Theory of low energy nucleon-nucleon scattering

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The nucleon-nucleon interaction at energies below the threshold for inelastic processes, is studied using a method based on the Cini—Fubini approach to the Double Integral Representation technique.

The essential idea is to treat all the singularities of the amplitudes, for values of the variables lying near their physical region, taking full advantage of the symmetry of the Mandelstam representation. The spectral functions are calculated using unitarity, both in the nucleon-nucleon and in the nucleon-anti-nucleon channels. In the latter the two pion contribution is calculated in terms of pion-nucleon scattering and contains amplitudes characterising the low waves of the $\bar{N}N \rightarrow 2\pi$ process. The construction of integral equations for the partial wave amplitude is discussed.

The nucleon-nucleon amplitude is developed in terms of linear combinations of the Fermi operators chosen to have simple properties under crossing, and whose invariant coefficients are shown to satisfy the Mandelstam representation in 4th order perturbation theory.

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I. Introduction.

We shall develop in this paper a theoretical study of the nucleon-nucleon interaction at low energies; i.e. valid for energies below the threshold of pion production, and perhaps for energies at which the inelastic scattering is still negligible.

The method we shall use is based on the two dimensional spectral representation proposed by Mandelstam ¹⁾ and follows a more simplified approach introduced recently by Cini and Fubini ²⁾. This approach consists mainly in taking correctly into account only the contributions to the causal amplitudes due to singularities near the physical region of the variables; the contribution of higher singularities being expressed by some free parameters in the final results. In the Cini-Fubini paper the case of the scattering of scalar neutral nucleons is discussed; the physical case of interaction of two fermions through a charged pseudoscalar meson field is dealt with in the present paper by a suitable extension of the Cini-Fubini method.

Application of dispersion relation techniques to N-N scattering was first developed by Goldberger, Nambu and Oehme ³⁾. They made use of dispersion relations, for the coefficients of a particular set of invariant operators, at fixed momentum transfer. The two pion contribution to such dispersion relations was recently worked out by Goldberger and Oehme ⁴⁾ in a way such that the dependence on one of the momentum transfer variables has been explicitly exhibited. However, because their starting point is a one dimensional dispersion relation, the dependence on the other momentum transfer variable (which dependence contributes to the angular dependence of the scattering amplitude) is hidden in an arbitrary unknown function. Alternatively, such a method only exhibits the properties of exchange forces (in lowest order due to the exchange of a charged pion between neutron and proton) and not the properties of the direct ones (neutral pion exchange in lowest order). We shall discuss more extensively the differences between our approach and that of Goldberger, Oehme and Nambu at the end of this paper.

The two dimensional dispersion relation approach to nucleon-nucleon scattering has been applied by Chew ⁵⁾ to the determination of the pion-nucleon coupling constant g from the angular distribution in N-N scattering; by Cini, Fubini and Stanghellini ⁶⁾ in order to obtain fixed angle dispersion relations, which are used both for the determination of g and for an analysis of the low energy deviation from the shape independent approximation in proton-proton scattering; the latter problem has also been treated by Noyes and Wong ⁷⁾.

We shall proceed as follows : in Section II we shall define the kinematics and separate the causal amplitude into a suitable set of invariant operators (which we shall choose as linear combinations of Fermi invariants, symmetric or antisymmetric under crossing). In Section III we shall write down the two dimensional representation for the scalar functions which multiply those invariant operators, and express them later on by means of dispersion relations at fixed energy. This will allow us to use unitarity in the nucleon-antinucleon channel and determine the corresponding two pion contributions. The way of dealing with the intermediate states with energy higher than two pion masses is then analysed. In Section IV the way leading to integral equations for the individual angular momentum amplitudes is discussed, and the results obtained are briefly summarised and discussed. Appendix I contains a general analysis of the problem of the choice of a set of invariant operators and Appendix II contains some technical details on the separation of the s and p -wave two pion intermediate states from the states with higher angular momentum.

II. The Scattering Matrix.

II.1 Kinematics

We wish to discuss the elastic scattering of two nucleons, with initial and final 4-momenta n_1, p_1 and n_2, p_2 respectively [see Fig.1]. Our process will be characterized by three independent 4-vectors (because energy-momentum conservation gives : $n_1 + p_1 = n_2 + p_2$), which we shall choose as :

$$N = \frac{1}{2}(n_1 + n_2); \quad P = \frac{1}{2}(p_1 + p_2); \quad \Delta = (n_1 - n_2) = (p_2 - p_1) \quad (\text{II.1})$$

with $N \cdot \Delta = P \cdot \Delta = 0$.

As the initial and final particles are on the mass shell, only two independent scalars can be constructed from these three vectors; we shall define the three scalars :

$$\begin{aligned} w &= -(p_1 + n_1)^2 \\ t &= -(n_1 - n_2)^2 \\ \bar{t} &= -(p_1 - n_2)^2 \end{aligned} \quad (\text{II.2})$$

where e.g. $p_1^2 = \underline{p}^2 - p_0^2 = -m^2$, where m is the nucleon mass.

These are connected through the relation :

$$w + t + \bar{t} = 4m^2 \quad (\text{II.3})$$

We shall also need in the following a set of four orthogonal 4-vectors, which we shall choose as :

$$M = N + P; \quad K = N - P; \quad \Delta; \quad L_\lambda = \epsilon_{\lambda\mu\nu\rho} M_\mu K_\nu \Delta_\rho \quad (\text{II.4})$$

Definitions (II.1), (II.2), (II.4) give :

$$N^2 = P^2 = -\frac{1}{4}(w + \bar{t}); \quad \Delta^2 = -t; \quad M^2 = -w; \quad K^2 = -\bar{t}; \quad L^2 = -w \, t \, \bar{t} \quad (\text{II.5})$$

In the C.M. system of the two nucleons (II.2) reduces to :

$$\begin{aligned} w &= 4(k^2 + m^2) = 4E_k^2 \\ t &= -2k^2(1 - \cos\theta) \\ \bar{t} &= -2k^2(1 + \cos\theta) \end{aligned} \quad (\text{II.6})$$

where k , E_k and θ are, respectively, the momentum and the energy of each nucleon and the angle of scattering in the C.M. system.

The same 4-vectors n_1, n_2, p_1 and p_2 will describe also the process of elastic nucleon-antinucleon scattering; for instance, that process in which n_1, p_2 and $-n_2, -p_1$ are the initial and final momenta of the nucleon and anti-nucleon respectively. In the nucleon-antinucleon C.M. system (II.2) now gives :

$$\begin{aligned} w &= -2p^2(1 + \cos\psi) \\ t &= 4(p^2 + m^2) = 4E_p^2 \\ \bar{t} &= -2p^2(1 - \cos\psi) \end{aligned} \quad (\text{II.7})$$

where p , E_p and ψ are, respectively, the momentum and the energy of each nucleon or antinucleon and the angle of scattering in the C.M. system.

II.2 Scattering and causal matrices

We define a transition amplitude D through its connection with the S matrix :

$$S_{\beta\alpha} = \delta_{\beta\alpha} + i \int^4 (p_\beta - p_\alpha) D_{\beta\alpha} \quad (\text{II.8})$$

In the case of elastic nucleon-nucleon and nucleon-antinucleon scattering the transition amplitude is connected with the differential cross-section in the C.M. system by the relation :

$$\sigma_{\beta\alpha}(\theta, \varphi) = \pi^2 E^2 |D_{\beta\alpha}|^2 \quad (\text{II.9})$$

where E is the C.M. energy of each particle.

The scattering amplitude T is such that

$$\sigma_{\beta\alpha}(\theta, \varphi) = |T_{\beta\alpha}|^2 \quad (\text{II.10})$$

so that

$$D_{\beta\alpha} = 1/(\pi E) T_{\beta\alpha} \quad (\text{II.11})$$

The covariant Feynman amplitude F is given by

$$F_{\beta\alpha} = (4\pi) E/m^2 T_{\beta\alpha} \quad (\text{II.12})$$

The causal amplitude M , with which the rest of this paper is concerned, can be defined as in G.N.O. (form. 2-14) and coincides with the Feynman amplitude for physical scattering processes.

We shall deal in the following also with the nucleon-antinucleon absorption into two pions ($N\bar{N} \rightarrow \pi\pi$); for this process it will be convenient to introduce an absorption amplitude $\tau^{(8)}$, which is connected with the scattering amplitude by :

$$T_{\beta\alpha}(N\bar{N} \rightarrow \pi\pi) = -\frac{1}{8\pi} \frac{m}{E} \sqrt{\frac{q}{p}} \tau_{\beta\alpha} \quad (\text{II.13})$$

where p and q are the C.M. momentum of the nucleons and pions respectively. We note here that the relation (II.10) still holds, when $\sigma(\theta, \varphi)$ describes the differential cross-section for the process $N\bar{N} \rightarrow \pi\pi$. We have therefore, from (II.10), (II.13) :

$$\sigma_{\beta\alpha}(N\bar{N} \rightarrow \pi\pi) = \left| \frac{q}{p} \left| \frac{m}{2E} \frac{\tau_{\beta\alpha}}{4\pi} \right| \right|^2 \quad (\text{II.14})$$

The absorption amplitude is directly related to the Feynman amplitude for the process ($N\bar{N} \rightarrow \pi\pi$):

$$\tau_{\beta\alpha} = -2\mu F_{\beta\alpha}(N\bar{N} \rightarrow \pi\pi) \quad (\text{II.15})$$

where μ is the π -meson mass.

