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SCATTERING BY A GIVEN CLASS OF NON CENTRAL FORCES.*)

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A B S T R A C T

Previous work has shown the remarkably simple properties of the partial wave scattering amplitude and corresponding radial functions derived from a potential, which is a continuous superposition of exponential or Yukawa potentials. Here we extend these results to the case where non central forces are present in the scattering of two spin one half particles. We give a method of calculation of the S-matrix elements in terms of the inverse Laplace transforms of the various radial potentials, we establish the analytic properties of these matrix elements and get integral representations of the wave functions.

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I. Introduction

The analytic properties of the partial wave scattering amplitudes have been established recently by one of us (A.M.) ¹⁾ for the following class of central potentials

$$V(r) = \int_{\mu}^{+\infty} C(\alpha) e^{-\alpha r} d\alpha, \quad |C(\alpha)| < M \alpha^{1-\varepsilon} \quad (1)$$

In this paper, we wish to consider the properties of the radial wave functions of the Schroedinger equation for two spin $\frac{1}{2}$ particles in the presence of non central potentials:

$$V(\vec{\sigma}_1, \vec{\sigma}_2; r) = V_d(r) + V_{\sigma}(r) \vec{\sigma}_1 \cdot \vec{\sigma}_2 + V_T(r) S_{12} + V_o(r) \vec{L} \cdot \vec{S} \quad (2)$$

where the position-dependent parts $V_d, V_{\sigma}, V_T, V_o$ belong to the family (1).

At first sight, the extension of the methods used in ¹⁾ might be complicated by the fact that the radial equations are now coupled, but we shall see that we may still solve the problem by using procedures which are very close to those described in ¹⁾, provided a special choice of independent solutions of the Schroedinger equation is made.

In Sections II and III, we shall derive the expressions of the S-matrix, the eigenphase-shifts and the admixture parameters in terms of special types of irregular solutions. In Section IV, these solutions are shown to be directly obtained from the inverse Laplace transforms of the potentials by an iteration method. Finally, the analytic properties of the S-matrix are studied in Section V.

II. Expression of the S-matrix in the JLSM scheme

If we separate the Schroedinger equation (e.g. describing the N-P system interacting through $V(\vec{\sigma}_1, \vec{\sigma}_2; \vec{r})$) into partial waves according to the JLSM scheme, it is well-known ^{2), 3)} that, for the $L=J-1$ and $L=J+1$ triplet states, the radial wave functions u_J and w_J are mixed by the following system of coupled differential equations:

$$\left\{ \begin{array}{l} u_J''(r) + \left[k^2 - \frac{(J-1)J}{r^2} - V_C(r) - (J-1)V_O(r) - \frac{J-1}{2J+1} V_T(r) \right] u_J = - \frac{6 \sqrt{J(J+1)}}{2J+1} V_T(r) w_J \\ w_J''(r) + \left[k^2 - \frac{(J+1)(J+2)}{r^2} - V_C(r) + (J+2)V_O(r) + \frac{2J+2}{2J+1} V_T(r) \right] w_J = \\ = - \frac{6 \sqrt{J(J+1)}}{2J+1} V_T(r) u_J \end{array} \right. \quad (3)$$

where $V_C = V_d + V_\sigma$.

A general solution of (3) may be written as

$$\begin{pmatrix} u_J(r) \\ w_J(r) \end{pmatrix} = a_1 \psi_1^+ + a_2 \psi_2^+ + b_1 \psi_1^- + b_2 \psi_2^- \quad (4)$$

where the $\psi_\lambda^{+,-}(r, k)$, ($\lambda = 1, 2$), are a complete set of linearly independent solutions of (3) defined by:

$$\psi_\lambda^+ = \begin{pmatrix} f(r, -k) e^{ikr} \\ g(r, -k) e^{ikr} \end{pmatrix}, \quad \psi_\lambda^- = \begin{pmatrix} f(r, k) e^{-ikr} \\ g(r, k) e^{-ikr} \end{pmatrix} \quad (5)$$

