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Double dispersion relations for deuteron photo- and
electrodisintegration.

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A B S T R A C T

The deuteron photodisintegration matrix element, which exhibits, in the relativistic case, anomalous thresholds is studied in the non-relativistic case, where the nucleons interact through a supersposition of Yukawa or exponential potentials. For simplicity all spins are taken to be zero. A double dispersion relation is derived, in agreement with the latest results of Eden et al. The treatment is easily extended to electrodisintegration.

I. Introduction

Some interest has been recently devoted to the analytic properties of the deuteron photodisintegration matrix element, because the weakly bounded character of the deuteron is expected to give rise to "anomalous threshold" ^{1), 2), 3)}. Indeed, as was shown by de Alfaro and Rossetti ⁴⁾, anomalous thresholds appear in a perturbation field theoretical treatment where the deuteron is regarded as an elementary particle; a Mandelstam representation may be written for the graphs containing at most one meson line with branch points given by the anomalous threshold. De Alfaro and Rossetti have also shown that these features appear in a non-relativistic treatment of the deuteron photodisintegration at least for a given partial wave; in the same way one of us (A.M.) has shown that for fixed real energy the analytic properties with respect to the angle are those predicted by the above mentioned Mandelstam representation ⁵⁾.

We wish to derive in the present paper double dispersion relations for the matrix element under consideration in the non-relativistic framework. This is obtained by containing the analytic properties for fixed momentum transfer with the analytic properties for fixed, almost real, energy. It turns out that it is not difficult to extend these results to off energy shell photodisintegration, i.e. electrodisintegration.

II. The matrix element and its perturbation expansion with respect to the final state interaction.

As indicated in ref. ⁵⁾ the non-relativistic matrix element for deuteron photodisintegration may be written in momentum space as

$$M(k, q, \cos \theta) = \int \psi_q^* (\underline{p}) \psi_D(\underline{p} - \frac{\underline{k}}{2}) d^3 p$$

where q is the relative momentum of the final nucleons, k the momentum of the incident photon; k and q are related by the law of conservation of energy. ψ_q is the ingoing final wave function, ψ_D the deuteron wave function. The spins of all particles including the photon have been taken to be zero for simplicity.

For a neutron-proton potential of the type

$$V(r) = \int_{\mu}^{\infty} c(\alpha) e^{-\alpha r} d\alpha, \text{ with } c(\alpha) < M \alpha^{1-\epsilon}$$

it has been shown by one of the authors (A.M.)⁶⁾ (and independently by Blankenbecler and Nambu²⁾) that the deuteron wave function may be written

$$\psi_D(\underline{p} - \frac{\underline{k}}{2}) = \int_0^{\infty} \frac{\rho(\sigma) d\sigma}{(\underline{p} - \frac{\underline{k}}{2})^2 + \sigma^2}$$

with

$$\rho(\sigma) = \frac{1}{N} \int d(\sigma - \alpha) + f(\sigma) e[\sigma - \alpha - \mu],$$

$\alpha^2 = MB$ (M nucleon mass, B binding energy of the deuteron)

and $f(\sigma)$ is $\frac{\text{const.}}{\sigma^{1+\epsilon}}$ for $\sigma \rightarrow \infty$.

A typical term of the integral representation of ψ_D will be $\frac{1}{(\underline{p} - \frac{\underline{k}}{2})^2 + \sigma^2}$. So we shall first investigate the analytic properties of the quantity

$$M(k, q, \cos \theta, \sigma) = \int \frac{d^3 p}{(\underline{p} - \frac{\underline{k}}{2})^2 + \sigma^2} \psi_q^*(p)$$

Then, as $\rho(\sigma)$ decreases rapidly enough at infinity we shall be able to obtain, by superposition, the analytic properties of $M(k, q, \cos \theta)$.

We shall now expand the final wave function in a perturbation series with respect to the final state interaction. Then

$$M(k, q, \cos \theta, \sigma) = \frac{1}{(\underline{q} - \frac{\underline{k}}{2})^2 + \sigma^2} + \sum_{n=1}^{\infty} M_n(k, q, \cos \theta, \sigma)$$

with restricting ourselves for simplicity to the case of a pure Yukawa potential (*):

$$M_n(k, q, \cos \theta, \sigma) =$$

$$\int \frac{1}{(\underline{p}_1 - \frac{\underline{k}}{2})^2 + \sigma^2} \frac{1}{p_1^2 - q^2 - i\epsilon} \frac{1}{(\underline{p}_1 - \underline{p}_2)^2 + \mu^2} \frac{1}{p_2^2 - q^2 - i\epsilon} \cdots \frac{1}{(\underline{p}_n - \underline{q})^2 + \mu^2} \times \\ \times d^3 p_1 \cdots d^3 p_n$$

Except for the fact that the first μ is replaced by σ , M_n looks very much like an off energy shell n^{th} order scattering matrix element, with initial momentum $\underline{q}$ and final momentum $\underline{k}/2$. It is convenient to introduce the quantity $\Delta^2 = (\underline{q} - \frac{\underline{k}}{2})^2$ which will be taken to be, by

(*) The extension to a more general case is very easy

definition, the square of the momentum transfer. Actually it may be shown that except for additive and multiplicative constants the square of the four dimensional momentum transfer for the process $\gamma + D \rightarrow N + P$ reduces to Δ^2 in the non-relativistic limit ⁴⁾. Taking into account the existence of a definite relationship between q and k we may write

$$M_n(q, k, \cos \theta, \sigma) = F_n(q^2, \Delta^2, \sigma) = f_n(k, \Delta^2, \sigma).$$

