A Note on the Proof of Dispersion Relations *)

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ABSTRACT

Lehmann’s proof of dispersion relations is completed by showing that the absorptive part of the scattering amplitude is regular in a uniform neighbourhood of the real axis as a function of the meson mass. The same point is considered for production processes; one or two other difficulties in the latter case are also discussed.

*) Submitted for publication in the Nuovo Cimento.
I. Introduction

In the usual proof of dispersion relations, according to the technique of Bogoliubov\(^1\),\(^2\), the most important step is the analytic continuation of the absorptive part \(A\) of the amplitude from a non-physical value of the meson mass \(\sqrt{\xi}\) up to the physical value \(\mu\). Using the Jost-Lehmann-Dyson representation\(^3\), Lehmann\(^4\) was able to show that this continuation is possible provided that the momentum transfer \(\Delta\) is not too large, but may be larger than the values for which dispersion relations had been proved in\(^1\),\(^2\). But Lehmann's proof is not quite complete, because what is required is that an integral of \(A\) over the energy \(W'\), namely

\[
\int_{-\infty}^{\infty} \frac{dW'}{(m+\mu)^{\frac{3}{2}}} \frac{1}{W'^2 - W^2} + \frac{1}{W'^2 - 4 \Delta^2 - 2(m^2 + \xi)} + W^2
\]

is regular in a strip \(S\): (with \(\xi = \xi_1 + i \xi_2\))

\[-\Delta \leq \xi \leq \mu^2\]

\[|\xi_2| < S\]

and for this, \(A(W', \Delta, \xi)\) must be regular in \(S\) for each \(W' \geq m + \mu\), that is \(S\) must be independent of the energy. In his proof\(^4\), Lehmann showed that \(A(W, \Delta, \xi)\) is regular, for each \(W\), in a strip \(S(W)\), and so the integral over any finite range of \(W\) is regular in such a strip.

This does not mean that an infinite integral over the energy will be regular. As a well known example, the absorptive part \(A(\Delta, W)\) is regular in \(\Delta^\nu\) in a large ellipse\(^4\), but we cannot deduce any regularity in \(\Delta^\xi\) for the scattering amplitude from the dispersion relation, because the ellipse shrinks to the physical region as \(W \to \infty\).

In this paper we remedy this objection to Lehmann's method by showing that a uniform strip \(S\) exists. First we tidy up the proof of the \(\pi - n\) dispersion relations, and then the general case.
II. Fion-nucleon relations

In the region \( \xi_2 = 0, \xi_1 < -\Delta^2 \), the forward tube becomes the upper-half \( \omega \)-plane, and the backward tube the lower-half \( \omega \)-plane, where \( W^2 = 2\sqrt{\Delta^2 + m^2} + 2\Delta^2 + m^2 + \xi \). Then we can write

\[
T(\xi, W, \Delta) = \text{bound state terms} + \frac{1}{\pi} \left( \int \frac{dW'}{(m+\mu)^2} + \int_0^\infty \frac{dW'}{W^2} \right) \left\{ \frac{\mathcal{R}(W')}{W^2 - W'^2} + \frac{\mathcal{R}(W')}{{W'^2 - 4\Delta^2 - 2(m^2 + \xi) + W^2}} \right\}
\]

where \( W_1 \) is a suitable energy to be fixed later. The first integral can be shown to be regular in \( \xi \) in a suitable strip \( S \) by the method of Lehmann, the existence of \( S \) following from continuity, and because the range of \( W' \) integration is finite. It turns out that the maximum \( \Delta^2 \) for which the continuation can be done is

\[
\Delta^2_{\text{max}} = \frac{p^2}{k_C} + \frac{(m_1^2 - \mu^2)(m_2^2 - m_1^2)}{W^2 - (m_1 - m_2)^2}
\]

where

\[
k_C^2 = \frac{[W^2 - (m + \mu)^2][W^2 - (m - \mu)^2]}{4W^2}
\]

and \( m_1, m_2 \) are certain masses at which the spectrum begins. In the \( \pi - n \) case, the minimum of the r.h.s. of Eq. (2) occurs at \( W = m + \mu \). In other cases the minimum need not occur at the lower limit of the \( W' \) integration, but will always occur at some finite \( W \), say \( W_0 \). We will choose \( W_1 \) larger than \( W_0 \).

In order to tackle the other integral in Eq. (1) more simply, we will deliberately lose information, namely, the rotational invariance of the absorptive part, using the form \( 4 \)
\[
A(W, \Delta, \xi) = \frac{1}{8\pi} \\
\int \int \int \int \int d\mu_1 d\mu_2 d\kappa_1 d\kappa_2 d\chi d\theta_1 \varphi(\mu_{i_0}, \mu, \lambda, \theta_i, W)
\]

\[
\left[ x_1(\xi) - k_c(\xi) \cos(\theta - \chi) \right] \left[ x_2(\xi) - k_c(\xi) \cos(\chi - \lambda) \right]
\]

where

\[
x_i = \left[ -\frac{k^2_c + \mu_i^2 + \kappa_i^2 - (\mu_0 + (\mu^2 - 5)/2W)^2}{2\mu_i \sin \theta_i} \right] / 2
\]

We will discuss the two factors in the denominator separately as functions of \( \xi \). We will show that, for a choice of \( W_1 \) which depends on \( \Delta \),

\[
Re \left[ x_1(\xi) - k_c \cos(\theta - \chi) \right] \quad \text{and} \quad Re \left[ x_2(\xi) - k_c \cos(\chi - \lambda) \right]
\]

are always positive over the support of \( \varphi \), provided that \( \xi_2 \) lies in a strip \( |\xi_2| < 8 \).

