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Statistical Weights of Many-Particle Systems

in Spin or Isospin Space

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1. Introduction.

It has been pointed out earlier ¹⁾ that in many cases one does not need to know all quantum numbers characterizing the end-state of a production process, in order to check certain properties of the interaction, and that it is useful to know the statistical weights of states characterized by an incomplete set of parameters.

We shall develop the theory with the example in mind of the high-energy nucleon-nucleon collisions, where the end-particles have to combine to yield states of given total isospin and charge.

We shall use the notation j_i, m_i for the (iso)spin and the projection along a given direction of the i^{th} particle.

2. The Formalism.

Experimentally one observes end-states which one describes by stating the number and the charges of each kind of particle ($N, \mathcal{I}, K$, etc.).

It is therefore natural to describe the (iso)spin dependence of such end-states by a product wave function

$$|j_1, m_1\rangle |j_2, m_2\rangle \dots |j_n, m_n\rangle \equiv |m_1, m_2, \dots, m_n\rangle \quad (1)$$

where in the r.h.s. notation the j_i 's are supposed to be given.

(Iso)spin conservation states, however, that

$$J^2 = \left(\sum j_i \right)^2$$

¹⁾ Y. Yeivin and A. de-Shalit, Nuovo Cimento 1, 1147 (1955).

leads from the one to the other ²⁾. The coefficient $\langle m_1, m_2, \dots | j_{12}, \dots, J, M \rangle$ is called a recoupling coefficient. The recoupling coefficient is a product of Clebsch-Gordan coefficients

$$\langle j_{12}, \dots, J, M | m_1, \dots, m_n \rangle = C_{m_1 m_2 m_1 + m_2}^{j_1 j_2 j_{12}} \dots C_{m_1 + \dots + m_{n-1} m_n M}^{j_{12} \dots j_{n-1} j_n J} \quad (4)$$

In a statistical theory of production, where nothing is known about the dynamics of the production, it is assumed that all end-states of n specified particles and with a given J, M have equal probability. The probability to observe an end-state $|m_1 \dots m_n\rangle$ is then

$$\sum_{j_{12} \dots j_{12 \dots n-1}} \left| \langle j_{12} \dots J, M | m_1 \dots m_n \rangle \right|^2 \quad (5)$$

3. The Charge Distribution Coefficients; their Properties.

To shorten the notation, let us define

$$j_{12 \dots i} = J_i$$

$$m_1 + m_2 + \dots + m_i = M_i$$

The probability (5) is then given by

$$P_{m_1, \dots, m_n}^J \equiv \sum_{J_2, J_3, \dots, J_{n-1}} \left| \langle J_2, J_3, \dots, J_{n-1}, J, M | m_1, m_2, \dots, m_n \rangle \right|^2 \quad (6)$$

$$= \sum_{J_2, J_3, \dots, J_{n-1}} \left(C_{m_1 m_2 M_2}^{j_1 j_2 J_2} C_{m_2 m_3 M_3}^{J_2 m_3 J_3} \dots C_{m_{n-1} m_n M}^{J_{n-1} j_n J} \right)^2 \quad (7)$$

$$(j_1, \dots, j_i, j_{i+1}, \dots)_{P_{m_1, \dots, m_i, m_{i+1}, \dots}}^J = (j_i, \dots, j_{i+1}, j_i, \dots)_{P_{m_1, \dots, m_{i+1}, m_i, \dots}}^J \quad (11)$$

One goes from the l.h.s. of (8) to the r.h.s. by interchanging the particles i and $i+1$. This means going over from the coupling scheme

$$\left[(J_i \ j_i)_{J_{i+1}} \ j_{i+1} \right] \ J_{i+2}, \dots \quad (12)$$

to the alternate scheme

$$\left[(J_i \ j_{i+1})_{J'_{i+1}} \ j_i \right] \ J_{i+2}, \dots \quad (13)$$

and from $j_1, m_1, \dots, j_i, m_i, j_{i+1}, m_{i+1}, \dots$ to

$$j_i, m_i, \dots, j_{i+1}, m_{i+1}, j_i, m_i, \dots$$

The latter involves only a trivial permutation of columns in the transformation matrix from the $|j_1, m_1, \dots, j_n, m_n\rangle$ to the $|j_1, j_2(J_2), \dots, J, M\rangle$ system. The relation between (12) and (13) is given by a recoupling coefficient :

$$\begin{aligned} \langle \dots (J_{i-1}) j_i (J_i) j_{i+1} (J_{i+1}) \dots | &= \sum_{J'_i} \langle \dots (J_{i-1}) j_i (J_i) J_{i+1} (J_{i+1}) \dots | \dots (J_{i-1}) j_{i+1} \\ &\quad (J'_i) j_i (J_{i+1}) \dots \rangle \\ &\quad \langle \dots (J_{i-1}) j_{i+1} (J'_i) j_i (J_{i+1}) \dots | \end{aligned}$$

The $P_{m_1, \dots, m_i, m_{i+1}, \dots}^J$ is then equal to

$$\sum_{J_1, \dots, J_{n-1}} \left| \sum_{j_{13}} \langle j_{12}, j_3 | j_{13}, j_2 \rangle^{j_4} \langle \dots (J_{i-1}) j_{i+1} (J'_i) j_i (J_{i+1}) | m_1, \dots, m_n \rangle \right|^2$$

Performing the other summations, over J_1 to J_{n-1} (except $J_i = j_{13}$) one gets the P coefficient with the order of the particles interchanged.