II.3 Separation of the causal amplitude M into invariants

The causal amplitude M is a covariant matrix in the spin and isotopic spin space of the incoming and outgoing particles. When using matrix elements $M_{\beta\alpha}$ for the elastic nucleon-nucleon and nucleon-antinucleon scattering, we shall assume in the following that the spin and isotopic spin operators in M are saturated as follows :

$$\begin{aligned} M_{\beta\alpha}(NN \rightarrow NN) &= \bar{u}_i(n_2) \bar{u}_\kappa(p_2) m_{ij}^{\kappa\ell} u_j(n_1) u_\ell(p_1) \\ M_{\beta\alpha}(N\bar{N} \rightarrow N\bar{N}) &= \bar{v}_i(-n_2) \bar{u}_\kappa(p_2) m_{ij}^{\kappa\ell} u_j(n_1) v_\ell(-p_1) = \\ &= \bar{v}_i(n') \bar{u}_\kappa(p_2) m_{ij}^{\kappa\ell} u_j(n_1) v_\ell(p_1') \end{aligned} \quad (\text{II.16})$$

where M_{ij}^{kl} , by the general substitution law ⁹⁾, is the same for both processes. Note that the spinor saturation of M is between n_1 and n_2 and between p_1 and p_2 independent of whether they are nucleons or antinucleons.

Since the operator M will be written as a sum of products of spin operators, we shall simply indicate the saturation of these operators by an index (n) or (p) . The quantities $u(q)$ and $v(q)$ are positive energy spinors satisfying respectively the equations :

$$\begin{aligned} (i \not{q} + m) u(q) &= 0 \\ (i \not{q} - m) v(q) &= 0 \end{aligned} \quad (\text{II.17})$$

where $q_0 = +(\underline{q}^2 + m^2)^{\frac{1}{2}}$.

Besides the spin and isotopic spin operators, M , being a covariant matrix, will depend only on the scalars that can be formed by means of the three independent vectors (II.1); in particular, we shall take M as dependent on the scalars w , t and $\bar{t}$ defined in (II.2). Therefore M can in principle be written as a linear combination of a set of invariant operators, O_j , whose coefficients o_j will be scalar functions of w , t and $\bar{t}$:

$$M_{\beta\alpha} = \sum_j o_j(w, t, \bar{t}) \langle \beta | O_j | \alpha \rangle \quad (\text{II.18})$$

(or course O_j will contain, in the general case, not only the spin and isotopic spin operators but also the dynamical variables, as, for instance, in G.N.O.).

There is a large freedom in the choice of the O_j ; each problem requires a choice of invariant operators most suitable to it. We shall see that our calculations, where use of the two dimensional dispersion relations is made, will require in an essential way a choice of invariant operators such that their coefficients can be expressed by a Mandelstam representation. Further a simplification occurs if these scalar functions behave simply under crossing ($t \rightleftharpoons \bar{t}$), which corresponds in the nucleon-nucleon channel to the interchange of the two initial (or final) particles. In order to find out the most suitable set of invariant operators we have proceeded as follows.

Perturbation calculations up to the 4th order indicate that the scalar functions which are coefficients of a particular set of invariant operators (which we shall call "perturbative invariants") have the right analytic properties to be represented by a two dimensional dispersion relation. This set of O_j is given by (we disregard here the isotopic spin part of O_j) :

$$\begin{aligned} P_1 &= 1^n 1^P ; \quad P_2 = (i \gamma^n \cdot P + i \gamma^P \cdot N) ; \\ P_3 &= (i \gamma^n \cdot P) (i \gamma^P \cdot N) ; \quad P_4 = \gamma^n \cdot \gamma^P ; \quad P_5 = \gamma_5^n \gamma_5^P \end{aligned} \quad (\text{II.19})$$

These invariants, however, present no simple behaviour under crossing (which is equivalent to a Fierz transformation on them); we have therefore to find a new set of invariants, linear combinations of the P_j defined by (II.19), with suitable coefficients that do not spoil the analytic behaviour of their corresponding scalar functions, and which transform in the simplest way under a Fierz transformation. This has led us to the use of linear combinations of Fermi invariants.

Indeed, if, based on perturbative arguments, the coefficients of the perturbative invariants P_j satisfy the M representation, then we show that this is also true for the coefficients of the Fermi invariants.

There are two ways of demonstrating that Fermi invariants are very probably "good invariants", at least in the framework of perturbation theory. We can express the perturbative invariants in terms of the Fermi invariants F_j defined as usual :

$$\begin{aligned} F_1 &= S = 1^n 1^P ; \quad F_2 = V = \gamma^n \cdot \gamma^P ; \\ F_3 &= T = \sum_{\mu < \nu} (i \gamma_\mu^n \gamma_\nu^n) (i \gamma_\mu^P \gamma_\nu^P) ; \quad F_4 = A = (i \gamma_5^n \gamma_\mu^n) (i \gamma_5^P \gamma_\mu^P) ; \\ F_5 &= \gamma_5^n \gamma_5^P \end{aligned} \quad (\text{II.20})$$

[see Appendix I] :

$$\begin{aligned} P_1 &= S \\ P_2 &= \frac{1}{m} \left[\frac{(\bar{t}-w)}{4} S - m^2 V + \frac{t}{4} T + \frac{(\bar{t}-w)}{4} P \right] \\ P_3 &= \frac{1}{4} \left[-(\bar{t}-w) V - t A + 4 m^2 P \right] \\ P_4 &= V \\ P_5 &= P \end{aligned} \quad (\text{II.21})$$

and note that no additional singularities are introduced in going from the P_j to the F_j . A second reason which suggests that a suitable extension of the Mandelstam technique to the case of two spinors is, by means of the Fermi invariants, the following.

In a second order perturbation calculation for the scattering of two fermions (Born approximation) the causal amplitude M is given by a single term, which is the product of a scalar function $1/(t^0 - t)$ (where t^0 is the square of the mass of the intermediate meson responsible for the interaction) and an invariant operator. This invariant operator will be one of the Fermi operators S, V, T, A, P if the intermediate meson is, respectively, a scalar, vector, tensor, axial and pseudoscalar particle. The scalar function present in the Born approximation is obviously a "good" function for writing down our two dimensional representation; it seems therefore that the Fermi invariants come naturally, into our development of M , as "good" invariants.

We want to stress, however, that these two arguments depend heavily on the assumption that qualitative results obtained at a given perturbation order for M are still valid for the complete causal amplitude. Therefore our assumption that the coefficients of Fermi invariants can be expressed by means of a Mandelstam representation is an hypothesis, on the same level as the present status of the Mandelstam representation for the scalar case.

II.4 Crossing symmetry

It is easy to see that invariant operators which are symmetric or anti-symmetric under crossing [i.e. under a Fierz transformation $P(2,4)$ which goes from $[\bar{u}(n_2) O^{(n)} u(n_1)] [\bar{u}(p_2) O^{(p)} u(p_1)]$ to $[\bar{u}(n_2) O^{(n)} u(p_1)] [\bar{u}(p_2) O^{(p)} u(n_1)]$ and therefore exchanges t into $\bar{t}$] can be constructed as linear combinations of the Fermi invariants, combinations which we shall denote henceforth by C_j . The choice of C_j will eventually depend on arguments of simplicity.

Let us define the index j in such a way that :

$$\begin{aligned} C_j & \text{ is } \overset{\text{anti-}}{\text{symmetric}} \text{ under } P(2,4) \text{ if } j \text{ is odd } (j=1,3,5) \\ C_j & \text{ is symmetric under } P(2,4) \text{ if } j \text{ is even } (j=2,4) \end{aligned} \quad (\text{II.22})$$

so that :

$$C_j \xrightarrow{P(2,4)} (-1)^j C_j \quad (\text{II.23})$$

We shall now go back to (II.18) and separate the spin and isotopic spin parts in O_j . Let us introduce the two projection operators in the isotopic spin space, P_T ($T = 0, 1$ is the total isotopic spin of the NN system), such that :

$$P_T |NN\rangle = |NN (T = T')\rangle \quad (\text{II.24})$$

in terms of the isotopic spin matrices τ^n and τ^p , P_0 and P_1 are given by :

$$P_0 = \frac{1}{4}(1 - \tau^n \cdot \tau^p) ; \quad P_1 = \frac{1}{4}(3 + \tau^n \cdot \tau^p) \quad (\text{II.25})$$

where, as usual, τ^n is to be saturated between $\bar{u}(n_2)$ and $u(n_1)$ and τ^p between $\bar{u}(p_2)$ and $u(p_1)$.

We can therefore write (II.18) as :

$$M_{\beta\alpha} = \sum_1^5 \sum_j \sum_{T=0,1} c_j^T(wt\bar{t}) \langle \beta | P_T C_j | \alpha \rangle \quad (\text{II.26})$$

As a $P(2,4)$ Fierz transformation is equivalent to an exchange $n_1 \leftrightarrow p_1$, the Pauli principle applied to the initial NN state will give :

$$\langle n_2 p_2 | M | n_1 p_1 \rangle = - \langle n_2 p_2 | M | p_1 n_1 \rangle \quad (\text{II.27})$$

where position, spin and isotopic spin of the two incoming particles are interchanged.

By using the crossing symmetry (II.23) and by noticing that :

$$\langle n_2 p_2 | P_T | n_1 p_1 \rangle = -(-1)^T \langle n_2 p_2 | P_T | p_1 n_1 \rangle \quad (\text{II.28})$$

we obtain, from (II.23), (II.26), (II.27) and (II.28), the following crossing conditions on the c_j^T :

$$c_j^T(w\bar{t}\bar{t}) = (-1)^{j+T} c_j^T(\bar{w}t\bar{t}) = \xi_{jT} c_j^T(\bar{w}t\bar{t}) \quad (\text{II.29})$$

which defines the meaning of the symbol ξ_{jT} , which will be used in the following to simplify our notations.