III. Parametric representation of the n^{th} order matrix element.

In this section we closely follow the calculation of Bowcock and Walecka ⁷⁾ who derive, by Feynman's technique, a parametric representation of the scattering matrix element, the following result can be established

$$F_n(q^2, \Delta^2, \sigma) = \left(\frac{-1}{2\sqrt{\pi}} \right)^n \Gamma\left(\frac{n}{2} + 1\right) \int_0^1 dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{2n-1}} dx_{2n} \times$$

$$\times \frac{1}{(a_1 \dots a_n)^{3/2} (D_n)^{n/2+1}},$$

where

$$D_n = \left(\alpha + \frac{\gamma}{2}\right) q^2 + \left(\beta + \frac{\gamma}{2}\right) \frac{k^2}{4} - \frac{\gamma}{2} \Delta^2 + \mu^2 (x_n - x_{2n}) + \sigma^2 x_{2n}$$

All the quantities depend on the $2n$ parametric variables and are given explicitly in appendix I. $(a_1 a_2 \dots a_n)$ vanishes on the border of the integration domain but this singularity of the integrant may be shown

to be harmless. To study D_n we shall use the following properties of D_n derived in Appendix I:

$$1) \quad -(1-x_n) \left\langle \alpha + \frac{\gamma}{2} \right\rangle < -\frac{(1-x_n)^2}{2}; \quad 0 < \beta + \gamma/2 < x_{2n};$$

$$-(1-x_n) \left\langle \alpha + \beta + \gamma \right\rangle < -\frac{(1-x_n)^2}{2}; \quad \gamma < 0$$

$$2) \quad \frac{2}{|\gamma|} \left[\mu^2(x_n - x_{2n}) + \sigma^2 x_{2n} \right] \geq (n_M + \sigma)^2$$

IV. Analytic properties for fixed momentum transfer $\Delta^2_{\text{real}} > [n_M + \sigma]^2$.

Here, we have to make a choice for the kinematical relation between k and q . Since the non-relativistic approximation has been made, and since the dynamical recoil effects in the photon absorption have been neglected, there is some arbitrariness in this choice. We have considered three cases

a) non-relativistic kinematics, neglecting deuteron recoil

$$q^2 = Mk - MB$$

b) non-relativistic kinematics with deuteron recoil

$$q^2 = \frac{k^2}{4} \cdot \frac{2M}{M_D} + Mk - MB > \frac{k^2}{4} + Mk - MB$$

c) relativistic kinematics

$$k^2 = \frac{[4(q^2 + M^2) - M_D^2]^2}{16 [q^2 + M^2]}$$

All these formulae approximately coincide in the low energy region but they begin to differ strongly from each other for $|q|^2 \sim M^2$. Out of these we shall choose the one which permits the simplest derivation of the desired properties.

Case a):

$$D_n = (\alpha + \frac{\gamma}{2})Mk + (\beta + \frac{\gamma}{2}) \frac{k^2}{4} - \frac{\gamma}{2} \Delta^2 + \mu^2(x_n - x_{2n}) + x_{2n} \sigma^2 - MB(\alpha + \frac{\gamma}{2}) - \frac{\gamma}{2} \Delta^2 + \mu^2(x_n - x_{2n}) + x_{2n} \sigma^2 - MB(\alpha + \frac{\gamma}{2}) > 0$$

for $\Delta^2 \text{ real} > - (n\mu + \sigma)^2$.

The possible singularities are given by $D_n = 0$. Assume first that D_n for fixed values of the parametric variables, has two real roots in k ; then from the signs of the coefficients they are both positive, and further, since $D_n = (\alpha + \frac{\gamma}{2})q^2 + (\beta + \frac{\gamma}{2}) \frac{k^2}{4} + \text{positive quantity}$, they both correspond to real positive values of q^2 . Unfortunately, it is not possible to guarantee that the roots are real because the discriminant is not always positive. If the roots are complex, their common real part is given by

$$\text{Re } k = -2M \frac{\alpha + \gamma/2}{\beta + \gamma/2} \gg 2M$$

so we get a minimum analyticity domain given by fig. 1a. May be one could enlarge this domain. However, the branch point $q^2 = 0$ may be shown to persist; it is an end point singularity in the sense of Eden⁸⁾.

Case b):

Here a slight complication arises because we want to obtain, eventually, analytic properties with respect to the variable q^2 : to a given value of q^2 there corresponds two values of k . However, D_n is a well defined function of k :

$$D_n = \left[\left(\alpha + \frac{\gamma}{2} \right) \frac{2M}{M_D} + \left(\beta + \frac{\gamma}{2} \right) \right] \frac{k^2}{4} + Mk \left(\alpha + \frac{\gamma}{2} \right) + \text{positive quantity}$$

noting that $\frac{2M}{M_D} > 1$, and, hence that $\left(\alpha + \frac{\gamma}{2} \right) \frac{2M}{M_D} + \left(\beta + \frac{\gamma}{2} \right) < 0$, we easily see that D_n , as a function of k , has two real roots with opposite signs. Since, on the other hand, $D_n = \left(\alpha + \frac{\gamma}{2} \right) q^2 + \left(\beta + \frac{\gamma}{2} \right) \frac{k^2}{4} + \text{positive quantity}$, they both correspond to positive values of q^2 . We get two cuts in the k plane:

$$k_1 \longrightarrow +\infty, \quad -\infty \longrightarrow k_2,$$

with

$$k_1 = M_D \left[-1 + \sqrt{1 + \frac{2B}{M_D}} \right] > 0$$

$$k_2 = M_D \left[-1 - \sqrt{1 + \frac{2B}{M_D}} \right] < -2M_D$$

(see fig. 2)

Let us now come back to the q^2 plane. We notice that from the relation

$$\frac{k^2}{4} \frac{2M}{M_D} + Mk - MB - q^2 = 0$$

we get that the sum $k' + k''$ of the two values of k corresponding to a given q^2 is equal to $-2M_D$. One of the roots is to the right of the

line $k = -M_D + i\lambda$ (λ real), the other one to left. The line $k = -M_D + i\lambda$ correspond to $q^2 = -M \left[\frac{M_D}{2} + B \right] - \lambda^2 \frac{M}{2M_D} = q_0^2 - \lambda^2 \frac{M}{2M_D}$.