Therefore $(j, j, \dots, j)_{P_{m_1, \dots, m_n}}^J$, i.e. the P^J coefficient for a system of n equal particles, is independent of the order of the $m_1, \dots, m_n$ and depends only on their set of values. If the particles have $j=1$, $P_{m_1, \dots, m_n}^J$ can be written as P_{n_-, n_0, n_+}^J where n_- , n_0 , n_+ are the numbers of particles with $m = -1, 0, +1$ respectively.

4. Orthogonality Properties.

In the following tables one will find P^J coefficients for systems composed of spin $1/2$, 1 and $3/2$ and all possible combinations of m 's. They have been prepared for use in statistical theories of meson production. The number of spin $1/2$ and $3/2$ particles is at most two, the number of spin 1 particles is at most 8.

The tables for spin 1 particles were computed using formula (8) and the well known formulae for the Clebsch-Gordan coefficients. The tables for mixed systems ($1/2$ and $3/2$ plus spin 1) using formula (9) are derived from the former.

There exist two simple checks as to the accuracy of the tables. Both stem from the fact that the recoupling transformation is orthogonal. Suppose we label the columns of the transformation matrix by the sets of numbers $m_1, \dots, m_n$, and the rows by $J_1, J_2, \dots, J, M$. The transformation is unitary and real and therefore orthogonal. This property can easily be checked on formula (4), using the orthogonality of the Clebsch-Gordan coefficients.

In applying the check (15) to a table of $P_{m_1, \dots, m_n}^J$ one has to keep in mind that each set of $m_1, \dots, m_n$ is listed only once, although there are as many columns in the transformation matrix labelled by the numbers of this set (in different orders) as there are permutations of $m_1, \dots, m_n$. One can alternatively sum over different sets of m_1 only, and use as terms in the sum the numbers

$${}^*P_{m_1, \dots, m_n}^J = P_{m_1, \dots, m_n}^J \times (\text{nr. of permutations of } m_1, \dots, m_n)$$

For systems composed of spin 1 particles one has e.g.

$${}^*P_{n_-, n_0, n_+}^J = \frac{n!}{n_-! n_0! n_+!} P_{n_-, n_0, n_+}^J$$

Tables of the ${}^*P^J$ are also given, because one needs precisely those numbers in analysing the probabilities for multiple meson production into different charge states. As these ${}^*P^J$ are likely to be used for computational purposes they are listed in decimal form.

For systems of two baryons (spins 1/2 or 3/2) we give tables of the coefficients

$$(j_1, j_2, j_3) P_{m_1, m_2}^{1(1-m_1-m_2)}$$

and

$$(j_1, j_2, j_3) P_{m_1, m_2}^{0 \text{ or } 1(-m_1-m_2)}$$

TABLE 1
 P_{m_1, m_2}^J for 2 spin 1 particles

M	m_1	m_2	J		
			0	1	2
2	+	+	0	0	1
1	+	0	0	1/2	1/2
0	0	0	1/3	0	2/3
	+	-	1/3	1/2	1/6

TABLE 2
 P_{n_+, n_0, n_-}^J for 3 spin 1 particles

M	n_+	n_0	n_-	J			
				0	1	2	3
3	3	0	0	0	0	0	1
2	2	1	0	0	0	2/3	1/3
1	2	0	1	0	3/5	1/3	1/15
	1	2	0	0	2/5	1/3	4/15
0	1	1	1	1/6	2/5	1/3	1/10
	0	3	0	0	3/5	0	2/5

TABLE 3
 P_{n_+, n_0, n_-}^J for 4 spin 1 particles

M	n_+	n_0	n_-	J				
				0	1	2	3	4
4	4	0	0					1
3	3	1	0				3/4	1/4
2	3	0	1			5/7	1/4	1/28
	2	2	0			11/21	1/3	1/7
1	2	1	1		2/5	8/21	11/60	1/28
	1	3	0		3/10	5/14	1/5	1/7
0	0	4	0	1/5	0	4/7	0	8/35
	1	2	1	2/15	3/10	13/42	1/5	2/35
	2	0	2	1/5	2/5	2/7	1/10	1/70

TABLE 4

 P_{n_+, n_0, n_-}^J for 5 spin 1 particles

M	n_+	n_0	n_-	J					
				0	1	2	3	4	5
5	5	0	0						1
4	4	1	0					4/5	1/5
3	4	0	1				7/9	1/5	1/45
	3	2	0				11/18	3/10	4/45
2	3	1	1			15/28	1/3	4/35	1/60
	2	3	0			3/7	1/3	6/35	1/15
1	3	0	2		3/7	5/14	1/6	3/70	1/210
	2	2	1		11/55	1/3	7/30	1/10	2/105
	1	4	0		9/35	2/7	4/15	4/35	8/105
0	2	1	2	2/15	11/35	13/42	8/45	2/35	1/126
	1	3	1	1/10	9/35	2/7	19/90	4/35	2/63
	0	5	0	0	3/7	0	4/9	0	8/63

