The field theoretical definition of nuclear potential II.

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S U M M A R Y

The results of Part I are extended to include the exchange forces introduced by charged pions. It is shown that the scattering amplitude for a proton and neutron between which potential with an exchange part acts, is related by symmetrisation to that for the scattering of non-identical particles through an isotopic spin dependent potential. This enables the amplitude to be split into two parts, each of which obeys a Khuri-type dispersion relation if the appropriate momentum transfer is kept fixed. The method allows also the generalisation of the Mandelstam representation to the case where exchange forces are present. A similar separation of the field-theoretic amplitude is possible into parts which satisfy one-dimensional dispersion relations without the pion singularities for negative energies. A comparison of the dispersion relations and the unitarity condition for the field theoretic amplitudes with those of potential theory lead to a definition of the potential (which is partly direct, partly exchange) which is capable of simulating the field-theoretic amplitude for energies sufficiently below meson production threshold.

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1. Introduction.

In Part I ¹⁾ we considered the scattering of two scalar meson field. This model deliberately avoids all difficulties connected with exchange forces and with spin. We were able to show that a comparison of the dispersion relations and the unitarity conditions for the field theory on the one hand and the potential theory on the other leads to a definition of a potential adequate to reproduce the field-theoretic scattering amplitude for energies sufficiently below meson-production threshold.

In the present paper we shall extend this model to allow for the interaction involving charged mesons. The negative energy cut in the fixed momentum transfer dispersion relation now extends much further towards the origin, because proton and antineutron annihilation into pions is now permitted. It is clear that a direct local potential cannot simulate a scattering amplitude with these analytic properties for any reasonable energy range; the potential must be either non-local or, what comes to the same thing, strongly energy dependent. We are able to show that the non-locality required is of a very kind, so that a superposition of a local direct potential and a local exchange potential, each of them energy-independent, is sufficient to give an approximation to the scattering amplitude below meson production threshold.

We shall leave over to a later paper the final complications associated with the spins.

The notation throughout this paper will be the same as in Part I, but we recall some of the more important definitions. The centre of mass 3-momentum $\underline{k}$ has modulus η , and the scattering angle in this reference frame is θ .

$$\begin{aligned} s &= E^2 = M^2 + \eta^2 \\ t &= -\frac{1}{2} \eta^2 (1 - \cos\theta) = -\Delta^2 \\ n &= -\frac{1}{2} \eta^2 (1 + \cos\theta) = -K^2 \end{aligned} \tag{1.1}$$

The potential, V , leads to a scattering matrix s related to the T-matrix by

$$(\vec{k}_2 | s | \vec{k}_1) = \delta^{(3)}(\vec{k}_2 - \vec{k}_1) - 2\pi i \int (W_2 - W_1) (\vec{k}_2 | T | \vec{k}_1) \quad (1.2)$$

In field theory the S-matrix is related to the scattering amplitude G by

$$(\vec{p}_2 n_2 | S | \vec{p}_1 n_1) = \delta^{(3)}(\vec{p}_2 - \vec{p}_1) \delta^{(3)}(\vec{n}_2 - \vec{n}_1) - 2\pi i \delta^{(4)}(p_2 + n_2 - p_1 - n_1) \times \\ \times \frac{M^2}{E^2} (\vec{k}_2 | G | \vec{k}_1) \quad (1.3)$$

We shall express matrices like T and G in terms of the scalar quantities η^2, t :

$$(\vec{k}_2 | G | \vec{k}_1) = G(\eta^2, t) \\ (\vec{k}_2 | T | \vec{k}_1) = T(\eta^2, t) \quad (1.4)$$

or sometimes in terms of s, t .