Therefore, to make the correspondence $q^2 \longleftrightarrow k$ a one to one correspondence we have to make a cut in the q^2 plane and select the value of k connected with the physical region, i.e. the one with the largest real part.

$F_n(q^2, \Delta^2, \sigma)$ for Δ^2 real $\rangle - (n\mu + \sigma)^2$ has therefore two cuts, one from 0 to $+\infty$ and the other one from $-\infty$ to $q_0^2 \leq -M^2$ and no other singularity (fig. 2).

Case c):

Here D_n can be expressed as a function of q^2

$$D_n = \left(\alpha + \frac{\gamma}{2}\right) q^2 + \left(\beta + \frac{\gamma}{2}\right) \frac{\left[4(q^2 + M^2) - M_D^2\right]^2}{64(q^2 + M^2)} + \text{positive quantity}$$

One easily sees that D_n vanishes for two values of q^2 , one which is positive and the other one which is less than $-M^2$. So the analyticity domain in q^2 is exactly the same as in case b). (except for the replacement of $q_0^2 = -\frac{MM_D}{2} - MB$ by $-M^2$).

If we now compare these three cases, we see that the analytic properties for fixed momentum transfer in the region $-M^2 < \text{Re } q^2 < 2M^2$ do not depend on the details of the kinematics which is rather encouraging.

We shall now devote our attention to case b), because it is the one where it is easiest to get double dispersion relations. Indeed, we notice that in the k plane k^2 and q^2 are both real positive along the two cuts. for k^2 and q^2 real positive we know the analytic properties in Δ^2 . From a preceding paper ⁵⁾ and, as will be seen these properties may be extended to the case when q^2 and k^2 have a small imaginary part.

On the other hand one may prove (Appendix I) that for Δ^2 real $\rangle - (n\mu + \sigma)^2 + \varepsilon$ D_n is larger than $\left| \frac{\text{Im } k}{k} \right| \left[\frac{\varepsilon}{[n\mu + \sigma]^2} \sigma^2 x_n + \frac{(1-x_n)^2}{2} |k|^2 \right]$,

and from this one can deduce that we do not need any subtraction in the fixed momentum transfer dispersion relations.

V. Analyticity in Δ^2 for q^2 and k^2 nearly real, positive.

It has already been shown in ref. 5) that for q^2 and k^2 real and positive $M_k(k, q, \cos \theta, \sigma) = m_n(k^2, q^2, \Delta^2, \sigma)$ has the following analytic properties in Δ^2 : it has a cut from $-\infty$ to $-\left[n\mu + \sigma\right]^2$ and no other singularity. We want to extend in some way this result to the case where k^2 and q^2 have small imaginary parts (independent of each other). The following may be proved (Appendix II):

Provided the phases of k^2 and q^2 are such that

$$\cos \left[\text{Arg } k^2 \right] \geq \frac{|k|^2 - 4\sigma^2}{|k|^2 + 4\sigma^2} + \varepsilon$$

, $\varepsilon > 0$

$$\cos \left[\text{Arg } q^2 \right] \geq \frac{|q|^2 - \mu^2}{|q|^2 + \mu^2} + \varepsilon$$

One can show that m_n is holomorphic in Δ^2 outside a domain surrounding the cut which reduces to the cut as the arguments of k^2 and q^2 both go to zero (fig. 3). This result could also be used to improve the rigor of the known proofs of Mandelstam representation for potential scattering.

VI. Double dispersion relations for non-relativistic kinematics with deuteron recoil.

Let us first consider the matrix element as a function of k and Δ^2 . For $\Delta^2_{\text{real}} > -(nM + \sigma)^2$ we may write a dispersion relation in k , following the contour Γ_k . The behaviour at infinity is such that this contour integral reduces to the sum of the integrals around the two cuts.

So:

$$f_n(k, \Delta^2, \sigma) = \frac{1}{\pi} \oint_{\text{positive cut}} \frac{f_n(k', \Delta^2, \sigma) dk'}{k' - k} + \frac{1}{\pi} \oint_{\text{negative cut}} \frac{f_n(k', \Delta^2, \sigma) dk'}{k' - k}$$

if the path of integration is close enough to the cuts we may use the results of section V and write

$$f_n(k, \Delta^2, \sigma) = \frac{1}{\pi^2} \oint_{\text{positive cut}} dk' \oint_C d\Delta'^2 \frac{f_n(k', \Delta'^2, \sigma)}{(k' - k)(\Delta'^2 - \Delta^2)} + \frac{1}{\pi^2} \oint_{\text{negative cut}} dk' \oint_C d\Delta'^2 \frac{f_n(k', \Delta'^2, \sigma)}{(k' - k)(\Delta'^2 - \Delta^2)}$$

where C is indicated on fig. 3. The paths of integration can be taken arbitrarily close to the cuts. We define in this way an analytic function of k and Δ^2 in the product of the cut planes. Let us now restrict ourselves to $\text{Re} k > -M_D$. Then a one to one correspondence between k and q^2 may be established and $\frac{dk}{dq^2}$ is well defined and exists everywhere. Hence we deduce that $F_n(q^2, \Delta^2, \sigma)$ is holomorphic in the product of the