with the following asymptotic behaviour for $r \rightarrow \infty$:

$$\psi_1^+ \sim \begin{pmatrix} e^{ikr} \\ e^{ikr} \end{pmatrix}, \quad \psi_2^+ \sim \begin{pmatrix} e^{ikr} \\ -e^{ikr} \end{pmatrix}, \quad \psi_1^- \sim \begin{pmatrix} e^{-ikr} \\ e^{-ikr} \end{pmatrix}, \quad \psi_2^- \sim \begin{pmatrix} e^{-ikr} \\ -e^{-ikr} \end{pmatrix} \quad (6)$$

The question of existence and determination of this complete set will be examined in Section IV.

Now, if we choose from (4) the solutions (u_a, w_a) and (u_b, w_b) such that their respective asymptotic form contains only pure $L=J-1$ or $L=J+1$ incoming wave, i.e.

$$\begin{pmatrix} u_a \\ w_a \end{pmatrix} = a_1 \psi_1^+ + a_2 \psi_2^+ + c(\psi_1^- + \psi_2^-)$$

$$\begin{pmatrix} u_b \\ w_b \end{pmatrix} = a'_1 \psi_1^+ + a'_2 \psi_2^+ + c'(\psi_1^- - \psi_2^-)$$

then the S-matrix

$$S = \begin{pmatrix} S_{J-1,J-1} & S_{J-1,J+1} \\ S_{J+1,J-1} & S_{J+1,J+1} \end{pmatrix}$$

may be easily obtained :

$$\left. \begin{aligned} S_{J-1,J-1} &= -\frac{a_1 + a_2}{2c} & S_{J-1,J+1} &= -\frac{a'_1 + a'_2}{2c'} \\ S_{J+1,J-1} &= \frac{a_2 - a_1}{2c} & S_{J+1,J+1} &= \frac{a'_2 - a'_1}{2c'} \end{aligned} \right\} \quad (7)$$

the a_i and a'_i are determined by the regularity condition ^{*}):

$$\lim_{r \rightarrow 0} \begin{pmatrix} r^{J-1} u_{a,b} \\ r^{J+1} w_{a,b} \end{pmatrix} = 0$$

which gives rise to the systems:

$$\begin{cases} a_1 f_1(-k) + a_2 f_2(-k) + c(f_1(k) + f_2(k)) = 0 \\ a_1 g_1(-k) + a_2 g_2(-k) + c(g_1(k) + g_2(k)) = 0 \end{cases} \quad (8)$$

$$\begin{cases} a'_1 f_1(-k) + a'_2 f_2(-k) + c'(f_2(k) - f_1(k)) = 0 \\ a'_1 g_1(-k) + a'_2 g_2(-k) + c'(g_1(k) - g_2(k)) = 0 \end{cases} \quad (8')$$

where

$$f_\lambda(k) = \lim_{r \rightarrow 0} r^{J-1} f_\lambda(r, k)$$

$$g_\lambda(k) = \lim_{r \rightarrow 0} r^{J+1} g_\lambda(r, k)$$

If we normalize the incoming waves such that $c=c'=1$, we can readily show that the S-matrix is symmetric: $S_{J+1, J-1} = S_{J-1, J+1}$, by means of (8), (8') and the following relation:

$$f_1(k) g_1(-k) - f_1(-k) g_1(k) + f_2(k) g_2(-k) - f_2(-k) g_2(k) = 0 \quad (9)$$

which will be proved in the Appendix.

^{*}) It is known that this necessary condition is also sufficient to make u and w vanish at the origin provided the potentials are less singular than $1/r^2$, which is the case here.