We shall now briefly discuss the choice of the set of C_j 's which we shall use in what follows. As our method will eventually lead to integral equations for the transition amplitudes corresponding to definite initial and final spin states (helicity states or singlet and triplet states), a simplicity criterion can be provided by the relations between the coefficients of the C_j 's, c_j , and the above-mentioned transition amplitudes. A set which turns out to be convenient from this point of view is given by :

$$\begin{aligned} C_1 &= S - V - T + A + P \\ C_2 &= 2S + V + A - 2P \\ C_3 &= 2S - V - A - 2P \\ C_4 &= 3S + T + 3P \\ C_5 &= S + V - T - A + P \end{aligned} \quad (\text{II.30})$$

We recall that the C_j 's with odd index j are odd under a Fierz transformation $P(2,4)$ and those with even j are even.

The usefulness of the set (II.30) can be easily seen by expressing the C_j as Fermi operators, defined by a saturation between spinors different from the standard saturation (II.16). We shall here merely sketch the argument which leads to the connection of the C_j 's, as given by (II.30), with the transition amplitudes for definite helicity states. Let us symbolize for simplicity the

standard saturation (II.16) of the Fermi invariants between spinors by the expression $(1, 2; 3, 4)$. If now we define a new set of Fermi invariants, $\bar{F}_j$, by sandwiching the Fermi operators between initial-initial (and final-final) particles, we shall get, for instance, an expression of the type : $(2, 4; 3, 1)$. It is easy to see that this permutation of the spinors can be obtained by the consecutive operation of a $P(1,2)$ and a $P(2,4)$ (Fierz) transformation to the standard order $(1, 2; 3, 4)$. $P(1,2)$ is diagonal [see ref.¹⁴] and gives plus sign for S, A and P and minus sign for V and T ; if we now apply the product matrix $P(1,2) P(2,4)$ to the Fermi invariants we get the linear combinations given by (II.30). It is therefore demonstrated that those combinations reduce each to a single Fermi invariant, saturated between the two initial and final particles. With the saturation $(2, 4; 3, 1)$ we get :

$$\bar{S} = C_1 ; \quad \bar{V} = C_2 ; \quad \bar{T} = C_4 ; \quad \bar{A} = C_3 ; \quad \bar{P} = C_5 \quad (\text{II.31})$$

To go now from the C_j 's to the scattering amplitudes corresponding to definite spin states, we have to saturate the invariant operators with spinors with definite helicity or with combinations of them corresponding to singlet and triplet states. It is clear that this connection will be much simplified, in the nucleon-nucleon C.M. system, by our new saturation; our invariant operators will be represented by single Fermi invariants, even or odd under the exchange $t \rightleftharpoons \bar{t}$, and the γ matrices contained in them will now connect the two incoming (outgoing) particles, which have opposite momenta in the C.M..

III. Two Dimensional Dispersion Relations.

III.1 Mandelstam representation for $c_j^T(w, t, \bar{t})$

We shall now assume a Mandelstam representation for the 10 scalar functions $c_j^T(w, t, \bar{t})$:

$$c_j^T(w, t, \bar{t}) = \frac{D_j(t, \bar{t})}{w_D - w} \mathcal{O}_T + \frac{1}{\pi^2} \int_{4\mu^2}^{w_D} dt' \int_{4\mu^2}^{\bar{t}} d\bar{t}' \frac{\mathcal{E}_j^T(t', \bar{t}')}{(t' - t)(\bar{t}' - \bar{t})} +$$

$$+ \left[\frac{R_j^T}{\mu^2 - t} + \frac{1}{\pi^2} \int_{4m^2}^{w_D} dw' \int_{4\mu^2}^{t'} dt' \frac{\mathcal{E}_j^T(w', t')}{(w' - w)(t' - t)} \right] + \mathcal{E}_{jT} \left[\text{sym. } (t \rightarrow \bar{t}) \right] \quad (\text{III.1})$$

where $D_j(t, \bar{t})$ and $\mathcal{E}_j^T(t, \bar{t})$ transform under crossing [see (II.29)] :

$$D_j^T(t, \bar{t}) = \mathcal{E}_{j0} D_j^T(\bar{t}, t)$$

$$\mathcal{E}_j^T(t, \bar{t}) = \mathcal{E}_{jT} \mathcal{E}_j^T(\bar{t}, t) \quad (\text{III.2})$$

The singularity in w at the deuteron mass squared $w_D = (2m - B)^2$, where B is the magnitude of the binding energy of the deuteron, gives a residue $D_j(t, \bar{t})$. Formula (4.43) of G.N.O. gives the residue due to the deuteron pole evaluated in the non relativistic limit as coefficients of the two dimensional invariants.

The residues R_j^T at the poles in t and $\bar{t}$ (which correspond, in the nucleon-nucleon channel, to the graphs for the Born approximation and, in the nucleon-antinucleon channel, to those for the Born approximation to the annihilation scattering) can be easily calculated from the explicit expression of the invariants c_j (II.24), (II.25). By using the two isotopic spin projection operators (II.27) and introducing $a_T = -3$ ($T=0$); $a_T=1$ ($T=1$) we have :

$$\begin{aligned}
R_1^T &= a_T g^2 \\
R_2^T &= -2a_T g^2 \\
R_3^T &= -2a_T g^2 \\
R_4^T &= 3a_T g^2 \\
R_5^T &= a_T g^2
\end{aligned} \tag{III.3}$$

where g is the renormalised (rationalised) pion-nucleon coupling constant.

The cut at w begins at $(2m)^2$; the cut in both t and in $\bar{t}$ starts at $(2\mu)^2$, which corresponds to the lowest energy for the process $N\bar{N} \rightarrow 2\pi$. Of course, from $t=(2\mu)^2$ to $t=(2m)^2$, this represents an unphysical region of the $N\bar{N}$ annihilation, whose corresponding channels have t or $\bar{t}$ as the energy squared. It will be our aim in what follows to take into account only the lowest lying singularities in w , t and $\bar{t}$. It is clear that such an approximation can be valid at low energy in the nucleon-nucleon scattering and not for physical energies in the nucleon-antinucleon scattering.

III.2 Dispersion relations at fixed w and the use of unitarity

Let us write down the representation (III.1) in the form of a one dimensional dispersion relation at fixed w . We can write :

$$c_j^T(w, t, \bar{t}) = \frac{D_j(t, \bar{t})}{w_D - w} \oint_{OT} + \left[\frac{R_j^T}{\mu^2 - t} + \frac{1}{\pi} \int \frac{S_j^T(w, t')}{(t' - t)} dt' \right] + \xi_{jT} \left[\text{sym.}(t \rightarrow \bar{t}) \right] \tag{III.4}$$

where

$$S_j^T(wt) = \frac{1}{\pi} \int_{4m^2}^T \frac{(w't)}{w'-w} dw' + \frac{1}{\pi} \int_{4\mu^2}^T \frac{(t, \bar{t}')}{t+\bar{t}'+w-4m^2} d\bar{t}' \quad (\text{III.5})$$

Relation (III.4) is suitable for the use of unitarity in the nucleon-antinucleon channel in which t is the energy and w and $\bar{t}$ are the two momentum transfers [see (II.7)], where the relation between w , t and $\bar{t}$ and the momentum and scattering angle in the $\bar{N}N$ C.M. system is given. As t is in this case positive and w and $\bar{t}$ negative we obtain (for $t > 4\mu^2$): *)

*) We shall frequently use the variables w t etc. in place of the integration variables w' t' ; the important thing to remember is that when this is the case their values are restricted to be positive by the ranges of integration independent of the channel.

$$\text{Im } c_j^T(w\bar{t}t) = S_j^T(wt) \quad (\text{III.6})$$

The unitarity condition in terms of the transition matrix D defined by (II.8), :

$$i(D_{\beta\alpha}^+ - D_{\beta\alpha}) = \sum_{\gamma} \int^4(p_{\gamma} - p_{\alpha}) D_{\beta\gamma}^+ D_{\gamma\alpha} \quad (\text{III.7})$$

Relation (III.7) leads, when written in terms of the causal matrix M and of the absorption matrix τ for the process $\bar{N}N \rightarrow 2\pi$ [see (II.13), (II.14), (II.15)], to the following relation :

$$i \langle N\bar{N} | M^+ - M | N\bar{N} \rangle = \frac{1}{(2\pi)^2 t} \sum_{(2\pi)} \langle N\bar{N} | \tau^+ | \pi\pi \rangle \langle \pi\pi | \tau | N\bar{N} \rangle + J_n \quad (\text{III.8})$$

where the sum is carried out over 2π intermediate states that conserve four momenta.