Δ^2 cut plane (cut: $-\infty \rightarrow -(\mu+\sigma)^2$) and the q^2 cut plane (cuts: $-\infty \rightarrow +q_0^2 \simeq -M^2$, $0 \rightarrow +\infty$). Finally, assuming that no subtraction is needed as $|\Delta|^2 \rightarrow \infty$ ^{*)}, we may write

$$F_n(q^2, \Delta^2, \sigma) = \int_{-\infty}^{-(\mu+\sigma)^2} d\Delta'^2 \int_0^{\infty} dq'^2 \frac{\rho(q'^2, \Delta'^2)}{(\Delta'^2 - \Delta^2)(q'^2 - q^2)} + \int_{-\infty}^{-(\mu+\sigma)^2} d\Delta'^2 \int_{-\infty}^{-M[\frac{M_D}{2}+B]} dq'^2 \frac{\rho(q'^2, \Delta'^2)}{(\Delta'^2 - \Delta^2)(q'^2 - q^2)}$$

So we get a Mandelstam representation containing an additional contribution from the left hand cut in q^2 . In our opinion this additional contribution is physically meaningless because it corresponds to values of $|q|^2$ outside the non-relativistic region. The second term should be regarded, in fact, as a function of Δ^2 alone, because its dependence on q^2 in the non-relativistic region is negligible. One might wonder, on the other hand whether it is possible to make this spurious cut disappear by multiplying $F_n(q^2, \Delta^2, \sigma)$ by some convenient factor, say $\sqrt{q^2 + M[\frac{M_D}{2}+B]}$; we were not able to do this but it might be so. The weight functions appearing in the representation have to be real because for real Δ^2 , $F_n(q^2, \Delta^2, \sigma)$ and $F_n(q^{2*}, \Delta^2, \sigma^2)$ are imaginary conjugate.

If we disregard the problem of convergence of the Born series. ⁺⁾ , we may sum over n and integrate over σ . We then get for

^{*)} This was shown to be the case for smooth superpositions of Yukawa potentials in ref. 5).

⁺⁾ Notice that here, as in the case of scattering ⁹⁾ one can prove the convergence of the Born series for fixed real $\Delta^2 > -(\mu+\sigma)^2$,

$\left| \frac{\text{Im } k}{k} \right| > \epsilon$, and $|k|$ large enough.

the complete matrix element:

$$F(q^2, \Delta^2) = \frac{1}{N(\alpha^2 + \Delta^2)}$$

$$+ \int_{-\infty}^{-(\mu + \alpha)^2} d\Delta'^2 \int_0^{\infty} dq'^2 \frac{\rho(\Delta'^2, q'^2)}{(\Delta'^2 - \Delta^2)(q'^2 - q^2)}$$

$$+ \int_{-\infty}^{-(\mu + \alpha)^2} d\Delta'^2 \int_{-\infty}^{-M_D^2/2} dq'^2 \frac{\rho(\Delta'^2, q'^2)}{(\Delta'^2 - \Delta^2)(q'^2 - q^2)}$$

If we compare this formula with the Mandelstam representation proposed by de Alfaro and Rossetti ⁴⁾ we see that apart from the last term (which in fact only depends on Δ^2) and from the deuteron pole $\frac{1}{q^2 + \alpha^2}$, they coincide. The absence of the pole in the q^2 variable is due to the expansion in power series of final wave function. One can make it appear by replacing the power series expansion by the Fredholm expansion of the final wave function as done for instance by Jost and Pais ¹⁰⁾.

The Fredholm expression may be put in the form:

$$\frac{\sum_n F'_n(q^2, \Delta^2)}{D(q^2)}$$

The only singularities of $D(q^2)$ in the q^2 plane are the bound states of the N-P potential plus a cut from 0 to $+\infty$. $F'_n(q^2, \Delta^2)$ is a sum of the n first terms of the Born expansion with coefficients

depending only on q^2 holomorphic, bound in the q^2 plane. The Fredholm numerator is absolutely convergent at least for physical values of q^2 and Δ^2 $((q - \frac{k}{2})^2 < \Delta^2 < (q + \frac{k}{2})^2)$. If we assume that the convergence property persists for non-physical values of q^2, Δ^2 , we can extract the deuteron pole $q^2 = -\alpha^2$ coming from the vanishing of $D(q^2)$ and write

$$F(q^2, \Delta^2) = \frac{1}{N(\alpha^2 + \Delta^2)} + \frac{\lambda}{\alpha^2 + q^2} + \int_{-\infty}^{-(\mu + \alpha)^2} d\Delta'^2 \int_0^{\infty} dq'^2 \frac{\rho(\Delta'^2, q'^2)}{(q'^2 - q^2)(\Delta'^2 - \Delta^2)} + \int_{-\infty}^{-(\mu + \alpha)^2} d\Delta'^2 \int_{-\infty}^{-M(\frac{M_D}{2} + B)} dq'^2 \frac{\rho(\Delta'^2, q'^2)}{(q'^2 - q^2)(\Delta'^2 - \Delta^2)}$$

It should be noted that the analytic properties of partial waves obtained from this formula are exactly those derived by de Alfaro and Rossetti, in spite of the existence of the last term, because the cut $-\infty \rightarrow -M\left[\frac{M_D}{2} + B\right]$ is "hidden" by the cut $-\infty \rightarrow -(\mu + \alpha)^2$. The fixed angle dispersion relations would only be modified for strongly forward angles such that $(1 - \cos \theta) < \frac{(\mu + \alpha)^2}{M^2}$.

VII. Extension of the results to electrodisintegration.

Since we have already dealt with complicated kinematical relations we should expect that it is possible to still complicate slightly the problem by dealing with a virtual photon which appears for instance in the processes $e + D \rightarrow e + N + P$ (fig. 4) or $N + P \rightarrow D + e^+ + e^-$. In the first case (electrodisintegration) the "photon mass" is negative; in the second case it is positive. It turns out that it is possible to write double dispersion relations for fixed negative mass of the photon, but not for fixed positive mass.