TABLE 5

$$P_{n_+, n_0, n_-}^J \text{ for 6 spin 1 particles}$$

M	n_+	n_0	n_-	J						
				0	1	2	3	4	5	6
6	6	0	0							1
5	5	1	0						5/6	1/6
4	5	0	1					9/11	1/6	1/66
	4	2	0					37/55	4/15	2/33
3	4	1	1				28/45	16/55	7/90	1/110
	3	3	0				31/60	69/220	2/15	2/55
2	4	0	2			5/9	14/45	6/55	1/45	1/495
	2	4	0			23/63	14/45	78/385	4/45	16/495
	3	2	1			55/126	59/180	53/308	1/18	4/495
1	3	1	2		9/28	85/252	2/9	36/385	29/1260	1/396
	2	3	1		9/35	19/63	43/180	213/1540	17/315	1/99
	1	5	0		3/14	5/18	2/9	2/11	4/63	4/99
0	3	0	3	1/7	9/28	25/84	1/6	9/154	1/84	1/924
	2	2	2	11/105	9/35	2/7	19/90	81/770	2/63	1/231
	1	4	1	3/35	3/14	11/42	2/9	52/385	4/63	4/231
	0	6	0	1/7	0	10/21	0	24/77	0	16/231

TABLE 6

 P_{n_+, n_0, n_-}^J for 7 spin 1 particles

M	n_+	n_0	n_-	J							
				0	1	2	3	4	5	6	7
7	7	0	0								1
6	6	1	0							6/7	1/7
5	6	0	1						11/13	1/7	1/91
	5	2	0						28/39	5/21	4/91
4	5	1	1					15/22	10/39	13/231	1/182
	4	3	0					32/55	19/65	8/77	2/91
3	5	0	2				7/11	3/11	1/13	1/77	1/1001
	4	2	1				259/495	17/55	76/585	13/385	4/1001
	3	4	0				74/165	17/55	32/195	24/385	16/1001
2	4	1	2			4/9	161/495	9/55	32/585	38/3465	1/1001
	3	3	1			31/84	103/330	151/770	71/780	32/1155	4/1001
	2	5	0			20/63	29/99	16/77	14/117	32/693	16/1001
1	4	0	3		1/3	1/3	7/33	1/11	1/39	1/231	1/3003
	3	2	2		11/42	19/63	47/198	103/770	212/4095	17/1386	4/3003
	2	4	1		23/105	17/63	118/495	61/385	328/4095	20/693	16/3003
	1	6	0		4/21	5/21	8/33	12/77	32/273	8/231	64/3003
0	3	1	3	3/28	11/42	2/7	9/44	31/308	3/91	1/154	1/1716
	2	3	2	3/35	23/105	11/42	12/55	52/385	11/182	4/231	1/429
	1	5	1	1/14	4/21	5/21	43/198	12/77	68/819	8/231	4/429
	0	7	0	0	1/3	0	14/33	0	8/39	0	16/429

TABLE 7

 P_{n_+, n_0, n_-}^J for 8 spin 1 particles

M	n ₊	n ₀	n ₋	J						
				0	1	2	3	4	5	6
6	7	0	1							13/15
	6	2	0							79/105
5	6	1	1						66/91	8/35
	5	3	0						115/182	19/70
4	6	0	2					9/13	22/91	2/35
	5	2	1					84/143	157/546	233/2310
	4	4	0					367/715	136/455	52/385
3	5	1	2				35/66	87/286	34/273	8/231
	4	3	1				224/495	4/13	263/1638	13/210
	3	5	0				157/396	171/572	148/819	20/231
2	5	0	3			5/11	7/22	45/286	5/91	1/77
	4	2	2			37/99	14/45	138/715	361/4095	97/3465
	3	4	1			74/231	193/660	4159/20020	157/1365	8/165
	2	6	0			65/231	3/11	213/1001	12/91	16/231
1	4	1	3		4/15	10/33	7/30	37/286	2/39	16/1155
	3	3	2		31/140	251/924	13/55	783/5005	11/140	131/4620
	2	5	1		4/21	170/693	91/396	683/4004	82/819	158/3465
	1	7	0		1/6	5/22	7/33	27/143	4/39	4/55
0	4	0	4	1/9	4/15	28/99	98/495	14/143	4/117	4/495
	3	2	3	11/126	31/140	727/2772	107/495	12/91	193/3276	23/1260
	2	4	2	23/315	4/21	166/693	43/198	1523/10010	68/819	118/3465
	1	6	1	4/63	1/6	305/1386	7/33	162/1001	4/39	172/3465
	0	8	0	1/9	0	40/99	0	48/143	0	64/495

TABLE 8

$$*P_{m_1 m_2}^J \text{ for 2 spin 1 particles}$$

M	m_1	m_2	J			$\frac{2!}{n_+!n_0!n_-!}$
			0	1	2	
2	+	+			1.0000	1
1	+	0		1.0000	1.0000	2
0	0	0	0.3333	0.0000	0.6666	1
	+	-	0.6666	1.0000	0.3333	2
$\rho(J)$			1	1	1	

TABLE 9

$$*P_{n_+ n_0 n_-}^J \text{ for 3 spin 1 particles}$$

M	n_+	n_0	n_-	J				$\frac{3!}{n_+!n_0!n_-!}$
				0	1	2	3	
3	3	0	0				1.0000	1
2	2	1	0			2.0000	1.0000	3
1	2	0	1		1.8000	1.0000	0.2000	3
		1	2		1.2000	1.0000	0.8000	3
0	1	1	1	1.0000	2.4000	2.0000	0.6000	6
	0	3	0	0.0000	0.6000	0.0000	0.4000	1
$\rho(J)$				1	3	2	1	