III. Eigenstates and Eigenphase-shifts

Though all the information needed is contained in the non diagonal S-matrix obtained in Section II, it may be worthwhile to look at some direct way of obtaining the eigenphase-shifts $\delta_J^\alpha, \delta_J^\gamma$ and the admixture parameters $\eta_J^\alpha, \eta_J^\gamma$ of the "eigenstates": (u_J^α, w_J^α) and (u_J^γ, w_J^γ) defined by the following boundary conditions:

$$A) \quad u_J^{\alpha, \gamma} \sim 2i \sin(kr + \delta_J^{\alpha, \gamma}) \quad (r \rightarrow \infty)$$

$$w_J^{\alpha, \gamma} \sim 2i \eta_J^{\alpha, \gamma} \sin(kr - \pi + \delta_J^{\alpha, \gamma})$$

$$B) \quad \lim_{r \rightarrow 0} \begin{cases} r^{J-1} u^{\alpha, \gamma} \rightarrow 0 \\ r^{J+1} w^{\alpha, \gamma} \rightarrow 0. \end{cases}$$

Remembering $\begin{pmatrix} u \\ w \end{pmatrix} = a_1 \psi_1^+ + a_2 \psi_2^+ + b_1 \psi_1^- + b_2 \psi_2^-$,

the asymptotic behaviour A) tells us that:

$$\begin{aligned} b_1 &= -a_1^* = -r_1 e^{-i \delta_J^{\alpha, \gamma}} \\ b_2 &= -a_2^* = -r_2 e^{-i \delta_J^{\alpha, \gamma}} \end{aligned} \quad (10)$$

$$\eta_J^{\alpha, \gamma} = \frac{a_2 - a_1}{a_2 + a_1}$$

*) Note that in order to simplify the notations the eigenphase-shifts are defined here as the sum of the usual eigenphase-shifts and $\frac{J-1}{2} \pi$.

The regularity condition B) leads to the following system for real energies:

$$\begin{cases} r_1 \left[\cos \delta \operatorname{Im} f_1(k) + \sin \delta \operatorname{Re} f_1(k) \right] + r_2 \left[\cos \delta \operatorname{Im} f_2(k) + \sin \delta \operatorname{Re} f_2(k) \right] = 0 \\ r_1 \left[\cos \delta \operatorname{Im} g_1(k) + \sin \delta \operatorname{Re} g_1(k) \right] + r_2 \left[\cos \delta \operatorname{Im} g_2(k) + \sin \delta \operatorname{Re} g_2(k) \right] = 0 \end{cases} \quad (10')$$

where δ is the common phase of a_1 and a_2 . The compatibility condition for the system (10') gives rise to the quadratic equation:

$$\frac{\operatorname{Im} f_1(k) + \operatorname{tg} \delta \operatorname{Re} f_1(k)}{\operatorname{Im} g_1(k) + \operatorname{tg} \delta \operatorname{Re} g_1(k)} = \frac{\operatorname{Im} f_2(k) + \operatorname{tg} \delta \operatorname{Re} f_2(k)}{\operatorname{Im} g_2(k) + \operatorname{tg} \delta \operatorname{Re} g_2(k)}$$

the roots of which determine the eigenphase-shifts δ^α and δ^γ ; then the admixture parameters η^α and η^γ may be obtained by means of (10') and (10). Furthermore, from the relation (9) between the f_λ and g_λ , one can deduce the well-known relation: $\eta^\alpha \eta^\gamma = -1$.

At this stage the problem is therefore to solve the coupled differential equations satisfied by $f_\lambda(r, k)$ and $g_\lambda(r, k)$, in which the exponential factors disappear, because of special choice of solutions:

$$\left. \begin{aligned} f_\lambda'' - 2ikf_\lambda' - \left[\frac{(J-1)J}{r^2} + U_M^f \right] f_\lambda &= U_T g_\lambda \\ g_\lambda'' - 2ikg_\lambda' - \left[\frac{J(J+1)}{r^2} + U_M^g \right] g_\lambda &= U_T f_\lambda \end{aligned} \right\} \quad (11)$$