Let us consider the case of electrodisintegration. Since we are dealing with a non-relativistic $\chi + D \rightarrow N + P$ matrix element we have to use a non-covariant formalism where 3-momentum and mass are conserved in intermediate states. Hence $k = |p - p'|$ and the conservation of energy gives

$$\frac{q^2}{M} = \frac{k^2}{2M_D} + E_p - E_{p'} - B = \frac{k^2}{2M_D} + \delta - B$$

the "photon mass" which would appear in a covariant treatment is

$$\left(E_p - E_{p'} \right)^2 - |p - p'|^2 = \delta^2 - k^2 = -\lambda^2 < 0$$

The physical situation corresponds to $\delta > 0$. For fixed λ^2 :

$$q^2 = \frac{\delta^2}{4} \frac{2M}{M_D} + \delta M + \lambda^2 \frac{M}{2M_D} - MB$$

the matrix element can be considered as a function of δ and Δ^2 (defined as in photodisintegration) or as a function of q^2 provided

one makes a choice between the two possible corresponding values of δ . Obviously, one should choose $\text{Re } \delta > -M_D$. One can proceed exactly as in photodisintegration. One first looks at the analytic properties in δ for fixed real $\Delta^2 > -[n\mu + \sigma]^2$:

$$D_n = \left[\left(\alpha + \frac{\gamma}{2} \right) \frac{2M}{M_D} + \left(\beta + \frac{\gamma}{2} \right) \right] \frac{\delta^2}{4} + M \delta \left[\alpha + \frac{\gamma}{2} \right] + \\ + \left[\left(\beta + \frac{\gamma}{2} \right) + \left(\alpha + \frac{\gamma}{2} \right) \frac{2M}{2M_D} \right] \frac{\lambda^2}{4} + \left| \frac{\gamma}{2} \right| \Delta^2 + \mu^2 \left[x_n - x_{2n} \right] + \sigma^2 x_{2n}$$

The discriminant of D_n reads

$$M^2 \left[\alpha + \frac{\gamma}{2} \right]^2 - \left(\frac{M}{M_D} \right)^2 \lambda^2 \left[\alpha + \frac{\gamma}{2} + \frac{M_D}{2M} \left(\beta + \frac{\gamma}{2} \right) \right]^2 + \text{positive quantity}$$

For $\lambda^2 < M_D^2$ it is positive and D_n vanishes for two real values of δ . These two real values of δ correspond to positive values of $k^2 (\lambda^2 > 0)$ and eventually to positive values of q^2 . One of the roots in δ is less than $-M_D$, the other one is larger than $-M_D$. Therefore one can write double dispersion relation in δ and Δ^2 and then in q^2 and Δ^2 . The cut in the Δ^2 plane is the same as in photodisintegration. The cuts in the q^2 plane are $-\infty \rightarrow -\frac{MM_D}{2} + \lambda^2 \frac{M}{2M_D} - MB$ and $0 \rightarrow +\infty$. These two cuts do not overlap for $0 < \lambda^2 < M_D^2$. We should repeat here that the left hand cut in the q^2 plane has no real physical significance and that, on the other hand, one should not expect our treatment to be valid for values of λ^2 as large as M^2 . In the case $\lambda^2 < 0$, which would happen in $N+P \rightarrow D+e^+e^-$, the difficulty which cannot be overcome is that k^2 is not always positive for real δ^2 : one can show that for a parameter x_n small enough one of the roots of D_n is smaller than $|\lambda|$.

VIII. Concluding remarks.

We have obtained double dispersion relations for photo- and electrodisintegration of the deuteron (the latter provided the 4-momentum transfer of the electrons is not exceedingly large). These double dispersion relations coincide with the Mandelstam representation if one disregards a spurious left hand cut starting at $q^2 \simeq -M^2$, far beyond the domain of validity of a non-relativistic approach. At the time this work is completed we understand that it had to be so, because in spite of the presence of anomalous thresholds the Mandelstam representation is valid for the fourth order terms obtained in a field-theoretical treatment of deuteron photodisintegration ⁴⁾. According to R.J. Eden, Landshoff, Polkinghorne and Taylor ¹¹⁾ this ensures that it will be valid in all orders of perturbation theory.

Our result concerning electrodisintegration as well as the negative result concerning NP radiative capture with emission of Dalitz pairs are in agreement with the general feeling one gets from field theory ¹²⁾. This gives some support to the idea that whenever a Mandelstam representation is assumed for a photo-process one should consistently extend it to the case where the photon is replaced by an electron producing a virtual photon.

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APPENDIX I.

The purpose of this appendix is to show the properties of D_n listed in section III.

First, let us recall the definition of α, β, γ , rewritten here in a slightly different form than in ⁷⁾ and which is more convenient for the following calculations,

$$\alpha = - \sum_{i=1}^n c_i + b_{n+1} \left(1 - \frac{b_{n+1}}{a_n}\right)$$

$$\beta = b_1 - \sum_{i=1}^n a_i \left(\frac{b_1 \dots b_i}{a_1 \dots a_i} \right)^2$$

$$\gamma = -2b_{n+1} \prod_{i=1}^n \frac{b_i}{a_i}$$

with

$$b_1 = x_{2n}, \dots, b_{n+1} = x_n - x_{n+1}$$

$$c_1 = 1 - x_1, \dots, c_n = x_{n-1} - x_n$$

satisfying
the condition:

$$\sum_{i=1}^n c_i + \sum_{i=0}^n b_{i+1} = 1$$

and

$$a_1 = c_1 + b_1 + b_2, \quad a_i = c_i + b_{i+1} + b_i \left(1 - \frac{b_i}{a_{i-1}}\right)$$

I. Miscellaneous properties of the a_i 's and some of their combinations.

In this section, we shall prove some relations satisfied by the a_i 's which will be used in the next sections.