2. Exchange effects in potential scattering.

- A) Before we discuss the effect of exchange forces, it is instructive to look at the exchange effect introduced by symmetrisation when two particles scattering through a potential are identical ^{*)}. If the T-matrix for non-identical particles scattering through the potential is $T(s, t)$, that for identical particles is

^{*)} We recall that all particles are treated as bosons in the present paper.

$$\bar{T}(s,t) = \frac{1}{2} \left\{ T(s,t) + T(s,u) \right\} \quad (2.1)$$

Now it is well-known that $T(s,t)$ obeys a dispersion relation, the Khuri relation, for fixed t ²⁾; evidently $\bar{T}$ obeys no such simple relation for fixed t . We may ask what are the analytic properties of $\bar{T}$, and to find them we use the Mandelstam representation for T , which has recently been proved rigorously³⁾, viz. :

$$T(s,t) = \int_{M^2}^{\infty} ds' \int_{\frac{\mu^2}{4}}^{\infty} dt' \frac{A(s',t')}{(s'-s)(t'-t)} \quad (2.2)$$

where the potential is supposed to be a superposition of Yukawa potentials with inverse ranges

$$V(t) = \int_{\frac{\mu^2}{4}}^{\infty} \frac{\varrho(t') dt'}{t'-t} \quad (2.3)$$

Then

$$\begin{aligned} \bar{T} &= \frac{1}{2} \int_{M^2}^{\infty} ds' \int_{\frac{\mu^2}{4}}^{\infty} dt' \frac{A(s',t')}{(s'-s)(t'-t)} \\ &+ \frac{1}{2} \int_{M^2}^{\infty} ds' \int_{\frac{\mu^2}{4}}^{\infty} du' \frac{A(s',u')}{(s'-s)(u'-u)} \end{aligned} \quad (2.4)$$

The one-dimensional dispersion relation obtained from this at fixed t is

$$\bar{T} = \int_0^{\infty} d\eta'^2 \frac{B(\eta'^2, t)}{\eta'^2 - \eta^2 - i\varepsilon} + \int_{\frac{\mu^2}{4}}^{\infty} d\eta'^2 \frac{C(\eta'^2, t)}{\eta'^2 + \eta^2 + t} \quad (2.5)$$

where

$$B(\eta'^2, t) = \frac{1}{2} \int_{\frac{\mu^2}{4}}^{\infty} dt' A(\eta'^2 + M^2, t') \left\{ \frac{1}{t' - t} + \frac{1}{t' + t + \eta'^2} \right\} \quad (2.6)$$

$$C(\eta'^2, t) = \frac{1}{2} \int_{\frac{\mu^2}{4}}^{\infty} dt' \frac{A(t', \eta'^2)}{\eta'^2 + t + t'}$$

This dispersion relation is formally identical with that of Goldberger, Nambu and Oehme ⁴⁾, in that there is in addition to the cut for $\eta'^2 > 0$, a cut on the left-hand side for $\eta'^2 < -\frac{\mu^2}{4} - t$.

In this simple case we may summarise by observing that the scattering amplitude may be decomposed into two parts as in (2.1), the first satisfying a Khuri-type dispersion relation for fixed t , the second for fixed u . The more complicated form of the dispersion relation (2.5) for $\bar{T}$ itself originates in the fact that for fixed t the second term on the right hand side of (2.1) depends implicitly on s through $u = M^2 - s - t$.

B) Consider now the scattering of two particles, p and n , through a potential containing an exchange part. We wish to show that the analytic properties of the scattering amplitude T_{pn} are quite analogous to the ones discussed in subsection A) : first, it is possible to separate

$$T_{pn} = T_D + T_E \quad (2.7)$$

so that T_D satisfies a Khuri-type relation for fixed t and T_E for fixed u . Secondly, for fixed t , T_{pn} satisfies a GNO type of dispersion relation of the form (2.5).

To do this it is convenient to make use of the isotopic spin formalism. This enables us to consider p and n as two different charge states N_1 and N_2 of the same particle N . Our problem is thus reduced to identical particle (N-N) scattering through an isotopic spin dependent potential. This is closely analogous to that treated in subsection A), the only additional complication being the charge degree of freedom possessed by each particle. We will show that the extra singularities coming from the exchange forces are just a consequence of symmetrisation.

