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On the uniqueness of a potential fitting a scattering
amplitude at a given energy. *)

A. Martin and Gy. Targonski

CERN - Geneva

A B S T R A C T

It is shown that when a scattering amplitude is known to be produced by a superposition of Yukawa potentials, the exact knowledge of the scattering amplitude at a given energy determines in a unique way the potential. This is established in two steps: i) by proving that, once the physical scattering amplitude is known, the discontinuity across the cut in the complex plane of the momentum transfer is uniquely defined. ii) by establishing the rigorous connection between this discontinuity and the potential. In order to be fitted by a superposition of Yukawa potentials at a given energy a scattering amplitude has to satisfy certain conditions which are investigated. The treatment can be modified in order to include the case of exchange forces.

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I. Introduction

The problem of the determination of a potential fitting a given partial wave amplitude has been considered some time ago by various authors ¹⁾. Up to now the only work on the construction of a potential from the knowledge of the scattering amplitude at all angles, for a given energy is that of Regge ²⁾, who makes use of complex angular momenta. Regge shows that once the possible interpolating phase shifts for non integral values of the angular momentum are known, a unique potential corresponds to a given interpolation. Here we want to use another approach to this problem. We shall be mainly interested by scattering amplitudes which can be produced by a potential of the type

$$\int_{\mu}^{\infty} c(\alpha) \frac{e^{-\alpha r}}{r} d\alpha \quad (1)$$

In this special case we prove that the analytic continuation of the scattering amplitude for complex angles is unique; the discontinuity across the cut in the $\cos \theta$ plane is unique, and from it one can construct the potential. We also look at the necessary conditions which must be fulfilled by a scattering amplitude to be produced by a potential of family (1); these conditions are obtained by making conformal mapping of the regularity domain of the scattering amplitude on a circle. Our treatment can be extended to the case of exchange forces.

II. Uniqueness of the continuation of the scattering amplitude for complex angles

Let us define $t = -2k^2(1 - \cos \Theta)$, where k is the momentum and Θ the scattering angle. The physical region for the scattering amplitude $T(t)^{*})$ is $-4k^2 \leq t \leq 0$. This is the region where $T(t)$ is supposed to be known.

Let us now make the assumption that the potential $V(r)$ which produces this scattering amplitude is exponentially decreasing, i.e. that there exist μ such that

$$\lim_{r \rightarrow \infty} e^{+\mu r} V(r) = \text{finite.} \quad (2)$$

Then it is known ³⁾ that $T(t)$ has an analytic continuation which is regular inside the ellipse with foci $t=0$, $t=-4k^2$ and which cuts the real axis at $t=\mu^2$ and $t=-4k^2-\mu^2$. When $f(t)$ is known in the physical domain the analytic continuation inside the ellipse is unique, because of the well-known theorem that when two functions coincide on a line inside a common regularity domain they coincide everywhere inside the regularity domain. If the continuation of $T(t)$ can be made in the whole complex plane (with possible singularities: cuts, poles, essential singularities etc.) the continuation is unique for the same reason.

Let us now make the more specific assumption that one of the potentials (if it exists) fitting, in the physical range, $T(t)$, has the form

$$V(r) = \int_{\mu}^{\infty} \frac{c(\alpha) e^{-\alpha r} d\alpha}{r} \quad (3)$$

with $\int_{\mu}^{\infty} |c(\alpha)| d\alpha$ finite, $\mu > 0$.

*) We drop the energy variable, which is kept fixed in the whole problem.

Then the analytic properties of $T(t)$ are known to be very simple 3), 4); it is regular in the whole complex plane, except along a cut on the real axis from $t=\mu^2$ to $t=+\infty$ (Fig. 1). Then $T(t)$ can be continued in a unique way in the whole complex t plane, because this is so in an arbitrarily large finite domain avoiding the cut. In particular, the discontinuity of $T(t)$ across the cut is uniquely defined when $T(t)$ is known in the physical region. As it will appear clearly in the next section when a potential of family (1) is used, the discontinuity is finite if $C(\alpha)$ is a finite function. It is a distribution if $C(\alpha)$ is a distribution.

fig. 1

We see that if the analytic continuation of $T(t)$ is as indicated on fig. 1 (the conditions for this will be investigated in section IV) the discontinuity of $T(t)$ across the cut is uniquely defined and, particularly μ itself is determined from the knowledge of $T(t)$ in the physical region.