First, from their definition, it is very easy to see by recursion that

$$a) \quad a_i \gg b_{i+1} + c_i \gg 0$$

and a fortiori

$$b) \quad 1 - \frac{b_{i+1}}{a_i} \gg 0$$

Now we want to show:

$$c) \quad 1 - \frac{b_{n+1}}{a_n} - \prod_{i=1}^n \frac{b_i}{a_i} \gg 0$$

Rewriting

$$\begin{aligned} 1 - \frac{b_{n+1}}{a_n} - \prod_{i=1}^n \frac{b_i}{a_i} &= \frac{1}{a_n} \left[a_n^{-b_{n+1}} - \frac{b_1 \dots b_n}{a_1 \dots a_{n-1}} \right] \\ &= \frac{1}{a_n} \left[c_n^{+b_n} \left(1 - \frac{b_n}{a_{n-1}} - \frac{b_1 \dots b_{n-1}}{a_1 \dots a_{n-1}} \right) \right] \end{aligned}$$

(the latter equality is obtained simply by considering the definition of a_n).

Now, if we rearrange the bracket in the same manner, we may obtain immediately by recursion:

$$1 - \frac{b_{n+1}}{a_n} - \prod_{i=1}^n \frac{b_i}{a_i} = \frac{c_n}{a_n} + \frac{b_n}{a_n} \frac{c_{n-1}}{a_{n-1}} + \frac{b_n}{a_n} \frac{b_{n-1}}{a_{n-1}} \frac{c_{n-2}}{a_{n-2}} + \dots + \frac{b_n}{a_n} \dots \frac{b_2}{a_2} \frac{c_1}{a_1}$$

Since this is a series of positive terms, the above mentioned result follows.

II. Properties of α, β, γ .

$$a) \quad \underline{\gamma \leq 0, \beta + \frac{\gamma}{2} \leq x_{2n}}$$

The a_i 's and b_i 's being positive, these results are obvious from the definition of β and γ .

$$b) \quad \underline{\alpha + \frac{\gamma}{2} \geq -(1-x_n)}$$

By definition

$$\alpha + \frac{\gamma}{2} = - \sum_{i=1}^n c_i + b_{n+1} \left(1 - \frac{b_{n+1}}{a_n} - \prod_{i=1}^n \frac{b_i}{a_i} \right)$$

But the last term is positive according to c) of section I and we have:

$$\alpha + \frac{\gamma}{2} \geq - \sum_{i=1}^n c_i = -(1-x_n)$$

$$c) \quad \underline{\beta + \frac{\gamma}{2} \geq 0, \alpha + \beta + \gamma \leq -\frac{(1-x_n)^2}{2}, \alpha + \frac{\gamma}{2} \leq -\frac{(1-x_n)^2}{2}}$$

Let us introduce here $\alpha_n, \beta_n, \gamma_n$ defined by :

$$\alpha(c_i, b_i) = \alpha_n(c_1, \dots, c_n; b_1, \dots, b_{n+1})$$

$$\beta(c_i, b_i) = \beta_n(c_1, \dots, c_n; b_1, \dots, b_{n+1})$$

$$\gamma(c_i, b_i) = \gamma_n(c_1, \dots, c_n; b_1, \dots, b_{n+1})$$

where now b_{n+1} may run from 0 to ∞ , the range of the other b_i 's is kept to be (0,1).

Now it is easy to verify, from the definition of α, β, γ and the results of Section I

$$i) \quad \lim_{b_{n+1} \rightarrow \infty} \left(\beta_n + \frac{\gamma_n}{2} \right) = \beta_{n-1} + \frac{\gamma_{n-1}}{2}$$

$$i') \quad \lim_{b_{n+1} \rightarrow \infty} (\alpha_n + \beta_n + \gamma_n) = \alpha_{n-1} + \beta_{n-1} + \gamma_{n-1}$$

$$ii) \quad \frac{\partial}{\partial b_{n+1}} \left(\beta_n + \frac{\gamma_n}{2} \right) = \prod_{i=1}^n \frac{b_i}{a_i} \left[\prod_{i=1}^n \frac{b_i}{a_i} - \left(1 - \frac{b_{n+1}}{a_n} \right) \right]$$

which is always negative (see Section I of this Appendix)

$$ii') \quad \frac{\partial}{\partial b_{n+1}} (\alpha_n + \beta_n + \gamma_n) = \left(1 - \frac{b_{n+1}}{a_n} - \prod_{i=1}^n \frac{a_i}{b_i} \right)^2 \geq \frac{c_n^2}{a_n^2}$$

$$iii) \quad \beta_1 + \frac{\gamma_1}{2} \geq 0$$

So we may conclude from i), ii), iii) that

$$\beta_n + \frac{\gamma_n}{2} \gg \beta_{n-1} + \frac{\gamma_{n-1}}{2} \dots \gg \beta_1 + \frac{\gamma_1}{2} \gg 0$$

From i'), ii') we have

$$\alpha_n + \beta_n + \gamma_n = \alpha_{n-1} + \beta_{n-1} + \gamma_{n-1} - \int_{b_{n+1}}^{\infty} \frac{\partial}{\partial b_{n+1}} (\alpha_n + \beta_n + \gamma_n) db_{n+1}$$

$$\leq \alpha_{n-1} + \beta_{n-1} + \gamma_{n-1} - \int_{b_{n+1}}^{\infty} \frac{c_n^2}{a_n} db_{n+1} \leq \alpha_{n-1} + \beta_{n-1} + \gamma_{n-1} - \frac{c_n^2}{a_n}$$