We have seen that the simplest way of discussing the analytic properties of the scattering amplitude for identical particles is first to consider the scattering of non-identical particles, and then to symmetrise. This leads us to consider the scattering of two particles, N and Y , each with two charge states, through a potential which depends on isotopic spin

$$V(t) = V_D(t) + V_E(t) P \quad (2.8)$$

where

$$P = \frac{1}{2}(1 + \vec{\tau}_1 \cdot \vec{\tau}_2) \quad (2.9)$$

The Schrödinger equation for such a process is a pair of coupled differential equations, with only direct potentials. These equations can be trivially separated by considering states symmetric or antisymmetric in the exchange $Y \rightleftharpoons N$ (corresponding of course to isotopic singlet or triplet states). This is achieved by means of the projection operators :

$$\Lambda_{(\pm)} = \frac{1}{2}(1 \pm P) \quad (2.10)$$

and the separation yields two uncoupled Schrödinger equations with the effective potentials

$$V_{(\pm)} = V_D \pm V_E \quad (2.11)$$

The resulting T-matrices, $T_{(\pm)}$ related by

$$T = T_{(+)} \Lambda_{(+)} + T_{(-)} \Lambda_{(-)} \quad (2.12)$$

will therefore have Mandelstam representations :

$$T_{(\pm)} = \int_{M^2}^{\infty} ds' \int_{\frac{\mu^2}{4}}^{\infty} dt' \frac{A_{(\pm)}(s', t')}{(s' - s)(t' - t)} \quad (2.13)$$

It is also possible to separate T into a direct and an exchange part

$$T = T_D + T_E P \quad (2.14)$$

$$T_D = \frac{1}{2}(T_{(+)} + T_{(-)}) \quad (2.15)$$

$$T_E = \frac{1}{2}(T_{(+)} - T_{(-)})$$

Evidently, T_D and T_E also have Mandelstam representations of the form (2.13) with weight functions $A_D(s', t')$, $A_E(s', t')$. In the iteration series for T , T_D and T_E are respectively the sums of terms in which the exchange potential acts an even or odd number of times.

Consider now the unitarity relation. Since the equations of motion are separated by the projection operators $\Lambda_{(\pm)}$, the unitarity relations for $T_{(\pm)}$ take the same form as in a neutral theory, which we may symbolise as

$$Im T_{(\pm)} = \pi T_{(\pm)}^+ * T_{(\pm)} \quad (2.16)$$

Thus

$$\begin{aligned}\text{Im } T_D &= \pi \left(T_D^+ * T_D + T_E^+ * T_E \right) \\ \text{Im } T_E &= \pi \left(T_D^+ * T_E + T_E^+ * T_D \right)\end{aligned}\quad (2.17)$$

which confirms the even-odd character of T_D and T_E in the number of exchanges.

This completes the summary of the general properties of the amplitudes $T_{(\pm)}$ relevant to the N-Y problem. It is now necessary to relate these quantities to the scattering amplitude for our original p-n problem. This is done by applying the appropriate symmetrisation. The N-N scattering amplitude $\bar{T}$ is

$$\bar{T} = \left[T_{(+)}(s,t) + T_{(+)}(s,u) \right] \Lambda_{(+)} \quad (2.18)$$

$$+ \left[T_{(-)}(s,t) - T_{(-)}(s,u) \right] \Lambda_{(-)}$$

$$= \left[T_D(s,t) + T_E(s,u) \right] \quad (2.19)$$

$$+ \left[T_E(s,t) + T_D(s,u) \right] P$$

The pp and pn scattering amplitudes are derived by taking matrix elements of P between the appropriate states