TABLE 10

$$*P_{n_+ n_0 n_-}^J \text{ for 4 spin 1 particles}$$

M	$n_+ n_0 n_-$	J					$\frac{4!}{n_+! n_0! n_-!}$
		0	1	2	3	4	
4	4 0 0					1.0000	1
3	3 1 0				3.0000	1.0000	4
2	3 0 1			2.8571	1.0000	0.1429	4
	2 2 0			3.1429	2.0000	0.8571	6
1	2 1 1		4.8000	4.5714	2.2000	0.4286	12
	1 3 0		1.2000	1.4286	0.8000	0.5714	4
0	2 0 2	1.2000	2.4000	1.7143	0.6000	0.0857	6
	1 2 1	1.6000	3.6000	3.7143	2.4000	0.6857	12
	0 4 0	0.2000	0.0000	0.5714	0.0000	0.2286	1
	$p(J):$	3	6	6	3	1	

TABLE 11

$$*P_{n_+ n_0 n_-}^J \text{ for 5 spin 1 particles}$$

M	$n_+ \ n_0 \ n_-$	J						$\frac{5!}{n_+! n_0! n_-!}$
		0	1	2	3	4	5	
5	5 0 0						1.0000	1
4	4 1 0					4.0000	1.0000	5
3	4 0 1				3.8889	1.0000	0.1111	5
	3 2 0				6.1111	3.0000	0.8889	10
2	3 1 1			10.7143	6.6667	2.2857	0.3333	20
	2 3 0			4.2857	3.3333	1.7143	0.6667	10
1	3 0 2		4.2857	3.5714	1.6667	0.4286	0.0476	10
	2 2 1		9.4286	10.0000	7.0000	3.0000	0.5714	30
	1 4 0		1.2857	1.4286	1.3333	0.5714	0.3810	5
0	2 1 2	4.0000	9.4286	9.2857	5.3333	1.7143	0.2381	30
	1 3 1	2.0000	5.1429	5.7143	4.2222	2.2857	0.6349	20
	0 5 0	0.0000	0.4286	0.0000	0.4444	0.0000	0.1270	1
	$\rho(J):$	6	15	15	10	4	1	

TABLE 12

$$*P_{n_+ n_0 n_-}^J \text{ for 6 spin 1 particles}$$

M	$n_+ n_0 n_-$	J							$\frac{6!}{n_+! n_0! n_-!}$
		0	1	2	3	4	5	6	
6	6 0 0							1.0000	1
5	5 1 0						5.0000	1.0000	6
4	5 0 1					4.9091	1.0000	0.0909	6
	4 2 0					10.0909	4.0000	0.9091	15
3	4 1 1				18.6667	8.7273	2.3333	0.2727	30
	3 3 0				10.3333	6.2727	2.6667	0.7273	20
2	4 0 2			8.3333	4.6667	1.6363	0.3333	0.0303	15
	3 2 1			26.1905	19.6667	10.3247	3.3333	0.4848	60
	2 4 0			5.4762	4.6667	3.0390	1.3333	0.4848	15
1	3 1 2		19.2857	20.2381	13.3333	5.6104	1.3810	0.1515	60
	2 3 1		15.4286	18.0952	14.3333	8.2987	3.2380	0.6061	60
	1 5 0		1.2857	1.6667	1.3333	1.0909	0.3810	0.2424	6
0	3 0 3	2.8571	6.4286	5.9524	3.3333	1.1688	0.2381	0.0216	20
	2 2 2	9.4286	23.1429	25.7143	19.0000	9.4675	2.8571	0.3896	90
	1 4 1	2.5714	6.4286	7.8571	6.6667	4.0519	1.9048	0.5195	30
	0 6 0	0.1429	0.0000	0.4762	0.0000	0.3117	0.0000	0.0693	1
	$\rho(J)$	15	36	40	29	15	5	1	

TABLE 13

$$*P_{n_+ n_0 n_-}^J \text{ for 7 spin 1 particles}$$

M	$n_+ n_0 n_-$									$\frac{7!}{n_+! n_0! n_-!}$
		0	1	2	3	4	5	6	7	
7	7 0 0								1.0000	1
6	6 1 0							6.0000	1.0000	7
5	6 0 1						5.9230	1.0000	0.0769	7
	5 2 0						15.0769	5.0000	0.9230	21
4	5 1 1					28.6364	10.7692	2.3636	0.2308	42
	4 3 0					20.3636	10.2308	3.6364	0.7692	35
3	5 0 2				13.3636	5.7273	1.6154	0.2727	0.0210	21
	4 2 1				54.9394	32.4545	13.6410	3.5454	0.4196	105
	3 4 0				15.6970	10.8181	5.7436	2.1818	0.5594	35
2	4 1 2			46.6667	34.1515	17.1818	5.7436	1.1515	0.1049	105
	3 3 1			51.6667	43.6970	27.4546	12.7436	3.8788	0.5594	140
	2 5 0			6.6667	6.1515	4.3636	2.5128	0.9697	0.3357	21
1	4 0 3		11.6667	11.6667	7.4242	3.1818	0.8974	0.1515	0.0117	35
	3 2 2		55.0000	63.3333	49.8485	28.0909	10.8718	2.5758	0.2797	210
	2 4 1		23.0000	28.3333	25.0303	16.6364	8.4104	3.0303	0.5594	105
	1 6 0		1.3333	1.6667	1.6970	1.0909	0.8204	0.2424	0.1492	7
0	3 1 3	15.0000	36.6667	40.0000	28.6364	14.0909	4.6154	0.9091	0.0816	140
	2 3 2	18.0000	46.0000	55.0000	45.8182	28.3637	12.6923	3.6364	0.4895	210
	1 5 1	3.0000	8.0000	10.0000	9.1212	6.5454	3.4872	1.4545	0.3916	30
	0 7 0	0.0000	0.3333	0.0000	0.4242	0.0000	0.2051	0.0000	0.0373	1
	$\rho(J):$	36	91	105	84	49	21	6	1	

