A Note on the Proof of Dispersion Relations *)

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ABSTRACT

Lehmann's proof of dispersion relations is completed by showing that the absorptive part of the scattering amplitude is regular in a uniform neighbourhood of the real axis as a function of the meson mass. The same point is considered for production processes; one or two other difficulties in the latter case are also discussed.

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I. Introduction

In the usual proof of dispersion relations, according to the technique of Bogoliubov^{1),2)}, the most important step is the analytic continuation of the absorptive part A of the amplitude from a non-physical value of the meson mass $\sqrt{s}$ up to the physical value μ . Using the Jost-Lehmann-Dyson representation³⁾, Lehmann⁴⁾ was able to show that this continuation is possible provided that the momentum transfer Δ is not too large, but may be larger than the values for which dispersion relations had been proved in^{1),2)}. But Lehmann's proof is not quite complete, because what is required is that an integral of A over the energy W' , namely

$$\int_{(m+\mu)^2}^{\infty} dW'^2 A(W', \Delta^2, \xi) \left\{ \frac{1}{W'^2 - W^2} + \frac{1}{W'^2 - 4\Delta^2 - 2(m^2 + \xi) + W^2} \right\}$$

is regular in a strip S : (with $\xi = \xi_1 + i\xi_2$)

$$-\Delta^2 \leq \xi_1 \leq \mu^2$$

$$|\xi_2| < \delta$$

and for this, $A(W', \Delta, \xi)$ must be regular in S for each $W' \geq m + \mu$, that is δ must be independent of the energy. In his proof⁽⁴⁾, Lehmann showed that $A(W, \Delta, \xi)$ is regular, for each W , in a strip $S(W)$, and so the integral over any finite range of W is regular in such a strip. This does not mean that an infinite integral over the energy will be regular. As a well known example, the absorptive part $A(\Delta, W)$ is regular in Δ^2 in a large ellipse⁽⁴⁾, but we cannot deduce any regularity in Δ^2 for the scattering amplitude from the dispersion relation, because the ellipse shrinks to the physical region as $W \rightarrow \infty$.

In this paper we remedy this objection to Lehmann's method by showing that a uniform strip S exists. First we tidy up the proof of the $\pi - \pi$ dispersion relations, and then the general case.

II. Pion-nucleon relations

In the region $\xi_2 = 0$, $\xi_1 < -\Delta^2$, the forward tube becomes the upper-half ω -plane, and the backward tube the lower-half ω -plane, where $W^2 = 2\omega/\Delta^2 + m^2 + 2\Delta^2 + m^2 + \xi$. Then we can write

$$T(\xi, W, \Delta) = \text{bound state terms} + \frac{1}{\pi} \left(\int_{(m+\mu)^2}^{W_1^2} dW'^2 + \int_{W_1^2}^{\infty} dW'^2 \right) \left\{ \frac{A(W')}{W'^2 - W^2} + \frac{A(W')}{W'^2 - 4\Delta^2 - 2(m^2 + \xi) + W^2} \right\} \quad (1)$$

where W_1 is a suitable energy to be fixed later. The first integral can be shown to be regular in ξ in a suitable strip S by the method of Lehmann, the existence of δ following from continuity, and because the range of W' integration is finite. It turns out that the maximum Δ^2 for which the continuation can be done is

$$\Delta_{\max}^2 = k_c^2 + \frac{(m_1^2 - \mu^2)(m_2^2 - m^2)}{W^2 - (m_1 - m_2)^2} \quad (2)$$

where

$$k_c^2 = \frac{[W^2 - (m+\mu)^2][W^2 - (m-\mu)^2]}{4W^2}$$

and m_1, m_2 are certain masses at which the spectrum begins. In the $\pi-n$ case, the minimum of the r.h.s. of Eq. (2) occurs at $W = m + \mu$. In other cases the minimum need not occur at the lower limit of the W' integration, but will always occur at some finite W , say W_0 . We will choose W_1 larger than W_0 .

In order to tackle the other integral in Eq. (1) more simply, we will deliberately lose information, namely, the rotational invariance of the absorptive part, using the form⁴⁾

$$A(W, \Delta, \xi) = \frac{1}{8\pi}.$$

$$\iiint du_{i0} u_i d\mu_i d\kappa_i d\alpha d\chi d\theta_i \varphi(u_{i0}, u, \alpha, \theta_i, W) \quad (3)$$

$$\frac{1}{[x_1(\xi) - k_c(\xi) \cos(\theta - \chi)][x_2(\xi) - k_c(\xi) \cos(\chi - \alpha)]}$$

where

$$x_i = [k_c^2 + u_i^2 + \kappa_i^2 - (u_0 + (u^2 - \xi)/2W)^2] / 2u_i \sin \theta_i$$