The contributions of the intermediate states in M^+M have been split into those coming from intermediate two pion states and those coming from higher energy intermediate states with baryonic number zero (J_n). As :

$$C_j^+ = C_j \quad (\text{III.9})$$

we can write (III.8) as :

$$\sum_{j=1}^5 \sum_{T=0,1} \text{Im } C_j^T \langle N\bar{N} | F_T C_j | N\bar{N} \rangle = \frac{\sum \langle N\bar{N} | T^+ | \pi\pi \rangle \langle \pi\pi | T | N\bar{N} \rangle}{2(2\pi)^2 t} + J_{3/2} \quad (\text{III.10})$$

J_n arise from intermediate states in (III.7) with mass higher than 3μ , this means that it will contribute to the cut in t and $\bar{t}$ in (III.4) only for $t' > 9\mu^2$. It will have, however, a cut in w beginning at the lowest limit, i.e. $w=4m^2$ as is clear from (III.5). We prefer then to write the contributions of J_n to C_j^T in a way which shows explicitly the strong dependence. We write

$$C_j^T(w, t, \bar{t}) = \frac{D_j(t, \bar{t})}{w_D - w} \oint_{OT} + R_j^T \left(\frac{1}{\mu^2 - t} + \xi_{jT} \frac{1}{\mu^2 - \bar{t}} \right) + \frac{1}{4\mu^2} \int_{t'-t}^{\bar{t}-t} \left[\rho_j^T(w, t') \left(\frac{1}{t'-t} + \xi_{jT} \frac{1}{t'-\bar{t}} \right) \right] dt' + \frac{1}{4\mu^2} \int \frac{\chi_j^T(w', t, \bar{t})}{w' - w} dw' \quad (\text{III.11})$$

where $\oint_j^T$ is the projection over the appropriate invariant of

$$\frac{\sum_{(2\pi)} \langle N\bar{N} | T^+ | 2\pi \rangle \langle 2\pi | T | N\bar{N} \rangle}{2(2\pi)^2 t} \quad (\text{III.12})$$

where χ_j^T has cuts in t and $\bar{t}$ starting only at $9\mu^2$. Following our general approach, we shall make the assumption that in $N-N$ scattering (in which t and $\bar{t}$ are negative) the dependence of the last integral in (III.11) on $t - \bar{t} = 4p^2 \cos\theta$ is sufficiently weak so as to contribute only to the lower partial waves. We shall discuss in Section IV the consequences of such a hypothesis. We stress here that owing to the crossing condition (II.29), χ_j^T is an even or odd function of $\cos\theta$ when $\xi_{jT} = 1$ or -1 respectively.

III.3 Evaluation of the two pion contribution

Let us call

$$\mathcal{M} = \sum_{(2\pi)} \langle \bar{p}_1 p_2 | T | 2\pi \rangle \langle 2\pi | T | n_1 \bar{n}_2 \rangle \quad (\text{III.13})$$

$\mathcal{M}$ corresponds to the graph of Fig.2 in which the intermediate pions are on the mass shell, where $n=n_1$, $n'=-n_2$ are the momenta of the incoming nucleon and antinucleon, respectively and $p=p_2$, $p'=-p_1$ those of the outgoing nucleon and antinucleon.

$\langle 2\pi | T | n_1 \bar{n}_2 \rangle$ is the matrix element for the process $\bar{N}N \rightarrow 2\pi$ and is an analytic continuation of that corresponding to pion nucleon scattering.

We write it in the usual way ⁸⁾ :

$$\langle \pi_\alpha \pi_\beta | T | n_1 \bar{n}_2 \rangle = \bar{v}(n') \left[-\left(A_n^+ \delta_{\alpha\beta} + \frac{1}{2} [\tau_\beta, \tau_\alpha] A_n^- \right) + i \gamma \cdot Q \left(B_n^+ \delta_{\alpha\beta} + \frac{1}{2} [\tau_\beta, \tau_\alpha] B_n^- \right) \right] u(n) \quad (\text{III.14})$$

where α, β are the isotopic spin indices of the two pions and

$$Q = \frac{q-q'}{2} \quad (\text{III.15})$$

A_n and B_n will be functions of

$$\begin{aligned} t &= -(n+n')^2 \\ s_n &= -(n-q)^2 = -(n^2 + q^2 - 2nq \cos \varphi_n) \\ \bar{s}_n &= -(n-q')^2 = -(n^2 + q^2 + 2nq \cos \varphi_n) \end{aligned} \quad (\text{III.16})$$

$$(t + s_n + \bar{s}_n = 2m^2 + 2\mu^2) \quad (\text{III.17})$$

where φ_n is the $\bar{N}N \rightarrow 2\pi$ scattering angle; n and q the $\bar{N}N$ modulus of the nucleon and pion momentum, respectively, in the C.M. system.

We see that the t defined here coincides with the t of (II.2), s_n and $\bar{s}_n$ depend on the scattering angle in the process $\bar{N}N \rightarrow 2\pi$. Similar reasonment can be made for

$$\langle \bar{p}_1 p_2 | T^+ | 2\pi \rangle$$

We split the sum over intermediate states in (III.13) into a sum over the isotopic indices of the pions (which is trivially done) and an integration over their momenta; we obtain then

$$\mathcal{M} = 3\mathcal{M}^+ + 2\tau^{(p)}\tau^{(n)}\mathcal{M}^- \quad (\text{III.18})$$

$$\mathcal{M}^\pm = \frac{1}{2} \int d^4Q \bar{u}(p) (-A_p^{\pm*} + i\gamma_0 Q B_p^{\pm*}) v(p') \cdot \bar{v}(m') (-A_n^\pm + i\gamma_0 Q B_n^\pm) u(m) \cdot \mathcal{J}((Q+A)^2, \mu^2) \mathcal{J}(Q^2, \mu^2) \quad (\text{III.19})$$

In order to perform the integration over the two $\mathcal{J}$ functions it is simpler to go into the $\bar{N}N$ centre of mass system. In such a system Δ has no spatial components, so that

$$\Delta_0^{c.m.} = \sqrt{-\Delta^2} = \sqrt{\frac{t}{4}} \quad (\text{III.20})$$

Writing $\mathcal{M}^\pm$ as an operator in spinor space, with our usual notations, we have

$$\mathcal{M}^\pm = \frac{1}{8} \left(\frac{t - 4\mu^2}{t} \right)^{1/2} (-A_p^{\pm*} + i\gamma_0 Q B_p^{\pm*}) (-A_n^\pm + i\gamma_0 Q B_n^\pm) d\Omega_Q \quad (\text{III.21})$$

where the integration is now extended only over the whole solid angle of Q , its modulus being given by

$$Q^2 = t - \frac{\mu^2}{4} \quad (\text{III.22})$$

In order to proceed in the evaluation of $\mathcal{M}$ we must introduce in (III.21) some knowledge about A and B. From their Mandelstam representation we know the location of their singularities, so we can write for them the following expression :

$$A_{(s\bar{s}t)}^{\pm} = \frac{1}{\pi} \int_{(m+\mu)^2}^{\infty} \sigma_A^{\pm}(s', t) \left(\frac{1}{s'-s} \pm \frac{1}{s'-\bar{s}} \right) ds' + \frac{1}{\pi} \int_{(2\mu)^2}^{\infty} \frac{\nu_A^{\pm}(t', s-\bar{s})}{t'-t} dt' \quad (\text{III.23})$$

$$B_{(s\bar{s}t)}^{\pm} = g^2 \left(\frac{1}{s-m^2} \mp \frac{1}{\bar{s}-m^2} \right) + \frac{1}{\pi} \int_{(m+\mu)^2}^{\infty} \sigma_B^{\pm}(s', t) \left(\frac{1}{s'-s} \mp \frac{1}{s'+\bar{s}} \right) ds' + \frac{1}{\pi} \int_{(2\mu)^2}^{\infty} \frac{\nu_B^{\pm}(t', s-\bar{s})}{t'-t} dt'$$

where the upper sign corresponds to isotopic index (+) and the lower to the index (-).

We distinguish for A and B three contributions; first the pole that represents the graph of Fig.3; next the cut in s and $\bar{s}$. This incorporates the π -N rescattering corrections to A and B and is that studied by Chew, Goldberger, Low and Nambu for π -N scattering⁸⁾; it is dominated by the ρ resonance. And lastly, the cut in t brings into the game the π - π interaction; this term has been considered by Frazer and Fulco¹⁰⁾ in the analysis of the electromagnetic structure of the nucleon, and recently by Bowcock, Cottingham and Lurié¹¹⁾ in the analysis of π -N scattering, under the hypothesis of p-wave π - π resonance. In these last investigations is discussed the location of the singularities of ν_A and ν_B in s and $\bar{s}$ as derived from the Mandelstam representation for A and B. These singularities are cuts in s and $\bar{s}$ that, however, begin only at $s=(m+2\mu)^2$ and $\bar{s}=(m+2\mu)^2$. The whole philosophy of our approach consists in making use of the weak dependence of every quantity in those variables for which the singularities are far from the region of interest. Or, alternatively, we lump the dependence of these variables into some constants which come from the lower terms of a development of the quantity in question.

The development parameter is just

$$s-\bar{s} = 4 pq \cos \varphi \quad (\text{III.24})$$

φ being the c.m. scattering angle in the reaction $N\bar{N} \rightarrow 2\pi$. Then a weak dependence on $s-\bar{s}$ means that only low waves of the two pions occur in the process $N\bar{N} \rightarrow 2\pi$. To fix better our ideas let us suppose that only s and p -waves occur in the development of the last terms in (III.23); it is clear, however, that higher waves of the two pions shall be present by virtue of the strong s and $\bar{s}$ dependence of the other terms in (III.23).

We stress that the restriction made is justified only for the values of t in which we are interested, i.e. a total energy of the two pion system of a few pion masses. The inclusion of higher waves coming from the integrals over t' in (23) would be a simple extension of the following arguments.