$$(pn | P | pn) = 0 \quad (2.20)$$

$$(pp | P | pp) = 1$$

so that

$$T_{pn} = T_D(s,t) + T_E(s,u) \quad (2.21)$$

$$T_{pp} = T_{nn} = T_{pn}(s,t) + T_{pn}(s,u) \quad (2.22)$$

Equations (2.21) shows clearly the striking analogy with the discussion of subsection A). The unsymmetrised amplitudes T_D and T_E have simple Mandelstam representations of the form (2.21), and satisfy a Khuri-type dispersion relation for fixed momentum transfer. The proton neutron scattering amplitude has the representation

$$T_{pn} = \iint \frac{A_D(s', t') ds' dt'}{(s' - s)(t' - t)} + \iint \frac{A_E(s', u') ds' du'}{(s' - s)(u' - u)} \quad (2.23)$$

From Eq. (2.23) it follows that for fixed t , T_{pn} satisfies a GNO type dispersion relation:

$$T_{pn}(\eta'^2, t) = \int_0^\infty \frac{d\eta'^2}{\eta'^2 - \eta^2} \alpha(\eta'^2, t) + \int_{\frac{\mu^2}{4}}^\infty \frac{d\eta'^2}{\eta'^2 + \eta^2 + t} \beta(\eta'^2, t) \quad (2.24)$$

where

$$\alpha(\eta'^2, t) = \int_{\frac{\mu^2}{4}}^\infty dt' \frac{A_D(M^2 + \eta'^2, t')}{t' - t} + \int_{\frac{\mu^2}{4}}^\infty du' \frac{A_E(M^2 + \eta'^2, u')}{u' + \eta'^2 + t} \quad (2.25)$$

$$\beta(\eta'^2, t) = \int_0^\infty dy \frac{A_E(M^2 + y, \eta'^2)}{\eta'^2 + y + t} \quad (2.26)$$

The separation of T_{pn} into parts corresponding to an even and odd number of exchanges has also been performed by Hamilton on the basis of perturbation theory⁵⁾. He also shows that each part satisfies a Khuri-type dispersion relation if the appropriate momentum transfer is kept fixed.

3. Exchange effects in field theory.

In this section we relax some of the unphysical restrictions imposed on the model treated in Part I by allowing $\pi^\pm$ to be exchanged between the nucleons. In this case $p + \bar{n} \rightarrow \pi s$ is permitted, and the field-theoretic dispersion relations have the form derived in GNO with a pole at $\eta^2 = -\frac{\mu^2}{4} - t$ and a cut for $\eta^2 < -\mu^2 - t$. It is clear that this cannot be derived from a direct potential, which leads to a Khuri-type of dispersion relation with no singularities for negative energies. But we saw in the previous section that a potential with an exchange part gives rise to just such singularities, and so it is still possible that the T-matrix from a potential with exchange might be able to reproduce that of field theory at low energies.

In order to prove that this is indeed the case, one would have to identify the low-lying singularities of the two kinds of dispersion relations. This is not easy, because in the GNO form the exchange and direct forces are treated on entirely different grounds. The simplest way is to demonstrate that the field-theoretic scattering amplitude can be split into two parts as in the previous section

$$G_{pn} = G_D(s, t) + G_E(s, u) \quad (3.1)$$

where G_D and G_E each satisfy dispersion relations similar to (I. 3.1)

$$G_{D,E}(\eta^2, t) = G_{D,E}(0, t) + \frac{1}{\pi} \int_0^\infty \frac{\eta'^2}{\eta'^2} \frac{\text{Im } G_{D,E}(\eta'^2, t) d\eta'^2}{(\eta'^2 - \eta^2 - i\epsilon)} - \frac{1}{\pi} \int_{\mu^2(u'+t)}^\infty \frac{\eta'^2}{\eta'^2} \frac{\text{Im } G_{D,E}^u(u', t) du'}{(u' + t + \eta'^2)} \quad (3.2)$$

This may be done in two ways. First, one may look at the perturbation expansion for G_{pn} ; then G_D is the sum of all graphs of the type of Fig.1a in which the net charge exchanged between the nucleons is zero, whilst G_E corresponds to graphs of the type of Fig.1b in which there is an over-all exchange of one unit of charge between the nucleons. Evidently this corresponds to an exchange of an even or an odd number of charged pions respectively, analogous to the occurrence of an even or an odd number of applications of the exchange potential in the corresponding treatment of the previous section.