(because a_n depends only linearly on b_{n+1})

and by recursion

$$\alpha_n + \beta_n + \gamma_n \leq - \sum_{i=1}^n \frac{c_i^2}{a_i} \leq - \sum_{i=1}^n \frac{c_i^2}{c_i + b_{i+1} + b_i}$$

Now minimizing the last term we may get out:

$$\alpha_n + \beta_n + \gamma_n \leq \frac{- \left(\sum_{i=1}^n c_i \right)^2}{\sum_{i=1}^n (c_i + b_{i+1} + b_i)} \frac{-(1-x_n)^2}{1+x_{n+1}-x_{2n}} \leq \frac{-(1-x_n)^2}{2}$$

and also a fortiori

$$\alpha_n + \frac{\gamma_n}{2} < \frac{-(1-x_n)^2}{2}$$

$$d) \underline{\alpha + \beta + \gamma \geq -(1-x_n)}$$

This is obtained by combining b) and c). Collecting then all the results we have

$$\gamma \leq 0$$

$$-(1-x_n) \leq \alpha + \frac{\gamma}{2} \leq -\frac{(1-x_n)^2}{2}$$

$$0 \leq \beta + \frac{\gamma}{2} \leq x_{2n}$$

$$-(1-x_n) \leq \alpha + \beta + \gamma \leq -\frac{(1-x_n)^2}{2}$$

$$III. \quad \underline{\Gamma = -\frac{2}{|\delta|} \left[\mu^2 (x_n - x_{2n}) + \sigma^2 x_{2n} \right]}.$$

Rewriting Γ in terms of the parameters b_i 's we get

$$\Gamma = \frac{-2}{|\delta|} \left(\mu^2 \sum_{i=1}^n b_{i+1} + \sigma^2 b_1 \right) \geq -\sum_{i=0}^n \frac{1}{b_{i+1}} \left[\mu^2 \sum_{i=1}^n b_{i+1} + \sigma^2 b_2 \right],$$

because $\frac{2}{|\delta|} \leq \sum_{i=0}^n \frac{1}{b_{i+1}}$ (see Appendix I of ref. 7) or

$$\Gamma \geq -\mu^2 \sum_{i=1}^n \frac{b_{i+1}}{b_1} - \sigma^2 - \left(\mu^2 \sum_{i=1}^n b_{i+1} + \sigma^2 b_1 \right) \sum_{i=1}^n \frac{1}{b_{i+1}}$$

Keeping first $\sum_{i=1}^n b_{i+1}$ and b_1 fixed, we may easily minimize this expression with respect to $b_2, \dots, b_{n+1}$ this gives

$$\sum_{i=1}^n \frac{1}{b_{i+1}} \leq \frac{n^2}{\sum_{i=1}^n b_{i+1}}$$

and

$$r \geq -\sigma^2 - \mu^2 n^2 - \mu^2 \sum_{i=1}^n \frac{b_{i+1}}{b_1} - n^2 \sigma^2 \frac{b_1}{\sum_{i=1}^n b_{i+1}}$$

Minimizing now with respect to $\sum_{i=1}^n \frac{b_{i+1}}{b_1}$ we obtain

$$r \geq -(n\mu + \sigma)^2$$

so that

$$\frac{|\gamma|}{2} \Delta^2 + \mu^2 (x_n - x_{2n}) + \sigma^2 x_{2n}$$

is always positive for $\Delta^2 \geq -(n\mu + \sigma)^2$.

IV. Lower limit of D_n and behaviour of f_n as $k^2 \rightarrow \infty$

Let us first recall expression of D_n as it is given in Section IV

$$D_n = A \frac{k^2}{4} + M(\alpha + \frac{\gamma}{2})k + C$$

with

$$A = (\alpha + \frac{\gamma}{2}) \frac{2M}{M_D} + \beta + \frac{\gamma}{2} < \alpha + \beta + \gamma$$

$$C = \frac{|\gamma|}{2} \Delta^2 + \mu^2(x_n - x_{2n}) + \sigma^2 x_{2n} - MB(\alpha + \frac{\gamma}{2})$$

When k is allowed to be complex we have:

$$\text{Re } D_n = -A \frac{|k|^2}{4} + \left[\frac{A}{4} 2\text{Re } k + M(\alpha + \frac{\gamma}{2}) \right] \text{Re } k + C$$

$$\text{Im } D_n = \left[\frac{A}{4} 2\text{Re } k + M(\alpha + \frac{\gamma}{2}) \right] \text{Im } k$$

and

$$|D_n|^2 = |k|^2 \left(Z + \frac{\text{Re } k}{|k|^2} Y \right)^2 + Y^2 \left| \frac{\text{Im } k}{k} \right|^2$$

with

$$Y = C - A \frac{|k|^2}{4}$$

$$Z = \frac{A}{2} \text{Re } k + M(\alpha + \frac{\gamma}{2})$$

so that

$$|D_n| \geq \left| \frac{\text{Im } k}{k} \right| \left(c - A \frac{|k|^2}{4} \right) > \left| \frac{\text{Im } k}{k} \right| \left[c - (\alpha + \beta + \gamma) \frac{|k|^2}{4} \right].$$

Now we have seen that $-(\alpha + \beta + \gamma) > \frac{(1-x_n)^2}{2}$ and it may be shown easily that

$$c > \frac{x_n \varepsilon \sigma^2}{(n\mu + \sigma)^2} \quad \text{if} \quad \Delta^2 - (n\mu + \sigma)^2 + \varepsilon$$