TABLE 14

$$*P_{n_+ n_0 n_-}^J \text{ for 8 spin 1 particles}$$

M	$n_+ \ n_0 \ n_-$	J							$\frac{8!}{n_+! n_0! n_-!}$
		0	1	2	3	4	5	6	
6	7 0 1							6.9333	8
	6 2 0							21.0667	28
5	6 1 1						40.6154	12.8000	56
	5 3 0						35.3846	15.2000	56
4	6 0 2					19.3846	6.7692	1.6000	28
	5 2 1					98.6853	48.3077	16.9455	168
	4 4 0					35.9301	20.9231	9.4545	70
3	5 1 2				89.0909	51.1049	20.9231	5.8182	168
	4 3 1				126.7071	86.1538	44.9572	17.3333	280
	3 5 0				22.2020	16.7413	10.1197	4.8485	56
2	5 0 3			25.4546	17.8182	8.8112	3.0769	0.7272	56
	4 2 2			156.9697	130.6667	81.0629	37.0256	11.7576	420
	3 4 1			89.6970	81.8788	58.1678	32.2051	13.5758	280
	2 6 0			7.8788	7.6363	5.9581	3.6923	1.9394	28
1	4 1 3		74.6667	84.8484	65.3333	36.2238	14.3590	3.8788	280
	3 3 2		124.0000	152.1212	132.3636	87.6084	44.0000	15.8788	560
	2 5 1		32.0000	31.2121	38.6061	28.6573	16.8205	7.6607	168
	1 7 0		1.3333	1.8182	1.6970	1.5105	0.8205	0.5818	8
0	4 0 4	7.7778	18.6667	19.7980	13.8586	6.8531	2.3932	0.5656	70
	3 2 3	48.8889	124.0000	146.8687	121.0506	73.8462	32.9914	10.2222	560
	2 4 2	30.6667	80.0000	100.6061	91.2121	63.9021	34.8718	14.3030	420
	1 6 1	3.5556	9.3333	12.3232	11.8788	9.0629	5.7436	2.7798	56
	0 8 0	0.1111	0.0000	0.4040	0.0000	0.3357	0.0000	0.1293	1
$\rho(J):$		91	232	284	238	154	76	28	

TABLE 15

$$(j_1 j_2 j_3)_{m_1, m_2, m_3}^0 \quad (\sum m_i = 0)$$

$j_1 \quad j_2$	$m_1 \quad m_2 \quad m_3$	j_3			
		0	1	2	3
$1/2 \quad 1/2$	$1/2 \quad 1/2 \quad -1$		$1/3$		
	$1/2 \quad -1/2 \quad 0$	$1/2$	$1/6$		
$3/2 \quad 1/2$	$3/2 \quad 1/2 \quad -2$			$1/5$	
	$3/2 \quad -1/2 \quad -1$		$1/4$	$1/20$	
	$1/2 \quad 1/2 \quad -1$		$1/12$	$3/20$	
	$1/2 \quad -1/2 \quad 0$		$1/6$	$1/10$	
$3/2 \quad 3/2$	$3/2 \quad 3/2 \quad -3$				$1/7$
	$3/2 \quad 1/2 \quad -2$			$1/10$	$1/14$
	$3/2 \quad -1/2 \quad -1$		$1/10$	$1/10$	$1/35$
	$3/2 \quad -3/2 \quad 0$	$1/4$	$3/20$	$1/20$	$1/140$
	$1/2 \quad 1/2 \quad -1$		$2/15$	0	$3/35$
	$1/2 \quad -1/2 \quad 0$	$1/4$	$1/60$	$1/20$	$9/140$

TABLE 16

$$(j_1 j_2 j_3)_{P_{m_1 m_2 m_3}}^{*P_0} \quad (\sum m_i = 0)$$

j_1	j_2	m_1	m_2	m_3	j_3				permut.
					0	1	2	3	
1/2	1/2	1/2	1/2	-1		0.3333			1
		1/2	-1/2	0	1.0000	0.3333			2
3/2	1/2	3/2	1/2	-2			0.2000		1
		3/2	-1/2	-1		0.2500	0.0500		1
		1/2	1/2	-1		0.0833	0.1500		1
		1/2	-1/2	0		0.1667	0.1000		1
3/2	3/2	3/2	3/2	-3				0.1429	1
		3/2	1/2	-2			0.2000	0.1429	2
		3/2	-1/2	-1		0.2000	0.2000	0.0572	2
		3/2	-3/2	0	0.5000	0.3000	0.1000	0.0143	2
		1/2	1/2	-1		0.1333	0.0000	0.0857	1
		1/2	-1/2	0	0.5000	0.3333	0.1000	0.1286	2