where

$$U_M^f(r) \equiv V_C(r) + (J-1) \left[V_O(r) + \frac{2}{2J+1} V_T(r) \right]$$

$$U_M^g(r) \equiv V_C(r) + (J+2) \left[V_O(r) + \frac{2}{2J+1} V_T(r) \right]$$

$$U_T(r) \equiv - \frac{6 \sqrt{J(J+1)}}{2J+1} V_T(r)$$

IV. Explicit determination of $f_\lambda(r, k)$ and $g_\lambda(r, k)$

For sake of simplicity, let us first treat the special case $J=1$ (3S_1 and 3D_1 states), then (11) becomes:

$$\left. \begin{aligned} f''_\lambda - 2ikf'_\lambda - V_C f_\lambda &= U_T g_\lambda \\ g''_\lambda - 2ikg'_\lambda - \left(\frac{6}{r} + U_M\right) g_\lambda &= U_T f_\lambda \end{aligned} \right\} \quad (12)$$

Now, because of the assumption:

$$V_C(r) = \int_M^{+\infty} C_C(\alpha) e^{-\alpha r} d\alpha$$

$$U_{T,M}(r) = \int_M^{+\infty} C_{T,M}(\alpha) e^{-\alpha r} d\alpha$$

and because of the asymptotic behaviour (6) of $f_\lambda(r, k)$ and $g_\lambda(r, k)$, it is natural to write

$$f_\lambda(r, k) = 1 + \int_0^{+\infty} f_{\lambda,k}(\alpha) e^{-\alpha r} d\alpha$$

$$g_\lambda(r, k) = \delta_{\lambda 1} - \delta_{\lambda 2} + \int_0^{+\infty} \sigma_{\lambda,k}(\alpha) e^{-\alpha r} d\alpha$$

Then it follows from (12)

$$\left\{ \begin{aligned} \alpha(\alpha + 2ik) f_{1,k}(\alpha) &= C_C(\alpha) + C_T(\alpha) + \int_0^{\alpha-\mu} d\beta \left[C_C(\alpha-\beta) f_{1,k}(\beta) + C_T(\alpha-\beta) \sigma_{1,k}(\beta) \right] \\ \alpha(\alpha + 2ik) \sigma_{1,k}(\alpha) - 6\alpha - 6 \int_0^\alpha (\alpha-\beta) \sigma_{1,k}(\beta) d\beta &= C_M(\alpha) + C_T(\alpha) + \\ &+ \int_0^{\alpha-\mu} d\beta \left[C_M(\alpha-\beta) \sigma_{1,k}(\beta) + C_T(\alpha-\beta) f_{1,k}(\beta) \right] \end{aligned} \right. \quad (13)$$

$$\left\{ \begin{array}{l} \alpha(\alpha+2ik) \rho_{2,k}(\alpha) = C_C(\alpha) + C_T(\alpha) + \int_0^{\alpha-\mu} d\beta \left[C_C(\alpha-\beta) \rho_{2,k}(\beta) + C_T(\alpha-\beta) \sigma_{2,k}(\beta) \right] \\ \alpha(\alpha+2ik) \sigma_{2,k}(\alpha) + 6\alpha - 6 \int_0^{\alpha} (\alpha-\beta) \sigma_{2,k}(\beta) d\beta = -C_M(\alpha) - C_T(\alpha) + \\ + \int_0^{\alpha-\mu} d\beta \left[C_M(\alpha-\beta) \sigma_{2,k}(\beta) + C_T(\alpha-\beta) \rho_{2,k}(\beta) \right] \end{array} \right. \quad (13')$$

We are going now to show that these systems (13) and (13') may be recast in such a form that they can be solved by an iteration procedure.