A practical way of obtaining this discontinuity $2i\pi D(t)$ would be to try to solve the following first kind Fredholm equations:

$$T(t) = \int_{\mu^2}^{\infty} \frac{D(t') dt'}{t' - t} \quad \text{with } -4k^2 \leq t \leq 0 \quad (4)$$

$$T(t) = a_0 + a_1 t + \dots + a_{n-1} t^{n-1} + t^n \int_{\mu^2}^{\infty} \frac{D(t') dt'}{(t'-t)t'^n}, \quad -4k^2 \leq t \leq 0 \quad (5)$$

where n has to be finite according to Regge's work²⁾, from what we said, at most one of these equations has a solution, or, more precisely, if one equation has a solution, those with larger member of subtractions give exactly the same solution. Here it is obvious that one needs to know $T(t)$ with infinite accuracy to draw such a conclusion. Otherwise a fit by a polynomial in t , with $D(t)=0$ is always possible when only n values of $T(t)$ are known.

III. Determination of the potential from the discontinuity along the cut in the t plane.

We have seen, in the preceding section, that when $T(t)$ is known in the physical region, and when the potential belongs to family (1) the discontinuity across the unphysical cut is uniquely determined. Let us now show in an explicit way how to get the potential from the discontinuity. The connection between potential and discontinuity in the t variable has been already established for low t values ($t < 9\mu^2$) by Charap and Fubini⁵⁾, but we wish to give here a general proof.

Let us write the scattering amplitude as

$$T = T_1 + \dots + T_n + R_n \quad (6)$$

where T_n is the n^{th} Born term. It has a cut starting at $t=n^2\mu^2$.

On the other hand, by writing $R_n = T_{n+1} + \dots + T_N + R_N$ and taking N large enough one can show that R_n is regular inside an ellipse with foci $0, -4k^2$, cutting the real axis at $t = (n+1)^2 \mu^2$. Therefore to compute the discontinuity across the cut in the region $\mu^2 < t < (n+1)^2 \mu^2$ it is sufficient to consider the n first Born terms.

Let us split the potential in various parts:

$$V = V_1 + V_2 + \dots + V_n + V_{n+1} \quad (7)$$

where

$$V_1 = \int_{1\mu}^{(1+1)\mu} \frac{c(\alpha) e^{-\alpha r_d} d\alpha}{r} \quad (8)$$

and

$$V_{n+1} = \int_{(n+1)\mu}^{\infty} \frac{c(\alpha) e^{-\alpha r_d} d\alpha}{r}$$

A typical contribution to the i^{th} Born term will be

$$V_{n_1} G_o V_{n_2} G_o \dots G_o V_{n_i} \quad (9)$$

From previous work (see for instance ref. ³⁾) it is known that this contribution will have a cut in the t plane starting at $t = (n_1 + n_2 + \dots + n_i)^2 \mu^2$. So, in the region $\mu^2 < t < (n+1)^2 \mu^2$:

- i) V_{n+1} will never contribute to the discontinuity
- ii) V_n will contribute to the discontinuity through the first Born term T_1 and nowhere else. More precisely, since

$$T_1(t) = \int \frac{c(\alpha) d\alpha}{t - \alpha^2}$$

$$T_1(t+i\varepsilon) - T_1(t-i\varepsilon) = 2i\pi \times \frac{c(\sqrt{t})}{2\sqrt{t}} \quad (10)$$

These remarks lead us to use an iterative procedure to determine $c(\alpha)$ from the discontinuity of T across that cut:

Assume that $c(\alpha)$ is known from $\alpha = \mu$ to $\alpha = n\mu$ then $V_1 \dots V_{n-1}$ are known; then in the region $n^2\mu^2 < t < (n+1)^2\mu^2$ we have:

$$2\pi i \frac{c(\sqrt{t})}{2\sqrt{t}} = 2\pi i D(t) \quad (11)$$

- discontinuity of $(T_2 + \dots + T_n)$

the discontinuity of $T_2 + \dots + T_n$ is known in terms of $V_1 \dots V_{n-1}$. Hence V_n is known and the iterative procedure can go on.