It is also clear that for graphs of the type of Fig.1a $p+n$ cannot give pions, whilst for graphs of the type of Fig.1b $p+\bar{p}$ cannot give pions. This shows that there are no negative energy singularities coming from pion states for the dispersion relation at fixed t for the first type of graph, nor for fixed u for the second. This proves that G_D and G_E satisfy the relation (3.2) keeping the appropriate momentum transfer fixed.

A more compact and convenient way of looking at this same point is, as in Section 2, to attribute the separation (3.1) to the effect of symmetrisation. Here we again introduce two different nucleon types N and Y , each having two different charge states, and interacting in the same way with an isotopic triplet pion field. Of course we do not allow transitions $N \rightleftharpoons Y + \pi$.

This model will allow us to consider both N - Y and N - N scattering. Since the Hamiltonian for the model is symmetric under $N \rightleftharpoons Y$ exchange, N - N scattering is obtained by properly symmetrising the N - Y scattering amplitude ^{*)}.

*) The N - N scattering in this model is not exactly the same as that in the case in which the Y -particle does not exist: this is because of the possibility of Y - $\bar{Y}$ pairs in intermediate states. However, this can make no difference to the low-lying singularities which are all that are important for our problem. The pair effects can only contribute to the extremely short range part of the potential (as indeed hyperon-antihyperon pairs certainly do) and about this region the present theory can in any case say nothing.

In this model the N-Y scattering amplitude can be written $\overline{\text{cf. (2.12)}}$

$$G = G_{(+)} \Lambda_{(+)} + G_{(-)} \Lambda_{(-)} \quad (3.3)$$

$G_{(+)}$ and $G_{(-)}$ satisfy dispersion relations of the form (3.2) for fixed t . The unitarity relation for elastic processes has the simple form

$$\text{Im}^{\text{el}} G_{(\pm)} = \frac{M}{E} G_{(\pm)}^+ * G_{\pm} \quad (3.4)$$

The identical particle amplitude for N-N scattering, $\overline{G}$, obtained by symmetrising G is

$$\overline{G} = \left[G_{(+)}(s, t) + G_{(+)}(s, u) \right] \Lambda_{(+)} + \left[G_{(-)}(s, t) - G_{(-)}(s, u) \right] \Lambda_{(-)} \quad (3.5)$$

$$= \left[G_D(s, t) + G_E(s, u) \right] + \left[G_E(s, t) + G_D(s, u) \right] P \quad (3.6)$$

where

$$G_D = \frac{1}{2} (G_{(+)} + G_{(-)}) \quad (3.7)$$

$$G_E = \frac{1}{2} (G_{(+)} - G_{(-)})$$

G_D and G_E evidently also satisfy dispersion relations of the form (3.2), and it is clear that (3.1) is a special case of (3.6).

We need hardly remark on the close analogy between the equations derived in this section and those of the previous section.

4. The definition of the potential.

In Section 4 or Part I we showed how the formal identity between the dispersion relation and the unitarity requirement for the field theory of proton-neutron scattering through neutral mesons and the scattering in a direct potential which obtains at sufficiently low kinetic energies, together with the requirement that the corresponding scattering amplitudes should be identical at zero kinetic energy, leads to a unique definition of the potential. An exactly parallel argument is used in this Section for the comparison between a field theory in which charged mesons also occur and a potential theory with exchange.