First, by introducing the auxiliary function

$$y(\alpha) = 1 + \int_0^{\alpha} \sigma_{1,k}(\beta) d\beta$$

the 2nd integral equation of (13) may be transformed into the following inhomogeneous differential equation:

$$\alpha(\alpha+2ik)y'' + 2(\alpha+ik)y' - 6y = F'(\alpha) \quad (14)$$

with

$$F(\alpha) = C_M(\alpha) + C_T(\alpha) + \int_0^{\alpha-\mu} d\beta \left[C_M(\alpha-\beta) \sigma_{1,k}(\beta) + C_T(\alpha-\beta) \rho_{1,k}(\beta) \right]$$

The associated homogeneous equation is a Legendre equation, so the solution of (14) satisfying the right boundary condition for $\alpha < \mu$ ($F(\alpha) = 0$ in this range) is:

$$y(\alpha) = P_2\left(1 + \frac{\alpha}{ik}\right) + C_1(\alpha) P_2\left(1 + \frac{\alpha}{ik}\right) + C_2(\alpha) Q_2\left(1 + \frac{\alpha}{ik}\right) \quad (15)$$

where P_2 and Q_2 are respectively Legendre functions of 1st and 2nd kinds, and C_1 and C_2 the following solution:

$$\begin{cases} c_1(\alpha) = - \int_0^\alpha Q_2 \left(1 + \frac{\beta}{ik}\right) F'(\beta) d\beta \\ c_2(\alpha) = \int_0^\alpha P_2 \left(1 + \frac{\beta}{ik}\right) F'(\beta) d\beta \end{cases}$$

of the system

$$\begin{cases} c_1' P_2 + c_2' Q_2 = 0 \\ c_1' P_2' + c_2' Q_2' = \frac{F'}{\alpha(\alpha + 2ik)} \end{cases}$$

Then, integrating (15) by parts and differentiating we get

$$\sigma_{1,k}(\alpha) \equiv y'(\alpha) = P_2' + (P_2 Q_2' - Q_2 P_2') F(\alpha) + \int_0^\alpha K(\alpha, \beta) F(\beta) d\beta \quad (16)$$

with

$$\begin{aligned} K(\alpha, \beta) &= P_2'(\alpha) Q_2'(\beta) - Q_2'(\alpha) P_2'(\beta) \\ K(\alpha, \beta) &= 0 \quad \text{for } \alpha < \beta \end{aligned}$$

Inserting in (16) the expressions for F and $P_2 Q_2' - Q_2 P_2'$ and inverting the order of the double integrals:

$$\begin{aligned} \sigma_{1,k}(\alpha) &= P_2' \left(1 + \frac{\alpha}{ik}\right) + \frac{1}{\alpha(\alpha + 2ik)} [c_M(\alpha) + c_T(\alpha)] \\ &+ \frac{1}{\alpha(\alpha + 2ik)} \int_0^{\alpha-\mu} d\beta [c_M(\alpha - \beta) \sigma_{1,k}(\beta) + c_T(\alpha - \beta) \rho_{1,k}(\beta)] \\ &+ \int_\mu^\alpha d\beta K(\alpha, \beta) [c_M(\beta) + c_T(\beta)] \\ &+ \int_0^{\alpha-\mu} d\gamma \int_{\gamma+\mu}^\alpha d\beta K(\alpha, \beta) [c_M(\beta - \gamma) \sigma_{1,k}(\gamma) + c_T(\beta - \gamma) \rho_{1,k}(\gamma)] \end{aligned} \quad (17)$$

So, the system (13) is equivalent to the system obtained by adding to the latter equation the following one

$$\alpha (\alpha + 2ik) \rho_{1,k}(\alpha) = c_C(\alpha) + c_T(\alpha) + \int_0^{\alpha-\mu} d\beta \left[c_C(\alpha-\beta) \rho_{1,k}(\beta) + c_T(\alpha-\beta) \sigma_{1,k}(\beta) \right]$$

By inspection, one sees immediately that this system of integral equations may be solved by iteration: when $\rho_{1,k}(\alpha)$ and $\sigma_{1,k}(\alpha)$ are known for $\alpha < n\mu$ one can get them for $\alpha < (n+1)\mu$ by inserting the known functions in the right hand sides.