TABLE 17

$$(j_1 j_2 j_3) \begin{matrix} 1 \\ P_{m_1 m_2 m_3} \end{matrix} (\sum m_i = 0)$$

j_1	j_2	m_1	m_2	m_3	j_3				
					0	1	2	3	4
$1/2$	$1/2$								
		$1/2$	$1/2$	-1		$1/2$	$3/10$		
		$1/2$	$-1/2$	0	$1/2$	$1/2$	$1/5$		
$3/2$	$1/2$								
		$3/2$	$1/2$	-2			$2/5$	$1/7$	
		$3/2$	$-1/2$	-1		$9/20$	$1/4$	$2/35$	
		$1/2$	$1/2$	-1		$21/60$	$3/20$	$6/35$	
		$1/2$	$-1/2$	0	$1/2$	$1/5$	$1/5$	$9/70$	
$3/2$	$3/2$								
		$3/2$	$3/2$	-3				$9/28$	$1/12$
		$3/2$	$1/2$	-2			$19/70$	$1/7$	$1/14$
		$3/2$	$-1/2$	-1		$3/10$	$13/70$	$17/140$	$1/28$
		$3/2$	$-3/2$	0	$9/20$	$7/20$	$27/140$	$9/140$	$1/105$
		$1/2$	$1/2$	-1		$1/5$	$9/35$	$3/140$	$3/28$
		$1/2$	$-1/2$	0	$1/20$	$7/20$	$19/140$	$9/140$	$3/35$

TABLE 18

$$(j_1 j_2 j_3)_{P_{m_1 m_2 m_3}}^* \quad (\sum m_i = 0)$$

j_1	j_2	m_1	m_2	m_3	j_3					permut.
					0	1	2	3	4	
1/2	1/2	1/2	1/2	-1		0.5000	0.3000			1
		1/2	-1/2	0	1.0000	1.0000	0.4000			2
3/2	1/2	3/2	1/2	-2			0.4000	0.1429		1
		3/2	-1/2	-1			0.2500	0.0571		1
		1/2	1/2	-1		0.3500	0.1500	0.1714		1
		1/2	-1/2	0	0.5000	0.2000	0.2000	0.1286		1
3/2	3/2	3/2	3/2	-3					0.0833	1
		3/2	1/2	-2			0.5428	0.2857	0.1429	2
		3/2	-1/2	-1		0.6000	0.3714	0.2429	0.0714	2
		3/2	-3/2	0	0.9000	0.7000	0.3857	0.1286	0.0190	2
		1/2	1/2	-1		0.2000	0.2571	0.0214	0.1071	1
		1/2	-1/2	0	0.1000	0.7000	0.2714	0.1286	0.1714	2

TABLE 19

$$(j_1 j_2 j_3) \quad P_{m_1 m_2 m_3}^1 \quad (\sum m_i = 1)$$

j_1	j_2	m_1	m_2	m_3	j_3				
					0	1	2	3	4
$1/2$	$1/2$	$1/2$	$1/2$	0	1	$1/2$	$1/10$		
		$1/2$	$-1/2$	1		$3/4$	$3/20$		
		$-1/2$	$-1/2$	2			$3/5$		
$3/2$	$1/2$	$3/2$	$1/2$	-1		$3/5$	$1/5$	$1/35$	
		$3/2$	$-1/2$	0	$3/4$	$9/20$	$3/20$	$3/140$	
		$1/2$	$1/2$	0	$1/4$	$7/20$	$1/4$	$9/140$	
		$1/2$	$-1/2$	1		$3/10$	$3/10$	$3/35$	
		$-1/2$	$1/2$	1		$3/10$	$3/10$	$3/35$	
		$-1/2$	$-1/2$	2			$3/10$	$3/14$	
		$-3/2$	$1/2$	2			$1/2$	$1/14$	
		$-3/2$	$-1/2$	3				$3/7$	
$3/2$	$3/2$	$3/2$	$3/2$	-2			$3/7$	$3/28$	$1/84$
		$3/2$	$1/2$	-1		$3/10$	$17/70$	$29/280$	$1/56$
		$3/2$	$-1/2$	0	$3/10$	$3/10$	$3/14$	$3/35$	$1/70$
		$3/2$	$-3/2$	1		$1/2$	$3/14$	$3/56$	$1/168$
		$1/2$	$1/2$	0	$2/5$	$1/5$	$1/7$	$9/70$	$3/70$
		$1/2$	$-1/2$	1		$3/10$	$9/70$	$39/280$	$3/56$
		$-1/2$	$-1/2$	2			$9/35$	$3/28$	$3/28$
		$-3/2$	$1/2$	2			$2/7$	$5/28$	$1/28$
		$-3/2$	$-1/2$	3				$15/56$	$1/8$
		$-3/2$	$-3/2$	4					$1/3$