We have now completed the proof that a given scattering amplitude at a given energy can be fitted by at most one potential which is a superposition of Yukawa potentials. However, the scattering amplitude in the physical region must fulfil certain requirements in order to ensure the existence of one solution. These will be investigated in the next section.

IV. Necessary conditions on the scattering amplitude

We shall now investigate the conditions to impose on the scattering amplitude in the physical region to make it have the analytic properties indicated in section II. This is obviously necessary (but not necessarily sufficient) to make it possible to reproduce the scattering amplitude by a superposition of Yukawa potentials. In the first place, we have to check that the starting point of the cut is $t = \mu^2$. This means that inside an ellipse with foci $-4k^2, 0$ cutting the real axis at $t = \mu^2$ a Legendre expansion of T is convergent and that outside the ellipse this expansion diverges. This means ⁶⁾:

$$\lim_{l \rightarrow \infty} \left| \int_{-4k^2}^0 T(t) P_l \left(1 + \frac{t}{2k^2} \right) \frac{dt}{2k^2} \right|^{1/l} = h \quad (12)$$

where h is a number such that

$$\frac{1}{2} \left(h + \frac{1}{h} \right) = 1 + \frac{\mu^2}{2k^2} \quad (13)$$

$$0 < h < 1$$

Equation (13) fixes μ . Let us now check that the whole analytic behaviour is correct. For this we shall make a conformal mapping of the regularity domain inside the unit circle ^{*)}. A possible mapping is

$$t = \frac{4\mu^2 Z}{(Z+1)^2} \quad (\text{see fig. 2}) \quad (14)$$

The upper (lower) half of the circle corresponds to the upper (lower) lip of the cut. $Z=1$ corresponds to $t = \mu^2$; $Z=-1$ corresponds to $t = \infty$.

*) A similar operation has been suggested independently by W. Frazer in a different connection. ⁷⁾

The physical domain lies on the real Z axis from

$$Z_{\min} = -\left(1 + \frac{\mu^2}{2k^2}\right) + \sqrt{\left(1 + \frac{\mu^2}{2k^2}\right)^2 - 1} \quad (t = -4k^2) \text{ to } Z=0 (t=0)$$

Then the condition to impose on $T\left(\frac{4\mu^2 Z}{(Z+1)^2}\right)$ is that the power series expansion of T around $Z=0$ has a radius of convergence $|Z| = 1$, namely, with

$$T = \sum a_n Z^n \quad (15)$$

$$\overline{\lim}(a_n)^{1/n} = 1$$

Of course this power series expansion must fit the values of T in the whole physical region.

These conditions are necessary but not sufficient. For instance, nothing has been said about the fact that $T(t)$ should not increase stronger than a polynomial at infinity. Even so it is not obvious that the iterative procedure for the construction of $C(\alpha)$ will give an acceptable potential. It might happen that $C(\alpha)$ increases too fast as $\alpha \rightarrow \infty$.

fig. 2

V. Extension to exchange forces

When exchange forces are present, with the same minimum range μ , T , for fixed energy has two cuts. Using the variable $\cos \theta$ instead of t we get two symmetrical cuts starting at

$$\cos \theta = \pm \left(1 + \frac{\mu^2}{2k^2}\right) \quad (\text{fig. 2})$$

it is useful to introduce variables

$$\begin{aligned} t &= -2k^2(1 - \cos \theta) \\ \bar{t} &= -2k^2(1 + \cos \theta) \end{aligned} \quad (16)$$

The right hand cut starts at $t = \mu^2$, while the left hand cut starts at $\bar{t} = \mu^2$ (3). Extending our preceding reasoning to this case we see that the knowledge of $T(\cos \theta)$ in the physical domain determines uniquely both discontinuities across the two cuts for potentials of the type

$$\int_{\mu}^{\infty} \frac{[C_D(\alpha) + PC_E(\alpha)] e^{-\alpha r_d}}{r} d\alpha$$

(P = Majorana exchange operator) .

One can write T as (3), 8):

$$T = \sum_{i=1}^n (T_{iD}(t) + T_{iE}(\bar{t})) + R_n \quad (17)$$

where the exchange potential appears an even (odd) number of times in T_{iD} (T_{iE}). T_{iD} as a function of t has a single cut from $t = \mu^2$ to $t = \infty$.