For instance,

$$\left. \begin{aligned} \rho_{1,k}(\alpha) &= 0 \\ \sigma_{1,k}(\alpha) &= P_1 \left(1 + \frac{\alpha}{ik} \right) \end{aligned} \right\} \text{ for } \alpha < \mu$$

etc.

Very similarly, by introducing $z(\alpha) = -1 + \int_0^{\alpha} \sigma_{2,k}(\beta) d\beta$ the second equation of (13') may be brought into the form:

$$\begin{aligned} \sigma_{2,k}(\alpha) &= P_2 \left(1 + \frac{\alpha}{ik} \right) - \frac{1}{\alpha(\alpha+2ik)} \left[c_M(\alpha) + c_T(\alpha) \right] \\ &+ \frac{1}{\alpha(\alpha+2ik)} \int_0^{\alpha-\mu} d\beta \left[c_M(\alpha-\beta) \sigma_{2,k}(\beta) + c_T(\alpha-\beta) \rho_{2,k}(\beta) \right] \\ &- \int_{\mu}^{\alpha} d\beta K(\alpha, \beta) \left[c_M(\beta) + c_T(\beta) \right] \\ &+ \int_0^{\alpha-\mu} d\gamma \int_{\gamma+\mu}^{\alpha} d\beta K(\alpha, \beta) \left[c_M(\beta-\gamma) \sigma_{2,k}(\gamma) + c_T(\beta-\gamma) \rho_{2,k}(\gamma) \right] \end{aligned} \quad (17')$$

which has the same structure as (17).

The general case $J \neq 1$ (3P_2 and 3F_2 , etc.) is slightly complicated by the need of transforming both integral equations of each system (13), (13') into the form (17), (17'), but the generalization is straightforward and one gets the system:

$$\left. \begin{aligned} \rho_{\lambda,k}(\alpha) \\ \sigma_{\lambda,k}(\alpha) \end{aligned} \right\} = \left\{ \begin{aligned} P_{J-1}'(1 + \frac{\alpha}{ik}) \\ P_{J+1}'(1 + \frac{\alpha}{ik}) \end{aligned} \right\} + \frac{1}{\alpha(\alpha + 2ik)} \left\{ \begin{aligned} c_M^{\rho}(\alpha) + c_T(\alpha) \\ [c_M^{\sigma}(\alpha) + c_T(\alpha)] \end{aligned} \right\} (\delta_{\lambda 1} - \delta_{\lambda 2})$$

$$+ \frac{1}{\alpha(\alpha + 2ik)} \int_0^{\alpha-\mu} d\beta \left\{ \begin{aligned} c_M^{\rho}(\alpha - \beta) \rho_{\lambda,k}(\beta) + c_T(\alpha - \beta) \sigma_{\lambda,k}(\beta) \\ c_M^{\sigma}(\alpha - \beta) \sigma_{\lambda,k}(\beta) + c_T(\alpha - \beta) \rho_{\lambda,k}(\beta) \end{aligned} \right\}$$

$$+ \int_{\mu}^{\alpha} d\beta \left\{ \begin{aligned} K_{J-1}(\alpha, \beta) [c_M^{\rho}(\beta) + c_T(\beta)] \\ K_{J+1}(\alpha, \beta) [c_M^{\sigma}(\beta) + c_T(\beta)] \end{aligned} \right\} (\delta_{\lambda 1} - \delta_{\lambda 2}) \quad (18)$$

$$+ \int_0^{\alpha-\mu} d\gamma \int_{\gamma+\mu}^{\alpha} d\beta \left\{ \begin{aligned} K_{J-1}(\alpha, \beta) [c_M^{\rho}(\beta - \gamma) \rho_{\lambda,k}(\gamma) + c_T(\beta - \gamma) \sigma_{\lambda,k}(\gamma)] \\ K_{J+1}(\alpha, \beta) [c_M^{\sigma}(\beta - \gamma) \sigma_{\lambda,k}(\gamma) + c_T(\beta - \gamma) \rho_{\lambda,k}(\gamma)] \end{aligned} \right\}$$