We notice first that not only the last term in (III.23) contributes to the s and p -waves; the other terms that we shall call $\mathcal{A}^\pm$ $\mathcal{B}^\pm$, where

$$\begin{aligned}\mathcal{A}_{(s\bar{s}t)}^\pm &= \frac{1}{\pi} \int_{(m+\mu)^2}^+ \bar{\sigma}_A(s',t) \left(\frac{1}{s'-s} \pm \frac{1}{s'-\bar{s}} \right) ds' \\ \mathcal{B}_{(s\bar{s}t)}^\pm &= \frac{1}{\pi} \int_{(m+\mu)^2}^+ \bar{\sigma}_B(s',t) \left(\frac{1}{s'-s} \mp \frac{1}{s'-\bar{s}} \right) ds'\end{aligned}\quad (\text{III.25})$$

with

$$\bar{\sigma}_B^\pm(s',t) = g^2 \delta(s'-m^2) + \sigma_B^\pm(s',t) \quad (\text{III.26})$$

also contribute to them. In Appendix 2 it is shown that the 2π s , p -wave contributions of $\mathcal{A}$ and $\mathcal{B}$ are given respectively by

$$\begin{aligned}\alpha^\pm &= \alpha_0^\pm(t) + \alpha_1^\pm(t) \frac{s-\bar{s}}{4} \\ \beta^\pm &= \beta_1^\pm(t)\end{aligned}\quad (\text{III.27})$$

where

$$\begin{aligned}\alpha_0^+(t) &= -\frac{1}{2} A_0 \\ \alpha_1^-(t) &= \frac{3}{2} \left(\frac{1}{p_9} \alpha_1^- + \frac{m}{p^2} \mathcal{D}_2^- \right) \\ \beta_1^-(t) &= \frac{1}{2} (\mathcal{D}_0^- - \mathcal{D}_2^-)\end{aligned}\quad (\text{III.28})$$

$$\alpha_n^- = \alpha_1^+ = \beta_1^+ = 0$$

having

$$\begin{aligned} \mathcal{A}_J^\pm(t) &= \int \mathcal{A}^\pm P_J(\cos \varphi) d\cos \varphi \\ \mathcal{D}_J^\pm(t) &= \int \mathcal{B}^\pm P_J(\cos \varphi) d\cos \varphi \end{aligned} \quad (\text{III.29})$$

Let us denote by A_{sp} and B_{sp} the whole s and p-wave contribution to A and B respectively. Then, because we supposed that the last term in (III.23) contributes only to such waves

$$\begin{aligned} A_{sp}^\pm &= \alpha^\pm + \frac{1}{\pi} \int \frac{\mathcal{V}_A^\pm(t', s-\bar{s})}{(2\mu)^2 t'-t} dt' \\ B_{sp}^\pm &= \beta^\pm + \frac{1}{\pi} \int \frac{\mathcal{V}_B^\pm(t', s-\bar{s})}{(2\mu)^2 t'-t} dt' \end{aligned} \quad (\text{III.30})$$

so we can write

$$\begin{aligned} A^\pm &= \mathcal{A}^\pm - \alpha^\pm + A_{sp}^\pm \\ B^\pm &= \mathcal{B}^\pm - \beta^\pm + B_{sp}^\pm \end{aligned} \quad (\text{III.31})$$

In analogy with (III.27) we can write for A_{sp} and B_{sp} the development

$$\begin{aligned} A_{sp}^\pm &= \lambda_0^\pm(t) + \lambda_1^\pm(t) \frac{(s-\bar{s})}{4} \\ B_{sp}^\pm &= \gamma_1^\pm(t) \end{aligned} \quad (\text{III.32})$$

with

$$\lambda_0^- = \lambda_1^+ = \gamma_1^+ = 0 \quad (\text{III.33})$$

We are now ready to go further in the evaluation of (III.21). For every A_n , B_n , A_p or B_p we shall use an expression of the form (III.31) with (III.27) and (III.32) where $s-\bar{s}$ will be either $s_n-\bar{s}_n$ and $s_p-\bar{s}_p$ respectively.

In Appendix 2 it is shown that the iteration of the terms containing A_{sp} and B_{sp} with those containing $A-\alpha$ and $B-\beta$ is zero. This is physically clear because interference between terms containing only s and p-waves (as A_{sp} and B_{sp}) with $A-\alpha$ and $B-\beta$, which by construction

do not contain them, must certainly vanish. Since α and A_{sp} (or β and B_{sp}) have the same form [(III.27), (III.32)] a consequence of the previous statement is that the iteration of α and β with $\mathcal{A}$ and $\mathcal{B}$ gives the same contribution as their iteration with α and β respectively ^{*)}.

^{*)} This reflects the fact that the interference of s and p-waves with the whole $\mathcal{A}$ and $\mathcal{B}$ is the same as their interference with only the s and p part of $\mathcal{A}$ and $\mathcal{B}$, i.e. α and β .

Then we can write for (III.21)

$$\mathcal{M}^{\pm} = \frac{1}{8} \left(t - 4\mu^2 \right)^{\frac{1}{2}} \left\{ \left(-\mathcal{A}_p^{\pm} + i\gamma \cdot \underline{Q} \mathcal{B}_p^{\pm} \right) \left(-\mathcal{A}_n^{\pm} + i\gamma \cdot \underline{Q} \mathcal{B}_n^{\pm} \right) + \right. \\ \left. + \left(-[A_{sp}^{\pm} - \alpha^{\pm}]_p + i\gamma \cdot \underline{Q} [B_{sp}^{\pm} - \beta^{\pm}]_p \right) \left(-[A_{sp}^{\pm} - \alpha^{\pm}]_n + i\gamma \cdot \underline{Q} [B_{sp}^{\pm} - \beta^{\pm}]_n \right) \right\} d\Omega_Q \quad (\text{III.34})$$

where we have used the fact that $\mathcal{A}, \mathcal{B}, \alpha$ and β are real ^{*)} as can be seen

^{*)} This is true because $t > 4\mu^2$ [we are calculating the $\mathcal{S}(wt)$ to be used in the integral of (III.11) with t substituted by t' and $s_n, \bar{s}_n, s_p$ and $\bar{s}_p$ over which we integrate in (III.33)] are momentum transfers (and therefore negative). This is the region (below threshold) for the process $N\bar{N} \rightarrow 2\pi$.

from (III.25), (III.27), (III.29). It is not difficult to develop (III.34) into invariants. We obtain readily

$$\mathcal{M}^{\pm} = (a^{\pm} + 2mc^{\pm} + m^2 e^{\pm}) - (b^{\pm} + md^{\pm}) [i\gamma \cdot N + i\gamma \cdot P] + \\ + e^{\pm} [i\gamma \cdot N i\gamma \cdot P] + f^{\pm} [i\gamma \cdot P i\gamma \cdot N] \quad (\text{III.35})$$

where

$$\begin{aligned}
a^{\pm} &= \frac{1}{8} \left(\frac{t-4u^2}{t} \right)^{1/2} \left[\mathcal{A}_p^{\pm} \mathcal{A}_n^{\pm} + A_{sp_p}^{\pm*} A_{sp_n}^{\pm} - \alpha_p^{\pm} \alpha_n^{\pm} \right] d\Omega_Q \\
b^{\pm} &= \frac{1}{8} \left(\frac{t-4u^2}{t} \right)^{1/2} \left[\mathcal{B}_p^{\pm} \mathcal{A}_n^{\pm} + \operatorname{Re}(B_{sp_p}^{\pm*} A_{sp_n}^{\pm}) - \beta_p^{\pm} \alpha_n^{\pm} \right] \left[\frac{Q \cdot \underline{M}}{M^2} + \frac{Q \cdot \underline{K}}{K^2} \right] d\Omega_Q \\
c^{\pm} &= \frac{1}{8} \left(\frac{t-4u^2}{t} \right)^{1/2} \left[\mathcal{B}_p^{\pm} \mathcal{A}_n^{\pm} + \operatorname{Re}(B_{sp_p}^{\pm*} A_{sp_n}^{\pm}) - \beta_p^{\pm} \alpha_n^{\pm} \right] \left[\frac{Q \cdot \underline{M}}{M^2} - \frac{Q \cdot \underline{K}}{K^2} \right] d\Omega_Q \\
d^{\pm} &= \frac{1}{8} \frac{\left(\frac{t-4u^2}{t} \right)^{1/2}}{M^2 K^2} \left[\frac{(Q \cdot \underline{M})^2}{M^2} (2K^2 + M^2) + \frac{(Q \cdot \underline{K})^2}{K^2} (2M^2 + K^2) - Q^2 (K^2 + M^2) \right] \\
&\quad \cdot \left[\mathcal{D}_p^{\pm} \mathcal{D}_n^{\pm} + B_{sp_p}^{\pm*} B_{sp_n}^{\pm} - \beta_p^{\pm} \beta_n^{\pm} \right] d\Omega_Q \\
e^{\pm} &= \frac{1}{8} \frac{\left(\frac{t-4u^2}{t} \right)^{1/2}}{M^2 K^2} \left[\frac{(Q \cdot \underline{M})^2}{M^2} (2K^2 - M^2) + \frac{(Q \cdot \underline{K})^2}{K^2} (-2M^2 + K^2) - Q^2 (K^2 - M^2) \right] \cdot \left[\mathcal{D}_p^{\pm} \mathcal{D}_n^{\pm} + B_{sp_p}^{\pm*} B_{sp_n}^{\pm} - \beta_p^{\pm} \beta_n^{\pm} \right] d\Omega_Q \\
f^{\pm} &= \frac{1}{8} \left(\frac{t-4u^2}{t} \right)^{1/2} \left[Q^2 - \frac{(Q \cdot \underline{M})^2}{M^2} - \frac{(Q \cdot \underline{K})^2}{K^2} \right] \left[\mathcal{D}_p^{\pm} \mathcal{D}_n^{\pm} + B_{sp_p}^{\pm*} B_{sp_n}^{\pm} - \beta_p^{\pm} \beta_n^{\pm} \right] d\Omega_Q
\end{aligned}$$

where $\underline{M}$ and $\underline{K}$ are the spatial components of M and K , as defined in (II.4) in the $\overline{NN}$ centre of mass system. In such a system the time component of M and K vanish, so that

$$\underline{M}^2 = -w \quad \underline{K}^2 = -\bar{t} \quad (\text{III.37})$$

Let us consider the integration over $\mathcal{A}$ and $\mathcal{B}$ in (III.36). We see that the dependence on Ω_Q for every $\mathcal{A}$ or $\mathcal{B}$ is explicitly exhibited in s and $\bar{s}$ ($s_n, \bar{s}_n$ and $s_p, \bar{s}_p$) in the denominators of (III.25). Then the integration over Ω_Q could be performed explicitly. We prefer, however, to carry out one integration contained in $d\Omega_Q$ and express the other in a parametric form