TABLE 20

$$(j_1 j_2 j_3)_{P_{m_1 m_2 m_3}}^{*1} (\sum m_i = 1)$$

j_1	j_2	m_1	m_2	m_3	j_3				
					0	1	2	3	4
1/2	1/2	1/2	1/2	0	1.0000	0.5000	0.1000		
		1/2	-1/2	1		1.5000	0.3000		
		-1/2	-1/2	2			0.6000		
3/2	1/2	3/2	1/2	-1		0.6000	0.2000	0.0286	
		3/2	-1/2	0	0.7500	0.4500	0.1500	0.0214	
		1/2	1/2	0	0.2500	0.3500	0.2500	0.0643	
		1/2	-1/2	1		0.6000	0.6000	0.1714	
		1/2	-1/2	2			0.3000	0.2142	
		-3/2	1/2	2			0.5000	0.0714	
		-3/2	-1/2	3				0.4285	
3/2	3/2	3/2	3/2	-2			0.4286	0.1071	0.0119
		3/2	1/2	-1		0.6000	0.4857	0.2071	0.0357
		3/2	-1/2	0	0.6000	0.6000	0.4286	0.1714	0.0286
		3/2	-3/2	1		1.0000	0.4286	0.1071	0.0119
		1/2	1/2	0	0.4000	0.2000	0.1429	0.1286	0.0429
		1/2	-1/2	1		0.6000	0.2571	0.2786	0.1071
		1/2	-3/2	2			0.5714	0.3571	0.0714
		-1/2	-1/2	2			0.2571	0.1071	0.1071
		-1/2	-3/2	3				0.5357	0.2500
		-3/2	-3/2	4					0.3333

TABLE 21

* P^1 for 2 spin $\frac{1}{2}$ and n spin 1 particles

n	m_1	m_2	n_+	n_0	n_-	* $P^1_{m_1, m_2; n_+, n_0, n_-}$ (fractional) (decimal)
0	1/2	1/2	0	0	0	1 1.000
1	1/2	1/2	0	1	0	1/2 0.500
	1/2	-1/2	1	0	0	3/2 1.500
2	1/2	1/2	0	2	0	2/5 0.400
			1	0	1	6/5 1.200
	1/2	-1/2	1	1	0	9/5 1.800
	-1/2	-1/2	2	0	0	3/5 0.600
3	1/2	1/2	1	1	1	12/5 2.400
			0	3	0	3/10 0.300
	1/2	-1/2	1	2	0	21/10 2.100
			2	0	1	3 3.000
	-1/2	-1/2	2	1	0	6/5 1.200
4	1/2	1/2	2	0	2	18/7 2.572
			1	2	1	132/35 3.771
			0	4	0	9/35 0.257
	1/2	-1/2	1	3	0	78/35 2.229
			2	1	1	60/7 8.571
	-1/2	-1/2	2	2	0	66/35 1.886
			3	0	1	12/7 1.714
5	1/2	1/2	2	1	2	135/14 9.643
			1	3	1	36/7 5.143
			0	5	0	3/14 0.214
	1/2	-1/2	1	4	0	33/14 2.357
			2	2	1	120/7 17.142
			3	0	2	15/2 7.500
	-1/2	-1/2	2	3	0	18/7 2.571
			3	1	1	45/7 6.429

TABLE 22

 P^1 for 1 spin $\frac{3}{2}$, 1 spin $\frac{1}{2}$ and n spin 1 particles

n	m_1	m_2	n_+	n_0	n_-	$(\frac{3}{2}, \frac{1}{2}, 1, \dots) * P^1$	
						$m_1, m_2; n_+, n_0, n_-$	
						(fractional)	(decimal)
0	3/2	-1/2	0	0	0	3/4	0.750
	1/2	1/2	0	0	0	1/4	0.250
1	3/2	1/2	0	0	1	3/5	0.600
	3/2	-1/2	0	1	0	9/20	0.450
	1/2	1/2	0	1	0	7/20	0.350
	1/2	-1/2	1	0	0	3/5	0.600
2	3/2	1/2	0	1	1	4/5	0.800
	3/2	-1/2	0	2	0	7/20	0.350
			1	0	1	1	1.000
	1/2	1/2	0	2	0	1/4	0.250
			1	0	1	3/5	0.600
	1/2	-1/2	1	1	0	6/5	1.200
	-1/2	-1/2	2	0	0	3/10	0.300
	-3/2	1/2	2	0	0	1/2	0.500
3	3/2	1/2	1	2	0	33/35	0.943
			2	0	1	9/7	1.286
	3/2	-1/2	0	3	0	39/140	0.279
			1	1	1	15/7	2.143
	1/2	1/2	0	3	0	33/140	0.236
			1	1	1	57/35	1.629
	1/2	-1/2	1	2	0	51/35	1.457
			2	0	1	12/7	1.714
4	-1/2	-1/2	2	1	0	57/70	0.814
	-3/2	1/2	2	1	0	15/14	1.071
	-3/2	-1/2	3	0	0	3/7	0.429
	3/2	1/2	1	3	0	36/35	1.029
			2	1	1	27/7	3.857
	3/2	-1/2	0	4	0	33/140	0.236
			1	2	1	24/7	3.429
			2	0	2	9/4	2.250
	1/2	1/2	0	4	0	27/140	0.193
			1	2	1	96/35	2.743
			2	0	2	45/28	1.607
	1/2	-1/2	1	3	0	12/7	1.714
			2	1	1	6	6.000
	-1/2	-1/2	2	2	0	48/35	1.371
			3	0	1	15/14	1.071
	-3/2	1/2	2	2	0	12/7	1.714
			3	0	1	3/2	1.500
	-3/2	-1/2	3	1	0	9/7	1.286

TABLE 22 (Ctd.)