As was pointed out in the Introduction, the essential new feature introduced by the presence of charged pions is the pion continuum and pole for proton-antineutron annihilation, which show up as additional singularities in the fixed t dispersion relation of GNO. The guiding principle of our method is to find a potential which leads to a scattering matrix with the same low-lying singularities as that of field theory. In Part I, when the negative energy cut in the field-theoretic fixed t dispersion relation began at $-M^2 - t$, it was reasonable that the amplitude could be approximated by a local direct potential, which leads to a Khuri-type dispersion relation. Now that we have a pole at $-\frac{\mu^2}{4} - t$ and a cut starting at $-\mu^2 - t$, a local direct potential, which cannot give rise to singularities on the negative axis, could only be useful in a very restricted energy range, so small in fact that effective range theory would have a wider validity. To reproduce these negative energy singularities we must introduce some kind of non-locality into the potential. In general, of course, this would be very difficult, and indeed not very useful, as the general non-local potential is a function of two variables, just like the scattering amplitude itself. However, the separation of the field theoretic amplitude achieved in the previous Section into parts which, with the appropriate momentum transfer fixed, have one-dimensional dispersion relations exactly as in Part I without the pion continuum, together with the discussion of Section 2 on exchange potentials, make it clear that the very special kind of non-locality in a potential which is a superposition of a pure direct and a pure exchange part is adequate.

We saw in Section 2 how the T-matrix may be separated into $T_{(\pm)}$, each of which satisfies a Khuri-type dispersion relation

$$T_{(\pm)}(\eta^2, t) = V_{(\pm)}(t) + \frac{1}{\pi} \int_0^\infty \frac{\text{Im } T_{(\pm)}(\eta'^2, t) d\eta'^2}{\eta'^2 - \eta^2 - i\varepsilon} \quad (4.1)$$

A similar separation of the field-theoretic amplitude into $G_{(\pm)}$ was shown in Section 3, and if we make the identification

$$T_{(\pm)}(0, t) = G_{(\pm)}(0, t) \quad (4.2)$$

we obtain

$$T_{(\pm)}(\eta^2, t) = G_{(\pm)}(0, t) + \frac{1}{\pi} \int_0^\infty \frac{\eta'^2}{\eta'^2} \frac{\text{Im } T_{(\pm)}(\eta'^2, t) d\eta'^2}{(\eta'^2 - \eta^2 - i\varepsilon)} \quad (4.3)$$

$$V_{(\pm)}(t) = G_{(\pm)}(0, t) - \frac{1}{\pi} \int_0^\infty \frac{\text{Im } T_{(\pm)}(\eta'^2, t) d\eta'^2}{\eta'^2} \quad (4.4)$$

We also have the unitarity condition

$$\text{Im } T_{(\pm)} = \pi T_{(\pm)}^+ * T_{(\pm)} \quad (4.5)$$

Exactly as in Part I, (4.3), (4.4) and (4.5) define the potential $V_{(\pm)}$ in terms of the zero-energy field-theoretic amplitude $G_{(\pm)}$: (4.3) and (4.5) are to be solved for $T_{(\pm)}$ which gives the potential through (4.4). Again we must show that this potential has a useful range of validity. A relation formally identical with (3.2) evidently exists for $G_{(\pm)}$, and this we may write as

$$G_{(\pm)}(\eta^2, t) = G_{(\pm)}(0, t) + \frac{1}{\pi} \int_0^\infty \frac{\eta'^2}{\eta'^2} \frac{\text{Im}^{el} G_{(\pm)}(\eta'^2, t) d\eta'^2}{(\eta'^2 - \eta^2 - i\varepsilon)} + O\left(\frac{\eta^2}{\eta_{\text{thresh}}^2}\right) \quad (4.6)$$

where

$$\eta_{\text{thresh}}^2 = M^2 + \frac{\mu^2}{4} \quad (4.7)$$

For momenta sufficiently below the meson-production threshold η_{thresh} , (4.6) and the unitarity relation (3.4) are formally identical with (4.3) and (4.5), so we may conclude that