$$\int \mathcal{L}(xt) \left(\frac{1}{x-\bar{t}} \pm \frac{1}{x-w} \right) dx$$

where, however, the dependence on x of $\mathcal{L}$ is determined. We shall see later the reason for this form. In order to integrate the terms in (III.36) containing $A_{sp_p}^{\pm} \alpha$ and β , we take for them their expressions (III.27) and (III.32) in which their dependence on Ω_Q appears explicitly in $s_n - \bar{s}_n = 4Q \cdot \underline{N}$ in one factor and $s_p - \bar{s}_p = 4Q \cdot \underline{P}$ in the other. The integration over $d\Omega_Q$ can then be performed easily. By collecting the results we find:

$$\begin{aligned}
a^{\pm} &= \frac{1}{\pi} \int_{x_0}^{\pm} \mathcal{H}_a^{\pm}(tx) \left(\frac{1}{x-t} \pm \frac{1}{x-w} \right) dx + \frac{\pi}{2} \left(\frac{t-4\mu^2}{t} \right)^{1/2} \left[\alpha_0^{\pm} + \frac{1}{3} (\mu_1^{\pm})^2 - \alpha_1^{\pm} \right] \left(\frac{w-t}{4} \right) \left(\frac{t-\mu^2}{4} \right) \\
b^{\pm} &= \frac{1}{\pi} \int_{x_0}^{\pm} \mathcal{H}_b^{\pm}(tx) \left(\frac{1}{x-t} \mp \frac{1}{x-w} \right) dx + \frac{\pi}{2} \left(\frac{t-4\mu^2}{t} \right)^{1/2} \left[\operatorname{Re}(\mu_1^{\pm}) \lambda_1^{\pm} - \alpha_1^{\pm} \beta_1^{\pm} \right] \left(\frac{t-\mu^2}{4} \right) \\
c^{\pm} &= \frac{1}{\pi} \int_{x_0}^{\pm} \mathcal{H}_c^{\pm}(tx) \left(\frac{1}{x-t} \pm \frac{1}{x-w} \right) dx \\
d^{\pm} &= \frac{1}{\pi} \int_{x_0}^{\pm} \mathcal{H}_d^{\pm}(tx) \left(\frac{1}{x-t} \mp \frac{1}{x-w} \right) dx \\
e^{\pm} &= \frac{1}{\pi} \int_{x_0}^{\pm} \mathcal{H}_e^{\pm}(tx) \left(\frac{1}{x-t} \pm \frac{1}{x-w} \right) dx \\
f^{\pm} &= \frac{1}{\pi} \int_{x_0}^{\pm} \mathcal{H}_f^{\pm}(tx) \left(\frac{1}{x-t} \mp \frac{1}{x-w} \right) dx + \frac{\pi}{3} \left(\frac{t-4\mu^2}{t} \right)^{1/2} (\mu_1^{\pm})^2 - \beta_1^{\pm} \left(\frac{t-\mu^2}{4} \right)
\end{aligned} \tag{III.38}$$

with *)

*) If we take $\sigma_A^{\pm} = \sigma_B^{\pm} = 0$, so that only the $\mathcal{J}$ functions of $\bar{\sigma}_B$ in (III.26) remain in (III.39), the expression for the $\mathcal{H}$ are just real analytic expressions for $t > 4\mu^2$ $x > 4m^2$. This is, on the other hand, just the perturbative fourth order (renormalized) contribution. We see then from (III.38), (III.35), (III.12) and (III.11) that the fourth order perturbative calculation gives for the coefficients of the "perturbative" invariants (III.) a Mandelstam two dimensional spectral representation. We have already used this result in Section II.

$$\begin{aligned}
\mathcal{H}_a^{\pm}(tx) &= \frac{2}{\sqrt{t}} \iint ds' ds'' \bar{\sigma}_A^{\pm}(s', t) \bar{\sigma}_A^{\pm}(s'', t) \mathcal{K}(s' s'' tx) \\
\mathcal{H}_b^{\pm}(tx) &= \frac{1}{\sqrt{t}} \iint ds' ds'' \bar{\sigma}_B^{\pm}(s', t) \bar{\sigma}_A^{\pm}(s'', t) \left[\frac{Z(s' s'' t)}{x-4m^2+t} + \frac{Y(s' s'')}{x} \right] \mathcal{K} \\
\mathcal{H}_c^{\pm}(tx) &= \frac{1}{\sqrt{t}} \iint ds' ds'' \bar{\sigma}_B^{\pm}(s', t) \bar{\sigma}_A^{\pm}(s'', t) \left[\frac{Z(s' s'' t)}{x-4m^2+t} - \frac{Y(s' s'')}{x} \right] \mathcal{K} \\
\mathcal{H}_d^{\pm}(tx) &= \frac{1}{2\sqrt{t}} \iint ds' ds'' \bar{\sigma}_B^{\pm}(s', t) \bar{\sigma}_B^{\pm}(s'', t) \left\{ \frac{t-4m^2}{x^2(x-4m^2+t)^2} \frac{1}{\mathcal{K}} + \left[\frac{Z^2}{(x-4m^2+t)^2} + \frac{Y^2}{x^2} \right] \mathcal{K} \right\} \\
\mathcal{H}_e^{\pm}(tx) &= \frac{1}{2\sqrt{t}} \iint ds' ds'' \bar{\sigma}_B^{\pm}(s', t) \bar{\sigma}_B^{\pm}(s'', t) \left\{ - \frac{2x-4m^2+t}{x^2(x-4m^2+t)^2} \frac{1}{\mathcal{K}} + \left[\frac{Z^2}{(x-4m^2+t)^2} - \frac{Y^2}{x^2} \right] \mathcal{K} \right\} \\
\mathcal{H}_f^{\pm}(tx) &= \frac{1}{2\sqrt{t}} \iint ds' ds'' \bar{\sigma}_B^{\pm}(s', t) \bar{\sigma}_B^{\pm}(s'', t) \left\{ \frac{1}{x(x-4m^2+t)} \frac{1}{\mathcal{K}} \right\}
\end{aligned} \tag{III.39}$$

where

$$K(s's''t) = \left[t(X - [\sqrt{s'} + \sqrt{s''}]^2)(X - [\sqrt{s'} - \sqrt{s''}]^2) - 4m^2(s' - s'')^2 - 4X \{ s's'' + [\mu^2 - m^2]^2 - (\mu^2 + m^2)(s' + s'') + \mu^2 X \} \right]^{-1/2} \quad (\text{III.40})$$

$$Z(s's''t) = s' + s'' - 2(\mu^2 + m^2)t, \quad Y(s's'') = s' - s'' \quad (\text{III.41})$$

and

$$X_0 = \frac{2}{t - 4\mu^2} \left\{ s's'' + (s' + s'') \left(\frac{t}{2} - \mu^2 - m^2 \right) + (\mu^2 - m^2)^2 + \sqrt{[s'^2 + s'(t - 2\mu^2 - 2m^2) + (\mu^2 - m^2)^2][s''^2 + s''(t - 2\mu^2 - 2m^2) + (\mu^2 - m^2)^2]} \right\} \quad (\text{III.42})$$

On the other hand, by making use of (III.35), (III.18), (III.12) and (II.24), (II.25) and (II.), we can write for the functions ρ_j^T of (III.11)

$$\rho_1^T = 1/(8t(2\pi)^2) \left[a^T/2 + \frac{b^T}{m}(w + 2t - 3m^2) + mc^T + d^T(w + 2t - 3m^2) + e^T(\bar{t} - m^2) + 2f^T \right]$$

$$\rho_2^T = 1/(16t(2\pi)^2) \left[a^T + 2mb^T + 2mc^T + 2m^2d^T + (w - 2m^2)e^T - 2f^T \right]$$

$$\rho_3^T = 1/(16t(2\pi)^2) \left[a^T - 2mb^T + 2mc^T - 2m^2d^T + (2m^2 - w)e^T + 2f^T \right]$$

$$\rho_4^T = 1/(16t(2\pi)^2) \left[a^T + \frac{b^T}{m}(w - 2m^2) + 2mc^T + (w - 2m^2)d^T + 2m^2e^T \right] \quad (\text{III.43})$$

$$\rho_5^T = 1/(8t(2\pi)^2) \left[a^T/2 + \frac{b^T}{m}(w + 2t + m^2) + mc^T + d^T(w + 2t + m^2) + e^T(\bar{t} + 3m^2) - 2f^T \right]$$

where

$$\begin{aligned} a^0 &= 3a^+ - 6a^- \\ a^1 &= 3a^+ + 2a^- \end{aligned} \quad (\text{III.44})$$

and similar equations for b^0, b^1 , etc.

This means, by looking to (III.38), (III.39)-(III.41) that we have obtained for all ρ_j^T expressions that depend on σ_A and σ_B on one side and on functions $\lambda_0^+(t)$, $\lambda_1^-(t)$, $\eta_1^-(t)$ on the other (α_0^+ , α_1^- and β_1^- are given by (III.28), (III.29) as functions of σ_A and σ_B). The former are given by π -N scattering; if for instance we take into account only the (3.3) resonance in π -N rescattering corrections as done by Chew, Goldberger, Low and Nambu⁸⁾ one finds

$$\begin{aligned}\sigma_A^+(st) &= -2\sigma_A^-(st) = \frac{8\pi}{3} \left[\frac{\sqrt{s+m^2}}{E+m} 3\left(1+\frac{t}{2k^2}\right) + \frac{\sqrt{s-m^2}}{E-m} \right] \frac{\sin^2 \delta_{33}(s)}{k} \\ \sigma_B^+(st) &= -2\sigma_B^-(st) = -\frac{8\pi}{3} \left[\frac{3}{E+m} \left(1+\frac{t}{2k^2}\right) - \frac{1}{E-m} \right] \frac{\sin^2 \delta_{33}(s)}{k}\end{aligned}\quad (\text{III.45})$$

where $E = \frac{s+m^2-\mu^2}{2\sqrt{s}}$, $k^2 = \frac{\frac{1}{4}(s-m^2-\mu^2)-\mu^2 m^2}{s}$ and δ_{33} is the usual $(\frac{3}{2}, \frac{3}{2}) \pi$ -N phaseshift.