T_{iE} has the same properties provided one replaces t by $\bar{t}$. Again, one can show that R_n is regular inside an ellipse with foci $\cos \theta = \pm 1$ cutting the real axis at $\cos \theta = \pm \left[1 + \frac{(n+1)^2 \mu^2}{2k^2} \right]$. The argument concerning the additivity of the ranges of the potentials entering in a matrix element of the Born series still holds. Then if the contribution to the potential

$$\int_{\mu}^{n\mu} \frac{C_D(\alpha) + PC_E(\alpha) e^{-\alpha r}}{r} d\alpha$$

is known the discontinuity in the region

$$1 + \frac{n^2 \mu^2}{2k^2} < \cos \theta < 1 + \frac{(n+1)^2 \mu^2}{2k^2}$$

consists of known terms plus the contribution of

$$\int_{n\mu}^{(n+1)\mu} \frac{C_D(\alpha) e^{-\alpha r}}{r} d\alpha$$

to the first Born approximation; in the region

$$- \left[1 + \frac{(n+1)^2 \mu^2}{2k^2} \right] < \cos \theta < - \left[1 + \frac{n^2 \mu^2}{2k^2} \right]$$

the discontinuity also consists of known terms plus the contribution of

$$P \int_{n\mu}^{(n+1)\mu} \frac{C_E(\alpha) e^{-\alpha r}}{r} d\alpha$$

to the first Born approximation. So the iteration procedure to determine $C_D(\alpha)$ and $C_E(\alpha)$ still works.

To look at the necessary conditions to impose on the scattering amplitude one can make the mapping of the cut plane on the unit circle:

$$\frac{\cos \theta}{1 + \frac{M^2}{2k^2}} = \frac{2Z}{Z^2 + 1} \quad (18)$$

Here the branch points correspond to $Z = \pm 1$, the physical domain becomes a symmetrical segment on the real Z axis, and $t = \infty$ (with $\text{Im } t \geq 0$) corresponds to $Z = \pm i$.

VI. Concluding remarks

We have solved the problem "in principle" of determining the potential from the scattering amplitude at a given energy when it is known that this potential (with or without exchange forces) is a superposition of Yukawa potentials. The statement that the discontinuity across the cuts is determined by the scattering amplitude is only valid when the scattering amplitude is known with infinite accuracy; to make this statement useful in practical cases one has to have some a priori idea of the maximum number of subtractions in the integral representation of the scattering amplitude. A systematic procedure to get the potential from the discontinuity of the scattering amplitude for fixed energy has been given. This is in a sense a generalisation of the treatment of Fubini and Charap⁵⁾ which was restricted to low values of t ; but from another point of view our treatment is much less deep than the one of Charap and Fubini who show, on a field theoretical basis, the energy-independence of the potential obtained in this way.

One might be tempted to apply our remarks to other objects having the same analytic properties as scattering amplitudes. This is assumed to be the case for the electromagnetic form factors of the nucleon. One then gets the curious result that knowing exactly a form factor in a limited range of values of the momentum transfer one knows it for any value of the momentum transfer. Unfortunately this statement is almost useless because of the limited accuracy of the experiments.

R E F E R E N C E S

- 1) R. Jost and W. Kohn, Phys. Rev. 87, 979 (1952)
(this contains reference to previous work)
- 2) T. Regge, Nuovo Cim. 14, 591 (1959)
- 3) See for instance J. Bowcock and A. Martin, Nuovo Cim. 14, 516 (1959)
- 4) R. Blankenbecler, M. Goldberger, N. Khuri and S. Treiman,
Annals of Physics, 10, 62 (1960)
A. Klein, Journal of Math. Phys. 1, 41 (1960)
- 5) J. Charap and S. Fubini, Nuovo Cim. 14, 540 (1959) and
15, 73 (1960)
- 6) See for instance A. Martin, Analytic properties of the deuteron
photodisintegration matrix element for fixed energy (Appendix)
Nuovo Cimento 19, 344 (1961)
- 7) W. Frazer, Bulletin of the American Physical Society Ser. II,
Vo. 6, 81 (1961)
- 8) J. Hamilton, Phys. Rev. 114, 1170 (1959)