n	m_1	m_2	n_+ n_0 n_-	$(\frac{3}{2}, \frac{1}{2}, 1, \dots)_{*1}^{*1}$ $P_{m_1, m_2}^{n_+, n_0, n_-}$ (fractional) (decimal)	
5	$3/2$	$1/2$	1 4 0	$23/21$	1.095
			2 2 1	$55/7$	7.857
			3 0 2	$10/3$	3.333
	$3/2$	$-1/2$	0 5 0	$17/84$	0.202
			1 3 1	$100/21$	4.762
			2 1 2	$35/4$	8.750
	$1/2$	$1/2$	0 5 0	$5/28$	0.179
			1 3 1	4	4.000
			2 1 2	$195/28$	6.964
	$1/2$	$-1/2$	1 4 0	$13/7$	1.857
			2 2 1	$90/7$	12.857
			3 0 2	5	5.000
	$-1/2$	$-1/2$	2 3 0	2	2.000
			3 1 1	$65/14$	4.642
	$-3/2$	$1/2$	2 3 0	$50/21$	2.379
			3 1 1	$35/6$	5.833
	$-3/2$	$-1/2$	3 2 0	$55/21$	2.619
			4 0 1	$5/3$	1.667

TABLE 23

 ${}^*P^1$ for 2 spin $\frac{3}{2}$ and n spin 1 particles

n	m_1	m_2	n_+ n_0 n_-	${}^*P^1_{m_1, m_2; n_+, n_0, n_-}$ (fractional)	n_+, n_0, n_- (decimal)
0	3/2	-1/2	0 0 0	3/5	0.600
	1/2	1/2	0 0 0	2/5	0.400
1	3/2	1/2	0 0 1	3/5	0.600
	3/2	-1/2	0 1 0	3/5	0.600
	3/2	-3/2	1 0 0	1	1.000
	1/2	1/2	0 1 0	1/5	0.200
	1/2	-1/2	1 0 0	3/5	0.600
2	3/2	3/2	0 0 2	3/7	0.429
	3/2	1/2	0 1 1	38/35	1.086
	3/2	-1/2	0 2 0	17/35	0.486
			1 0 1	8/7	1.143
	3/2	-3/2	1 1 0	10/7	1.429
	1/2	1/2	0 2 0	8/35	0.229
			1 0 1	18/35	0.514
	1/2	-1/2	1 1 0	6/7	0.857
	1/2	-3/2	2 0 0	4/7	0.571
	-1/2	-1/2	2 0 0	9/35	0.257
3	3/2	3/2	2 1 0	27/28	0.964
	3/2	1/2	1 2 0	48/35	1.371
			2 0 1	45/28	1.607
	3/2	-1/2	0 3 0	3/7	0.429
			1 1 1	3	3.000
	3/2	-3/2	1 2 0	12/7	1.714
			2 0 1	9/4	2.250
	1/2	1/2	0 3 0	6/35	0.171
			1 1 1	87/70	1.243
	1/2	-1/2	1 2 0	6/5	1.200
			2 0 1	39/28	1.393
	1/2	-3/2	2 1 0	3/2	1.500
	-1/2	-1/2	2 1 0	87/140	0.622
	-1/2	-3/2	3 0 0	15/28	0.536

TABLE 23 (Ctd.)

n	m_1	m_2	n_+ n_0 n_-	$\begin{matrix} *1 \\ P \\ m_1, m_2; \\ (fractional) \end{matrix}$ $\begin{matrix} n_+, n_0, n_- \\ (decimal) \end{matrix}$	
4	$3/2$	$3/2$	2 2 0	11/7	1.571
			3 0 1	4/3	1.333
	$3/2$	$1/2$	1 3 0	8/5	1.600
			2 1 1	39/7	5.571
	$3/2$	$-1/2$	0 4 0	13/35	0.371
			1 2 1	36/7	5.142
			2 0 2	3	3.000
	$3/2$	$-3/2$	1 3 0	40/21	1.905
			2 1 1	7	7.000
	$1/2$	$1/2$	0 4 0	6/35	0.171
			1 2 1	78/35	2.228
			2 0 2	9/7	1.286
	$1/2$	$-1/2$	1 3 0	48/35	1.371
			2 1 1	33/7	4.714
	$1/2$	$-3/2$	2 2 0	18/7	2.571
			3 0 1	2	2.000
	$-1/2$	$-1/2$	2 2 0	39/35	1.114
			3 0 1	6/7	0.857
	$-1/2$	$-3/2$	3 1 0	13/7	1.857
	$-3/2$	$-3/2$	4 0 0	1/3	0.333
5	$3/2$	$3/2$	2 3 0	31/14	2.214
			3 1 1	16/3	5.333
	$3/2$	$1/2$	1 4 0	37/21	1.762
			2 2 1	169/14	12.071
			3 0 2	14/3	4.666
	$3/2$	$-1/2$	0 5 0	1/3	0.333
			1 3 1	158/21	7.524
			2 1 2	13	13.000
	$3/2$	$-3/2$	1 4 0	43/21	2.048
			2 2 1	29/2	14.500
			3 0 2	6	6.000
	$1/2$	$1/2$	0 5 0	1/7	0.143
			1 3 1	23/7	3.286
			2 1 2	39/7	5.571
	$1/2$	$-1/2$	1 4 0	11/7	1.571
			2 2 1	21/2	10.500
			3 0 2	4	4.000
	$1/2$	$-3/2$	2 3 0	79/21	3.762
			3 1 1	26/3	8.666
	$-1/2$	$-1/2$	2 3 0	23/14	1.643
			3 1 1	26/7	3.714
	$-1/2$	$-3/2$	3 2 0	169/42	4.024
			4 0 1	7/3	2.333
	$-3/2$	$-3/2$	4 1 0	4/3	1.333