We will discuss the two factors in the denominator separately as functions of ξ . We will show that, for a choice of W_1 which depends on Δ ,

$$\operatorname{Re}[x_1(\xi) - k_c \cos(\theta - \chi)] \quad \text{and} \quad \operatorname{Re}[x_2(\xi) - k_c \cos(\chi - \alpha)]$$

are always positive over the support of ϕ , provided that ξ_2 lies in a strip

$$|\xi_2| < \delta.$$

Consider first the simplest case $\Delta = 0$. We know that $x_{1,2}$ is linear in ξ and that

$$(\operatorname{Re} x_{1,2})^2 \geq \left(\frac{W^2 + m^2 - \xi_1}{2W} \right)^2 - m^2 + \frac{(m_1^2 - \mu^2)(m_2^2 - m^2)}{W^2 - (m_1 - m_2)^2}$$

and so a sufficient condition that the denominator in Eq. (3) does not vanish is

$$\xi_2^2 < \left(\frac{4W^2}{W^2 + m^2} \right)^2 \left[\frac{8\mu^2 k_1^2 (2m\mu + \mu^2)}{W^2} + \frac{\{8\mu^2 (2m\mu + \mu^2)\}^2}{W^2} \right] \quad (4)$$

where

$$k_1^2 = \frac{(W^2 + m^2 - \xi_1)^2}{4W^2} - m^2$$

Now because the r.h.s. of Eq. (4) has a positive definite minimum when W ranges from W_1 to ∞ , and $\xi_1 \leq \mu^2$, it follows that there exists a

uniform strip around $\xi_1 \leq \mu^2$ where A is regular in ξ .

In the case $\Delta^2 \neq 0$ we will show that

$$(a) \operatorname{Re} x_1 > \operatorname{Max} \left[\left| \operatorname{Re} k_c \left(1 - \frac{2\Delta^2}{k_c^2} \right) \right| |\cos \alpha| + \left| \operatorname{Re} \left(\frac{2\Delta}{k_c} (k_c^2 - \Delta^2)^{\frac{1}{2}} \right) \right| |\sin \alpha| \right]$$

$$(b) \operatorname{Re} x_2 > \operatorname{Re} k_c$$

The ξ dependence is contained in

$$k_c^2 = \frac{[W^2 + m^2 - (\xi_1 + i\xi_2)]^2}{4W^2} - m^2 = k_1^2 + ik_2^2 \text{ say.}$$

It can be shown by elementary majorisations that sufficient for condition

(a) is

$$\mu^2 (2m\mu + \mu^2) > \frac{1}{4} k_2^2 + Q\left(\frac{1}{k_1^2}\right) \quad (5)$$

where $Q\left(\frac{1}{k_1^2}\right)$ is a polynomial in k_1^{-2} whose coefficients are polynomials in Δ^2 and ξ_2 . Provided that k_1 is sufficiently big, condition (5) leads to condition (4), which is the same as condition (b) as well. Hence, by choosing W_1 in Eq. (1) large enough, i.e. so that Eq. (5) is satisfied, the existence of a uniform strip S is proved.

III. Production processes

Dispersion relations for the process $a + b \rightarrow c + d$ have been proved⁵⁾ by the method of Lehmann for a certain range of Δ^2 and the

masses, which includes⁶⁾ the process $\Lambda + \pi \rightarrow \Sigma + \pi$, but not associated production. For the proof, analytic continuation in the masses $m_b^2 = \xi$ and $m_d^2 = \beta$ must be performed. It is preferable⁷⁾ to keep $\xi - \beta$ equal to its physical value $m_b^2 - m_d^2$, and to regard $\frac{\xi + \beta}{2} = \xi'$ as the variable in which the continuation is made. The forward tube (R) reduces to the form before Eq. (3.1) of reference⁵⁾ only if $\xi_1 = \beta_1$. But also without this simplification, (R) is the upper half plane if ξ' is sufficiently negative. As we continue in ξ' , a branch point in the momentum $K_1(\xi')$ occurs at some value, if the energy is below threshold. This was not treated completely in^{5), 6)}. However, the branch gives no trouble, since we can cut the ξ' -plane from the relevant value of ξ' , the cut lying everywhere below the real axis. Then a path of continuation in ξ' from large negative values up to the physical value can be chosen, with $\xi'_2 > 0$, lying in the forward tube (R). Just as above we can prove regularity in a strip S (except for the cut) and so dispersion relations are proved for a limited range of Δ^2 and the masses. If the initial and final particles have not got the same mass, $\Delta^2 < 0$ is a physical possibility for some energies. But a dispersion relation with $\Delta^2 < 0$, if it could be proved, would have an infinite unphysical region. This is because the smallest physical value of Δ^2 tends to zero as the energy tends to infinity (for then the mass-differences can be neglected, and we tend to the equal-particle case). If $\Delta^2 < 0$ the exponential $e^{i(k+k').x}$ is rapidly increasing for some x in the forward cone for any complex ω , and so dispersion relations cannot be proved^(*).

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