Finally

$$|D_n| > \left| \frac{\text{Im } k}{k} \right| \left[\frac{\varepsilon \sigma^2}{(n\mu + \sigma)^2} x_n + \frac{(1-x_n)^2}{8} |k|^2 \right]$$

On the other hand one may notice that the singularities occurring in the integrant due to the factor $(a_1, a_2, \dots, a_n)^{3/2}$ are harmless. Indeed one may easily see that

$$a_1 a_2 \dots a_n > (b_1 + c_1)(b_2 + c_2) \dots (b_n + c_n)$$

and hence

$$\int_0^1 dx_1 \int_0^{x_1} \dots \int_0^{x_{2n-1}} dx_{2n} \frac{1}{(a_1 a_2 \dots a_n)^{3/2}} < \frac{[\Gamma(\frac{1}{2})]^n}{\Gamma(n+\frac{1}{2})}.$$

Combining these two results one can prove

- i) that the expansion in perturbation with respect to the final state interaction is bound by a geometric series
- ii) that a given term of the expansion, for real $\Delta^2 > 0$ vanishes as $|k|^2 \rightarrow \infty$, and that for $|k|^2$ large enough (with $\frac{\text{Im } k}{k} > \epsilon$) the expansion is convergent, a result similar to the one obtained in a different way by Klein⁹⁾ for the case of scattering.

A P P E N D I X II.

We wish to extend the analytic properties in Δ^2 , proved in ref. 5), to the case where the initial and final energy are slightly complex. The procedure we used consisted in isolating the angular integrals. So for the n^{th} order term one has to compute

$$\int d\Omega_1 \dots d\Omega_n \frac{1}{\frac{k_f^2 + k_1^2 + \sigma^2}{2k_f k_1} - \cos \theta_{f1}} \frac{1}{\frac{k_1^2 + k_2^2 + \mu^2}{2k_1 k_2} - \cos \theta_{12}} \dots$$

$$\frac{1}{\frac{k_n^2 + k_i^2 + \nu^2}{2k_n k_i} - \cos \theta_{ni}}$$

when k_f and k_i are complex, it is convenient to integrate first on $d\Omega_2, \dots, d\Omega_{n-1}$. Then the integral becomes:

$$\int d\Omega_1 d\Omega_n dx \frac{1}{\frac{k_f^2 + k_1^2 + \sigma^2}{2k_f k_1} - \cos \theta_{f1}} \frac{1}{\frac{k_1^2 + k_n^2 + (n-1)^2 \lambda^2(x, k)}{2k_1 k_n} - \cos \theta_{1,n}}$$

$$\frac{1}{\frac{k_n^2 + k_i^2 + \nu^2}{2k_n k_i} - \cos \theta_{ni}}$$

with $\lambda(x, k) \gg \mu$; x is written for $n-1$ parametric variables.

We now make use of the formula (see ref. 5)

$$\int \frac{d\Omega_2}{(A - \cos \theta_{12})(B - \cos \theta_{23})} = 2\pi \int_1^\infty \frac{dz}{\sqrt{z^2 - 1}} \frac{1}{[AB + z\sqrt{A^2 - 1}\sqrt{B^2 - 1} - \cos \theta_{13}]}$$

and apply it to the first angular integration, and then to the second. We are led to an expression of the type

$$\frac{1}{(AB + z\sqrt{A^2 - 1}\sqrt{B^2 - 1})C + Y\sqrt{C^2 - 1}\sqrt{(AB + z\sqrt{A^2 - 1}\sqrt{B^2 - 1})^2 - 1} - \cos \theta}$$

We notice first that if A is complex but such that

$$\operatorname{Re} A > 1, \left| \frac{\operatorname{Arg}(\sqrt{A^2 - 1})}{\operatorname{Arg} A} \right| < \frac{\operatorname{Re} A}{\operatorname{Re} A - \frac{1}{\operatorname{Re} A}}$$

Here

$$A = \frac{k_f^2 + k_1^2 + \sigma^2}{2k_f k_1}$$

and $\operatorname{Re} A > 1$ for any k_1 provided

$$\cos^2 \phi_f > \frac{|k_f|^2}{|k_f|^2 + \sigma^2}$$

on the other hand $\operatorname{Arg} A < |\phi_f|$. So by taking ϕ_f small enough one can make the argument of $AB + z\sqrt{A^2 - 1}\sqrt{B^2 - 1}$ smaller than a fixed number, independent of k_1 , k_{n-1} and Z . Then one can show that the

real part of $AB+Z\sqrt{A^2-1}\sqrt{B^2-1}$ can be made arbitrarily close of the quantity obtained by replacing k_f by $|k_f|$, by choosing ϕ_f small enough (irrespective of the value of k_1 and k_n). The latter quantity is known to be larger than

$$\frac{|k_f|^2 + k_n^2 + ((n-1)\mu + \sigma)^2}{2|k_f|k_n} > 1.$$

One can then repeat the operation and prove than the integrant may be written as: $\frac{1}{X - \cos \Theta}$ with for

$$\cos^2 \phi_f > \frac{|k_f|^2}{|k_f|^2 + \sigma^2} \quad \cos^2 \phi_i > \frac{|k_i|^2}{|k_i|^2 + \mu^2}$$

$$|\text{Arg}(X)| < \lambda [|\phi_i| + |\phi_f|]$$

and

$$\text{Re } X > \frac{|k_i|^2 + |k_f|^2 + (n\mu + \sigma)^2}{2|k_i||k_f|} - \epsilon$$

with $\epsilon \rightarrow 0$ as ϕ_i and ϕ_f go to zero. The important point is that λ and ϵ do not depend on the integration variables (Z, Y, k_1, k_n etc.).

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Minimum analyticity domain for case a.
(deuteron recoil neglected)

Fig. 1

Fig 2

Fig. 3

Fig. 4