$$G_{(\pm)}(\eta^2, t) = T_{(\pm)}(\eta^2, t) + O\left(\frac{\eta^2}{\eta_{\text{thresh}}^2}\right) \quad (4.8)$$

i.e., that the potential is adequate to reproduce $G_{(\pm)}$ for energies sufficiently below meson-production threshold. Since the symmetrisations which lead to (2.18) on the one hand and to (3.5) on the other are equivalent, the potential

$$V = V_{(+)} \Lambda_{(+)} + V_{(-)} \Lambda_{(-)} \quad (4.9)$$

leads to a scattering matrix for identical particles which approximates well to the field-theoretic nucleon-nucleon amplitude.

In Part I we gave a method of explicit construction of the potential as a superposition of Yukawa potentials

$$V(t) = \frac{1}{(2\pi)^3} \int \frac{\mathcal{P}(\sigma^2) d\sigma^2}{\sigma^2 - 4t} \quad (4.10)$$

in the physically interesting case when we may write

$$G(0,t) = \frac{1}{(2\pi)^3} \int \frac{F(\sigma^2) d\sigma^2}{\sigma^2 - 4t} \quad (4.11)$$

This method still applies, *mutatis mutandis*, to the present case, and we will not bore the reader with its repetition. We shall, however, illustrate an application to the one- and two-meson exchange potentials of perturbation theory.

The one-meson exchange amplitude is now

$$\begin{aligned} G^{(1)}(\eta^2, t) &= \vec{\tau}_1 \cdot \vec{\tau}_2 G_0^{(1)}(\eta^2, t) \\ &= (\Lambda_{(+)} - 3\Lambda_{(-)}) G_0^{(1)}(\eta^2, t) \end{aligned} \quad (4.12)$$

where the suffix 0 indicates the corresponding quantity of the neutral theory. Similarly the amplitudes for the two-meson exchange uncrossed and crossed graphs are respectively

$$G^{(A)}(\eta^2, t) = (\Lambda_{(+)} + 9\Lambda_{(-)}) G_0^{(A)}(\eta^2, t) \quad (4.13A)$$

$$G^{(B)}(\eta^2, t) = (5\Lambda_{(+)} - 3\Lambda_{(-)}) G_0^{(B)}(\eta^2, t) \quad (4.13B)$$

In a direct application of perturbation theory, the potential is defined by a term-by-term identification of the T-matrix obtained by iterating a potential with the perturbation expansion of the field-theoretic scattering amplitude. This yields

$$V^{(1)} = G^{(1)} \quad (4.14)$$

$$V^{(2)} = G^{(A)} + G^{(B)} + I \quad (4.15)$$

where I is the iteration of $V^{(1)}$. In the neutral theory we saw that $G_O^{(1)}$, and hence $V_O^{(1)}$, is independent of η^2 , and that in the adiabatic limit ($\eta^2 \ll M^2$) the same is true for $G_O^{(B)}$. Moreover, there is a cancellation between the energy-dependent parts of $G_O^{(A)}$ and I_O which enables us to say that in the adiabatic limit $V_O^{(2)}$ is also energy-independent.

In the present case, it is clear from (4.12) and (4.14) that the one-meson exchange potential is energy-independent, being proportional to that of the neutral theory. Similarly in the adiabatic limit, $G^{(B)}$ is independent of η^2 . It remains to show that the cancellation between the energy-dependent part of $G^{(A)}$ and I still occurs. But from (4.12)

$$\begin{aligned} I &= (\not{A}_{(+)} - 3 \not{A}_{(-)})^2 I_O \\ &= (\not{A}_{(+)} + 9 \not{A}_{(-)}) I_O \end{aligned} \quad (4.16)$$

so that

$$\begin{aligned} V^{(A)} &= G^{(A)} + I \\ &= (\not{A}_{(+)} + 9 \not{A}_{(-)}) V_O^{(A)} \end{aligned} \quad (4.17)$$

is indeed energy-independent in the adiabatic limit.

