha) = \alpha^{1-\epsilon}$, $C > 0$ for $\alpha > A \gg \mu$. Of course, a limitation of our method is that too singular potentials (even repulsive ones) are excluded because they do not belong to family (1). However, in the case of a hard core, the modifications to be done are almost trivial.

Assume

$$V(r) = +\infty \quad \text{for } r < a,$$

$$V(r) = V'(r-a) \quad \text{for } r > a,$$

where $V'(r)$ belongs to family (1).

Then

$$\delta(k) = \delta'(k) - a$$

and

$$S(k) = S'(k) \exp[-2ika].$$

Hence $S(k)$ and $S'(k)$ have the same analytic properties, except for the behaviour at infinity. One can get the discontinuity $D'(\kappa)$ of $S'(k)$ by using the relation

$$D'(\kappa) = D(\kappa) \exp[-2\kappa a].$$

Then one obtains the inverse Laplace transform of $V'(r)$ in a range corresponding to the range where $D(\kappa)$ is known. Clearly things would not be so simple for higher waves.

Let us now come back to the case where the potential belongs to family (1) and say a few words about the *academic* problem of determining the potential when the discontinuity across the unphysical cut is completely known from $k^2 = -\mu^2/4$ to $-\infty$. It is not obvious that this problem has always a solution because the inverse Laplace transform $C(\alpha)$ obtained from $D(\kappa)$ might not have the property $|C(\alpha)| < M\alpha^{1-\epsilon}$ for large α . By using upper bounds for $\rho_{\text{in}}(\alpha)$ and $C(\alpha)$ one easily gets a *necessary* condition from eq. (10), namely:

$$|D(\kappa)| < \text{const } \kappa^{-\epsilon}.$$

As an example we give a *sufficient* condition which can be derived in a rather tedious way:

$$|D(\kappa)| < \frac{\mu}{8\kappa}.$$

However, these difficulties should not be taken too seriously, because, in actual cases, the left-hand cut discontinuity is not known up to infinity.

5. - Concluding remarks.

We believe that the present paper brings some clarification of the connection between the scattering amplitude, the left-hand cut discontinuity and an assumed two body potential. Of course, it would be desirable to extend this treatment to all partial waves. Then appears the problem of finding a unique potential; this problem has been solved by CHARAP and FUBINI in the framework of meson theory for one and two meson intermediate states.

In our opinion, the results we obtained bring some justification for the search for a nucleon-nucleon potential (or its inverse Laplace transform) though it might look old-fashioned since it is at present possible to get directly the scattering amplitude from its left-hand cut discontinuity. The reason for this is our ignorance of the left-hand cut discontinuity for large negative values of k^2 . One may prefer to feed in phenomenological assumptions on the high components of the inverse Laplace transform of the potential, which can be easily interpreted, than on the left-hand cut discontinuity of the scattering amplitude.

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RIASSUNTO (*)

Presupponendo fra due corpi un potenziale, sovrapposizione continua di potenziali esponenziali, si mostra dapprima in maniera diretta come ottenere senza ambiguità, dalla discontinuità del taglio sinistro dell'ampiezza di scattering dell'onda S , l'ampiezza di scattering stessa. D'altra parte si stabilisce una corrispondenza univoca fra la discontinuità del taglio non fisico e la trasformazione di Laplace inversa del potenziale. Quando la discontinuità è nota in un certo campo di valori la trasformazione di Laplace inversa del potenziale è esattamente nota in un campo di valori corrispondente. Si dà una estensione del trattamento per il caso in cui sia presente a breve distanza un nocciolo repulsivo. Si discute l'utilità della trasformazione di Laplace inversa del potenziale quale strumento intermedio per il calcolo dell'ampiezza di scattering.

(*) Traduzione a cura della Redazione.