The functions $\lambda_0^+(t)$, $\lambda_1^-(t)$, $\eta_1^-(t)$ come instead from the $\bar{N}\bar{N} \rightarrow 2\pi$ process; unhappily we have not much information about it. The fact that they represent the whole 2π s and p-wave contribution to such a reaction would allow, however, the use of unitarity in order to specify their properties. For its discussion we refer to the work of Bowcock, Lurié and Cottingham¹¹⁾ in which (under the assumption of a sharp resonance in p-wave pion-pion scattering), $\lambda_1^-(t)$ and $\eta_1^-(t)$ ($\lambda_0^+ = 0$) are obtained in terms of certain constants. In a future paper, consequences of the π - π resonance in N-N scattering, will be investigated.

IV. Determination of χ_j^T and discussion.

The previous argument makes clear that we have obtained explicit formula to evaluate the integrals over ρ_j^T in the expressions for c_j^T in (III.11) in terms of certain parameters that come from the analysis of simpler processes. What remains, is to evaluate the functions χ_j^T of (III.11). This can be accomplished in a straightforward way by using unitarity in the N-N channel. In this channel $w > 4m^2$, while t and $\bar{t}$ are negative. Let us consider, for the moment, energies for the NN scattering system below the threshold for meson production (i.e. $w < (2m+\mu)^2$). Let us now take the imaginary part of both sides of Eq. (III.11) where for the ρ_j^T we put their values as given by (III.43) and (III.38). Clearly a contribution to the imaginary part of the r.h.s. of Eq. (III.11) comes from the last integral in it; another contribution comes, however, from the first integral. Indeed, if we look at the expressions for ρ_j as given from (III.43) and (III.38), we see that it has an imaginary part coming from the zero in the denominator $x-w$ in (III.38). Every integral over $\mathcal{H}_j(tx)$ in (III.38) will then contribute $(\mathcal{H}_j(tw))$ to the imaginary part of a, b , etc. The $\mathcal{H}_j(tw)$ are now given by (III.39); it is not difficult, however, to realise that owing to the fact that we consider for the moment $w < (2m+\mu)^2$ only the $\bar{\mathcal{O}}$ function contained in $\bar{\mathcal{O}}_B$ (III.26) contributes to the integrals over σ_A and $\bar{\mathcal{O}}_B$ in (III.39). This can be seen by the fact that if $\sqrt{s'} + \sqrt{s''} > (2m+\mu)^2$, x_0 turns out to be greater than $(2m+\mu)^2$ so that no imaginary part can come from (III.38).*) This means that for the

*) This implies that we have succeeded in separating in ρ_j that part which besides contributing to the lower cut in t and $\bar{t}$ in (III.11), has also a cut in w starting from $4m^2$. This part is just the re-normalised fourth order contribution as is clearly recognisable from the fact that only the $\mathcal{O}$ function in B (III.23) contributes to it.

range of variables we are now interested in,

$$\operatorname{Im} a^{\pm} = \operatorname{Im} b^{\pm} = \operatorname{Im} c^{\pm} = 0$$

$$\operatorname{Im} d^{\pm} = \mp \frac{g^4}{2\sqrt{t}} \left(\frac{t-4m^2}{w^2(w-4m^2+t)} \frac{1}{\chi} + \frac{(t-2\mu^2)^2}{(w-4m^2+t)^2} \chi \right)$$

$$\operatorname{Im} e^{\pm} = \pm \frac{g^4}{2\sqrt{t}} \left(-\frac{2w-4m^2+t}{w^2(w-4m^2+t)} \frac{1}{\chi} + \frac{(t-2\mu^2)^2}{(w-4m^2+t)^2} \chi \right)$$

(IV.1)

$$\operatorname{Im} f^{\pm} = \mp \frac{g^4}{2\sqrt{t}} \frac{1}{w(w-4m^2+t)} \frac{1}{\chi}$$

where now

$$|\chi| \equiv |\chi(m^2, m^2, t, w)| = \left[x t (x-2m^2) - 4\mu^2 x (x-4m^2+\mu^2) \right]^{-1/2} \quad (\text{IV.2})$$

From (IV.1) and (IV.2) we obtain immediately the explicit value of $\operatorname{Im} \mathcal{F}_j^T$.

Finally from (III.11) we write

$$\chi_j^T(w, t-\bar{t}) = \operatorname{Im} \mathcal{F}_j^T(w, t) - \int \operatorname{Im} \mathcal{F}_j^T(w, t') \left(\frac{1}{t'-t} + \mathcal{E}_j^T \frac{1}{t'-\bar{t}} \right) dt' \quad (\text{IV.3})$$

where now $\operatorname{Im} \mathcal{F}_j^T$ are given (III.43) and (IV.1).

As discussed previously, the cuts $\chi_j^T(w, t-\bar{t})$ in t and $\bar{t}$ begin only at $9\mu^2$ so that, according to the philosophy of our approach, it will have a weak dependence on $t-\bar{t} = 4p^2 \cos\theta$ and so in its development into partial waves, we shall keep only the lower waves ^{*)}.

*) By "low" waves we mean those for which the imaginary part of the amplitude is not negligible with respect to its real part.

Then developing the r.h.s. of (IV.3) into partial waves and retaining only the lower ones, we shall be able to express χ_j^T in terms of the phase shifts and coupling parameters of these lower waves.

We can now go back to (III.11), perform a similar development and insert the expressions for χ_j^T into the r.h.s. integral. This will give us a set of coupled linear, inhomogeneous integral equations relating the phase shifts and coupling parameters for the lower waves. The solution of such integral equations present no conceptual difficulty and are of a form similar to that used by Noyes and Wong⁷⁾. But now we see that the inhomogeneous part of the integral equation (coming from all the r.h.s. of (III.11) except the last integral) contain not only the terms coming from the deuteron and the one pion exchange, but also the integral over $\int_j \bar{\Gamma}$ that is given by (III.43), (III.38), (III.39) in which the π -N rescattering and the π - π interaction play an important role. The higher waves are given then just by such contributions plus the pole terms.

In a future investigation the above-mentioned integral equation will be further examined.

We note that in the previous arguments we have approximated χ_j^T in (III.11) by the expression for it found from unitarity (IV.3) which is, however, correct only in the low energy region. This means that we have taken into account badly the high energy contributions to the last integral of (III.11). However, these contributions depend weakly on $t-\bar{t}$ and, because they come from high values of w' , also, are weakly dependent on w . This means that they can be characterized by constants or, at most, by constants arising from the lower terms in a development in the variables. Such constants enter as parameters into our theory. The number of them will clearly depend on the region of the variables in which we wish to reach definite conclusions, i.e. the energy up to which one is interested in. The theory is however meaningful only if the number of parameters is rather restricted: this is the whole philosophy of our approach. This philosophy has as background the assumption that the weight functions (of the Mandelstam representations) in all the phenomena we are interested in have no "strange" behaviour, so that the main features are still understandable just from the positions of the singularities. This is a theoretical assumption whose consequences are reflected in our results in which the whole aspect of low energy N-N scattering is given in terms of only a few parameters.

We want to discuss here briefly the difference between our approach and that of Goldberger, Oehme and Nambu ^{3),4)}. They used dispersion relations at fixed momentum transfer t of the form

$$F(t, w) = \frac{R_1}{m_0 - w} + \frac{R_2}{\mu^2 - \bar{t}} + \frac{1}{\pi} \int_{(2\mu)^2}^{\infty} \frac{\text{Im } F(x, t)}{x - w} dx + \frac{1}{\pi} \int_{(2\mu)^2}^{\infty} \frac{\text{Im } F(x, \bar{t})}{x - \bar{t}} dx \quad (\text{IV.4})$$

Goldberger and Oehme have analysed the integral from $(2\mu)^2$ to $(2m)^2$ by considering the two pion contributions to it.

Let us analyse the relation (IV.4) in some detail. First we note that it is surely most appropriate to use it in the case of forward scattering ($t=0$) in which, by virtue of the optical theorem, the imaginary part of the amplitudes in the first and second integrals in (IV.4) is connected to $N-N$ and $N-\bar{N}$ total scattering cross-sections respectively. In this case (IV.4) is transformed in a sum rule similar to that analysed by Puppi and Stanghellini for $\pi-N$ scattering ¹³⁾.

But if we want to obtain more information from the theory (i.e. phase shifts) we are forced to use the dispersion relations also for $t \neq 0$, for instance, by doing a development in partial wave. This means to integrate t over the appropriate Legendre functions. It is clear, however, that in regions in which there are singularities in t (poles or cuts) such developments have no meaning.

This means that in some way one must identify in (IV.4) the terms which, having singularities in t , would give contribution to many waves in $N-N$ scattering. For instance, neglect of the high energy contribution in the first integral in (IV.4) can be misleading if one does not single out the strong t dependence that such contributions have; i.e. the high energy contribution would generate a function of t (not developable) that would prevent a development in partial waves. This is also true for the low energy contribution we have seen in Section III that the fourth order perturbative contribution of Fig.4 is just that term which is strongly dependent both in t and w (low lying singularities in both variables).