where $c_M^{\rho, \sigma}$ and c_T are defined by

$$U_M^{\rho}(r) = \int_{\mu}^{+\infty} c_M^{\rho}(\alpha) e^{-\alpha r} d\alpha$$

$$U_M^{\sigma}(r) = \int_{\mu}^{+\infty} c_M^{\sigma}(\alpha) e^{-\alpha r} d\alpha$$

$$U_T(r) = \int_{\mu}^{+\infty} c_T(\alpha) e^{-\alpha r} d\alpha$$

and

$$K_L(\alpha, \beta) = P_L'(\alpha) Q_L'(\beta) - Q_L'(\alpha) P_L'(\beta)$$

The iteration method may still be applied for solving (18).

V. Analytic properties of the radial wave functions and of the S-matrix

It is readily seen from the system (18) that $\varphi_{\lambda,k}(\alpha)$ and $\sigma_{\lambda,k}(\alpha)$ are well defined quantities in the k -plane cut along the imaginary axis from $k=i\frac{\mu}{2}$ to $i\infty$; due to the fact that the $|C_{G,M,T}(\alpha)|$ are assumed to be bound by $M\alpha^{1-\xi}$, one may show that $f_{\lambda}(r,k)$, $g_{\lambda}(r,k)$, $f_{\lambda}(k)$ and $g_{\lambda}(k)$ are analytic functions in the k -plane cut from $+i\frac{\mu}{2}$ to $+i\infty$, by using the same kind of arguments as those presented in ref. 1).

According to the equations (7), (8) and (8'), the elements of the S-matrix in the JLS scheme ^{*)} have the same analytic properties as $f_{\lambda}(k)$ and $g_{\lambda}(k)$ except additional poles introduced by algebraic operations, actually these poles occur when

$$\frac{f_1(-k)}{g_1(-k)} = \frac{f_2(-k)}{g_2(-k)} \quad (19)$$

and, in the upper half k -plane, they correspond to bound states: indeed, when (19) holds, from (8) and (8') one must have $c=c'=0$, so that a regular solution of (3) is:

$$\begin{pmatrix} u(r) \\ w(r) \end{pmatrix} = a_1 \psi_1^+ + a_2 \psi_2^+ \underset{r \rightarrow \infty}{\sim} \begin{pmatrix} (a_1 + a_2)e^{ikr} \\ (a_1 - a_2)e^{ikr} \end{pmatrix} \quad (20)$$

Now, due to the reality of the potentials, one can deduce very easily from (3) the following relation:

$$(u^*u' - uu'^* + w^*w' - ww'^*) \int_0^r (k^2 - k'^2) \int_0^r (|u|^2 + |w|^2) dr' = 0$$

which, in the present case, writes:

$$(k+k^*) \left[2i(|a_1|^2 + |a_2|^2) + (k-k^*) \int_0^{+\infty} (|u|^2 + |w|^2) dr \right] = 0$$

this shows that, for $\text{Im}k > 0$, $\text{Re}k=0$ and the above mentioned result follows.

*) It is not convenient to study the analytic properties of the eigenphase-shifts because they are artificially complicated by the fact that $\delta^{\alpha\lambda}(k)$ has two determinations.

In the very special case when either the system (8) or (8') is undetermined, it is easily seen (from the symmetry property of the S-matrix) that both systems must be undetermined: then, this means that the regular solution is a 3-parameter dependent solution and it would behave as an usual 2-parameter dependent regular solution only for $k=0$.