Consider first the simplest case \( \Delta = 0 \). We know that \( x_{1,2} \) is linear in \( \xi \) and that

\[
(Re x_{1,2})^2 \geq \frac{(W^2 + m_1^2 - 5)}{2W} - m^2 + \frac{(m_1^2 - \mu^2)(m_2^2 - m^2)}{W^2 - (m_1 - m_2)^2}
\]

and so a sufficient condition that the denominator in Eq. (3) does not vanish is

\[
\xi_2^2 < \left( \frac{4W^2}{W^2 + m_1^2} \right)^2 \left[ \frac{8\mu^2 k_1^2 (2m_1 + \mu)}{W^2} + \frac{8\mu^2 (2m_1 + \mu)}{W^2} \right] \quad (4)
\]

where

\[
K_1^2 = \frac{(W^2 + m_1^2 - 5)^2}{4W^2} - m^2
\]

Now because the r.h.s. of Eq. (4) has a positive definite minimum when \( W \) ranges from \( W_1 \) to \( \infty \), and \( \xi \leq \mu^2 \), it follows that there exists a
uniform strip around $\tilde{\xi}_1 \leq \mu^2$ where $A$ is regular in $\tilde{\xi}$.

In the case $\Delta^2 \neq 0$ we will show that

(a) $\text{Re} \ x_1 > \text{Max} \left[ \left| \text{Re} \ k_c \left( 1 - \frac{2\Delta^2}{k_c^2} \right) \right| \left| \cos \alpha \right| + \left| \text{Re} \left( \frac{2\Delta}{k_c} \left( \frac{1}{k_c^2 - \Delta} \right)^z \right| \left| \sin \alpha \right| \right]$

(b) $\text{Re} \ x_2 > \text{Re} \ k_c$

The $\tilde{\xi}$ dependence is contained in

$$\frac{\tilde{\xi}^2}{k_c^2} = \frac{\left[ W^2 + m^2 - (\tilde{\xi}_1 + i\tilde{\xi}_2) \right]^2}{4W^2} - m^2 = \kappa_1^2 + i\kappa_2^2 \text{ say.}$$

It can be shown by elementary majorisations that sufficient for condition (a) is

$$\mu^2 \left( 2m\mu + \mu^2 \right) > \frac{1}{4} \kappa_2^2 + Q \left( \frac{1}{\kappa_1^2} \right)$$

where $Q \left( \frac{1}{\kappa_1^2} \right)$ is a polynomial in $\kappa_1^{-2}$ whose coefficients are polynomials in $\Delta^2$ and $\tilde{\xi}_2$. Provided that $\kappa_1$ is sufficiently big, condition (5) leads to condition (4), which is the same as condition (b) as well. Hence, by choosing $W_1$ in Eq. (1) large enough, i.e. so that Eq. (5) is satisfied, the existence of a uniform strip $S$ is proved.

III. Production processes

Dispersion relations for the process $a + b \rightarrow c + d$ have been proved by the method of Lehmann for a certain range of $\Delta^2$ and the
masses, which includes the process \( \Lambda + \pi \to \Sigma + \pi \), but not associated production. For the proof, analytic continuation in the masses \( m_b^2 = \xi' \) and \( m_d^2 = \beta' \) must be performed. It is preferable to keep \( \xi' - \beta' \) equal to its physical value \( m_b^2 - m_d^2 \), and to regard \( \frac{\xi' + \beta'}{2} = \xi' \) as the variable in which the continuation is made. The forward tube (R) reduces to the form before Eq. (3.1) of reference only if \( \xi' = \beta' \).

But also without this simplification, (R) is the upper half plane if \( \xi' \) is sufficiently negative. As we continue in \( \xi' \), a branch point in the momentum \( K_1(\xi') \) occurs at some value, if the energy is below threshold. This was not treated completely in reference. However, the branch gives no trouble, since we can cut the \( \xi' \)-plane from the relevant value of \( \xi' \), the cut lying everywhere below the real axis. Then a path of continuation in \( \xi' \) from large negative values up to the physical value can be chosen, with \( \xi'_2 > 0 \), lying in the forward tube (R). Just as above we can prove regularity in a strip S (except for the cut) and so dispersion relations are proved for a limited range of \( \Delta^2 \) and the masses. If the initial and final particles have not got the same mass, \( \Delta^2 < 0 \) is a physical possibility for some energies. But a dispersion relation with \( \Delta^2 < 0 \), if it could be proved, would have an infinite unphysical region. This is because the smallest physical value of \( \Delta^2 \) tends to zero as the energy tends to infinity (for then the mass-differences can be neglected, and we tend to the equal-particle case). If \( \Delta^2 < 0 \) the exponential \( e^{i(\omega \cdot k - \omega' \cdot k')} \) is rapidly increasing for some \( x \) in the forward cone for any complex \( \omega \), and so dispersion relations cannot be proved.

Acknowledgements

The second named author would like to thank Drs. R. Omnès, M. Froissart and R. Stora for many helpful discussions.

(*) This remark is due to R. Stora.
References