There is a term analogous to this one that has instead a strong dependence in w and $\bar{t}$; this is given by the perturbative graph of Fig.5. This contribution is considered by Goldberger and Oehme and comes from the two pion contribution to the last integral in (IV.4). The reason why in such an approach the graph of Fig.5 is taken into account while that of Fig.4 is not included, comes from the asymmetry of Eq.(IV.4) in the treatment of t and $\bar{t}$.

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A P P E N D I X I

The invariant operators in the nucleon-nucleon and nucleon-antinucleon elastic scattering

In order to build up all the possible independent invariant operators for NN and $\bar{N}N$ elastic scattering, we have at our disposal two γ matrices γ^n and γ^p , to be saturated between spinors as in (II.16) and the three independent vectors (II.1) : P, N and Δ . It is clearly possible to construct from these five 4-vectors quite a few scalar quantities (matrices in the spin space of the two fermions) but many of them can be further reduced, by using the Dirac equation (II.17), to simpler ones. At the end of this process of reduction, we are left with a set of 16 possible invariants [this can be most easily seen by using the separation, (as given by Michel ¹⁴), of all the forms that can be obtained by using two γ matrices, into five "classes" characterised by "class" number Q (Q = 0,1,2,3,4 indicating that the two γ matrices combine to give, respectively, a scalar, vector, tensor, axial pseudoscalar matrix)]. These 16 invariants belong to three different classes [as we have no independent pseudo-vector at our disposal, no combination of γ matrices present in the orders with Q=3 and Q=4 can be saturated in order to give eventually a scalar]; five of them are the Fermi invariants (Q=0) and the others are :

$$\begin{aligned}
 Q = 1 : & \quad (1^n \cdot \gamma^p \cdot N) ; (i \gamma^n \cdot P \cdot 1^p) ; (\gamma^n \gamma^p \cdot \gamma^p \cdot N) ; \\
 & \quad (\gamma^n \cdot \gamma^n \cdot P \cdot \gamma^p) ; (\gamma^n \gamma_5^n \cdot \gamma^p \gamma_5^p \cdot \gamma^p \cdot N) ; \\
 & \quad (\gamma^n \gamma_5^n \cdot \gamma^n \cdot P \cdot \gamma^p \gamma_5^p) ; (\gamma_5^n \gamma_5^p \cdot \gamma^p \cdot N) ; (AI.1) \\
 & \quad (\gamma_5^n \cdot \gamma^n \cdot P \cdot \gamma_5^p) \\
 Q = 2 : &
 \end{aligned}$$

$$(i \gamma^n \cdot P \cdot \gamma^p \cdot N) ; (\gamma^n \cdot \gamma^n \cdot P \cdot \gamma^p \cdot \gamma^p \cdot N) ; (\gamma_5^n \cdot \gamma^n \cdot P \cdot \gamma_5^p \cdot \gamma^p \cdot N)$$

The requirement of time reversal invariance (as expressed, for instance, in G.N.O.) allows us to rule out $(\gamma_5^n \gamma_5^p \cdot \gamma^p \cdot N)$ and $(\gamma_5^n \cdot \gamma^n \cdot P \cdot \gamma_5^p)$; time reversal invariance plus use of the Dirac equation helps us to reduce

$$(\gamma^n \gamma^p \cdot \gamma^p \cdot N), (\gamma^n \cdot \gamma^n \cdot P \cdot \gamma^p), (\gamma^n \gamma_5^n \gamma^p \gamma_5^p \cdot \gamma^p \cdot N) \text{ and } (\gamma^n \gamma_5^n \cdot \gamma^n \cdot P \cdot \gamma^p \gamma_5^p).$$

The invariant $(\gamma^n i \gamma^{n.P} \gamma^P i \gamma^P.N)$ can be expressed in terms of the remaining ones by developing γ^n and γ^P in terms of the four orthogonal 4-vectors defined by (II.4): M , K , Δ and L . We are therefore left with the five Fermi invariants and four other invariants :

$$Q = 1 : \quad (1^n i \gamma^{P.N}) ; \quad (i \gamma^{n.P} 1^P) \quad (AI.2)$$

$$Q = 2 : \quad (i \gamma^{n.P} i \gamma^{P.N}) ; \quad (\gamma_5^n i \gamma^{n.P} \gamma_5^P i \gamma^{P.N})$$

If we assume now charge symmetry, only the symmetric combination $(1^n i \gamma^{P.N} i \gamma^{n.P} 1^P)$ can be present in the development of M into invariant operators. Thus we now have the five Fermi invariants and three other invariants, which latter coincide with three of the invariants introduced and used by G.N.O. (the other two being S and P). These invariants cannot be reduced further by the use of the Dirac equation; they are however not linearly independent and can be expressed always in terms of only five linearly independent invariant operators. This can be done in two ways : either expressing $(1^n i \gamma^{P.N} i \gamma^{n.P} 1^P)$, $(i \gamma^{n.P} i \gamma^{P.N})$ and $(\gamma_5^n i \gamma^{n.P} , \gamma_5^P i \gamma^{P.N})$ in terms of Fermi invariants, or expressing V , T and A in terms of the G.N.O. invariants. The former way is, at least in principle, trivial, but gives rise to a rather cumbersome calculation; as an even number of Fierz transformations can be applied to an invariant without altering it, a repeated application of the Michel tables for Fierz transformations of invariants belonging to different Q can be made, followed by a suitable application of the Dirac equation. At the end, our three G.N.O. invariants will be reduced to linear combinations of Fermi invariants. To go the other way round, we can express each γ matrix in V , T and A in terms of the four orthogonal 4-vectors (II.4); a repeated application of the Dirac equation will now allow us to express the three Fermi invariants we started from in terms of the G.N.O. invariants.

It is therefore clear that the NN and $\overline{NN}$ elastic scattering can be always described in terms of only five independent invariant functions, which we shall take, as explained in the text, as linear combinations of Fermi invariants. The connection between the Fermi and the G.N.O. sets is given by the following table :

$$S = G_1$$

$$V = 1/(wt) \left[m^2(\bar{t} - w) G_1 - m(\bar{t} + w) G_2 + (\bar{t} - w) G_3 + t G_4 \right]$$

$$T = -4 \left[\frac{(\bar{t} - w)}{4t} \left(1 - \frac{4m^2}{wt} \right) \zeta_1 + \frac{m}{t} \left(\frac{m^2(w + \bar{t})}{wt} - 1 \right) \zeta_2 - \frac{m^2(\bar{t} - w)}{wt\bar{t}} \zeta_3 + \right. \\ \left. - \frac{m^2}{wt\bar{t}} \zeta_4 - \frac{(w - \bar{t})}{4t} \zeta_5 \right] \quad (AI.3)$$

$$A = - \left[\frac{m^2(\bar{t} - w)^2}{wt\bar{t}} \zeta_1 - \frac{m(\bar{t}^2 - w^2)}{wt\bar{t}} \zeta_2 + \frac{(w + \bar{t})^2}{wt\bar{t}} \zeta_3 + \frac{(\bar{t} - w)}{wt\bar{t}} \zeta_4 - \frac{4m^2}{t} \zeta_5 \right]$$

$$P = \zeta_5$$

$$\text{where } G_2 = (1^n i \gamma^{P.N+i} \gamma^{n.P} 1^P); \quad G_3 = (i \gamma^{n.P} i \gamma^{P.N}) \quad \text{and} \quad G_4 = (\gamma_5^n i \gamma^{n.P} \gamma_5^P i \gamma^{P.N})$$

From table (AI.3) and the definition (II.19) of the perturbative invariants P_j it is easy to find out the transformation table (II.20) between the perturbative and the Fermi set of invariant operators.

A P P E N D I X I I

Separation of the S and P-wave two pion contribution to the unitarity condition

The absorption matrix (II.13) can be expressed in terms of two independent scalar functions A and B :

$$\tau = -A + i \gamma \cdot Q B \quad (\text{AII.1})$$

We want now to separate, out of A and B, those parts that contribute to the nucleon-antinucleon absorption into S and P two pion states, from the rest, which gives absorption into two pion states with angular momentum higher than 1. In order to do so, let us write A and B in the form

$$\begin{aligned} A &= (\mathcal{A} - \alpha) + A_{\text{sp}} \\ B &= (\mathcal{B} - \beta) + B_{\text{sp}} \end{aligned} \quad (\text{AII.2})$$

where $\mathcal{A}$ and $\mathcal{B}$ are defined as in (III.25) and A_{sp} and B_{sp} contain all of the S and P-wave two pion contribution. The definition of α and β has therefore to be such that the S and P-waves coming from them have to cancel exactly those coming from $\mathcal{A}$ and $\mathcal{B}$.

The S and P-wave contributions of $(-\mathcal{A} + i \gamma \cdot Q \mathcal{B})$ to τ will be proportional to :

$$\begin{aligned} &\int (-\mathcal{A} + i \gamma \cdot Q \mathcal{B}) d\Omega \\ &\int (-\mathcal{A} + i \gamma \cdot Q \mathcal{B}) Q_{\mu} d\Omega \end{aligned} \quad (\text{AII.3})$$

is indeed required on physical grounds. It is enough for this to note that relation (AII.5b) can also be written as :

$$\int (-A - \alpha, \frac{s-\bar{s}}{4}) Q_{\mu} dR = \int (\beta - \bar{\beta}) i \gamma \cdot Q Q_{\mu} dR \quad (\text{AII.7})$$

We have insofar disregarded isotopic spin. It is easy to see that the Pauli principle for the two pion states implies :

$$\alpha_0^{(-)} = \beta_0^{(+)} = 0 \quad (\text{AII.8})$$

$$\alpha_1^{(+)} = \beta_1^{(-)} = 0$$

R E F E R E N C E S

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FIG. 1

FIG. 2

FIG. 3

FIG. 4

FIG. 5