VI. Conclusions

We have given a method of calculation of phase shifts and admixture parameters directly from the inverse Laplace transform of the potential in the case of non central forces; this is of some interest because it often happens that a potential is given in terms of its inverse Laplace transform, for instance when it is derived from field theoretical considerations ⁴⁾. This method may look somewhat complicated for practical purposes, but there is no really simple method of solution of the Schroedinger equation when tensor forces are present.

At the same time, we have shown that the S-matrix has, in the present case, the same analytic features as in the case of central forces.

Finally, we obtain, as a by product, integral representations of the wave functions which might be useful to study the analytic properties of non relativistic matrix elements.

A P P E N D I X

We wish to prove here the relation:

$$f_1(r,k)g_1(r,-k)-f_1(r,-k)g_1(r,k)+f_2(r,k)g_2(r,-k)-f_2(r,-k)g_2(r,k) = 0 \quad (A1)$$

which has been used several times.

Clearly, this relation is equivalent to:

$$u_1^+ w_1^- - u_1^- w_1^+ + u_2^+ w_2^- - u_2^- w_2^+ = 0 \quad (A2)$$

where we have put, for sake of brevity

$$\begin{pmatrix} u_{\lambda}^+ \\ w_{\lambda}^+ \end{pmatrix} = \begin{pmatrix} f_{\lambda}(r,-k)e^{ikr} \\ g_{\lambda}(r,-k)e^{ikr} \end{pmatrix}, \quad \begin{pmatrix} u_{\lambda}^- \\ w_{\lambda}^- \end{pmatrix} = \begin{pmatrix} f_{\lambda}(r,k)e^{-ikr} \\ g_{\lambda}(r,k)e^{-ikr} \end{pmatrix} \quad (\lambda=1,2)$$

Now, from (3) we may deduce:

$$(u_2^+ u_1^+ - u_1^+ u_2^+ + w_2^+ w_1^+ - w_1^+ w_2^+) \Big|_r^{\infty} = 0$$

i.e.

$$u_2^+ u_1^+ - u_1^+ u_2^+ + w_2^+ w_1^+ - w_1^+ w_2^+ = 0 \quad (A3)$$

and in the same way, the following relations may be obtained

$$u_2^- u_1^- - u_1^- u_2^- + w_2^- w_1^- - w_1^- w_2^- = 0 \quad (A4)$$

$$u_2^- u_1^+ - u_1^+ u_2^- + w_2^- w_1^+ - w_1^+ w_2^- = 0 \quad (A5)$$

$$u_2^+ u_1^- - u_1^- u_2^+ + w_2^+ w_1^- - w_1^- w_2^+ = 0 \quad (A6)$$

Multiplying (A3), (A4), (A5) and (A6) respectively by $u_2^- u_1^-$, $-u_2^+ u_1^-$, $u_2^+ u_1^+$ and $-u_2^- u_1^+$ and adding them together we get

$$(u_2^- w_2^+ - u_2^+ w_2^-) (u_1^- w_1^+ - u_1^+ w_1^-) - (u_1^- w_1^+ - u_1^+ w_1^-) (u_2^- w_2^+ - u_2^+ w_2^-) = 0 \quad (A7)$$

Furthermore, noticing that the relations (A3) - (A6) are invariant by $u_\lambda^+ \longleftrightarrow w_\lambda^+$ and $u_\lambda^- \longleftrightarrow w_\lambda^-$ we may obtain

$$(u_2^- w_2^+ - u_2^+ w_2^-) (w_1^- u_1^+ - w_1^+ u_1^-) - (u_1^- w_1^+ - u_1^+ w_1^-) (w_2^- u_2^+ - w_2^+ u_2^-) = 0 \quad (A8)$$

Subtracting (A8) from (A7) and then integrating we get

$$\frac{u_1^+ w_1^- - u_1^- w_1^+}{u_2^+ w_2^- - u_2^- w_2^+} = c \quad (A9)$$

the constant c is readily determined by (A7) or (A8) for $r \rightarrow \infty$:
actually $c = -1$ and the relation (A2) is proved.

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