It would, of course, have been possible to express the whole argument of this section in terms of a decomposition into direct and exchange parts than by using the projection operators $\not{A}_{(\pm)}$. The reason that the direct and exchange potentials can be separated in perturbation theory is that at each stage the field-theoretic amplitude is separated, and that since the lower-order potentials are already known and separated, so are their combinations in the iteration series.

We conclude this Section by stating the results of applying the method of Part I for obtaining the spectral-function for the potential, modified to deal with the separation by $\Lambda_{(\pm)}$. For $\sigma^2 < 4\mu^2$

$$\rho(\sigma^2) = -\frac{g^2}{4M^2} \int (\sigma^2 - \mu^2) (\Lambda_{(+)} - 3\Lambda_{(-)}) \quad (4.18)$$

For $4\mu^2 < \sigma^2 < 9\mu^2$

$$\begin{aligned} \rho(\sigma^2) = & \frac{g^4}{64\pi^2} \frac{1}{M^3\mu^2} \frac{1}{\sigma} (\Lambda_{(+)} + 9\Lambda_{(-)}) \theta \\ & - (5\Lambda_{(+)} - 3\Lambda_{(-)}) \frac{\sin^2 \theta}{2} + F_{\text{rescattering}}(\sigma^2) \end{aligned} \quad (4.19)$$

$$\theta(\sigma) = \tan^{-1} \left\{ \frac{\mu}{M} \frac{\mu}{\sqrt{\sigma^2 - 4\mu^2}} \right\} \quad (4.20)$$

The potential obtained from these equations by (4.10) yields a T-matrix which when symmetrised reproduces for $\eta^2 < \eta_{\text{thresh}}^2$ the one- and two-meson exchange parts of the field-theoretic problem.

5. Conclusions.

In this second Part we have generalised the results of Part I to include the case of charged meson exchange, and have shown that apart from the appearance of exchange forces, the conclusions as to the existence of a potential are the same. Up to now all our considerations have been for spinless particles, and the discussion of the physically interesting case has been deferred to a further paper.

The reason for this is not entirely a consequence of the disinclination of the authors to tackle the considerable complexities introduced by the spin. The methods developed in Parts I and II cannot be extended immediately to the treatment of particles with spin because the mathematical tools required are not yet available. In particular the Khuri result has not yet been generalised to, for example, the spin orbit potential. Nor is it clear what choice of relativistic invariants gives the simple analytic properties required for the definition of the potential.

Only when these difficulties have been resolved can our formalism be extended to cover the scattering of particles with spin. If no unexpected difficulties arise, we anticipate that a comparison of the ten field-theoretical dispersion relations with the corresponding ones for potential theory will lead to the definition of a potential

$$V = V_D + V_E P \quad (5.1)$$

$$V_{D,E} = V_{D,E}^{(1)} + V_{D,E}^{(2)} (\vec{\sigma}_1 \cdot \vec{n}) + V_{D,E}^{(3)} (\vec{\sigma}_1 \cdot \vec{n}) (\vec{\sigma}_2 \cdot \vec{n}) + V_{D,E}^{(4)} (\vec{\sigma}_1 \cdot \vec{n}) (\vec{\sigma}_2 \cdot \vec{n}) + V_{D,E}^{(5)} (\vec{\sigma}_1 \cdot \vec{n}) (\vec{\sigma}_2 \cdot \vec{n}) \quad (5.2) \quad 6)$$

where the ten potentials $V_{D,E}^{(i)}$ are functions of the distance r alone. Since the $\frac{\mu}{M}$ corrections are not small, there is no reason to suppose that any of these ten potentials may be ignored.

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Fig. 1a

Fig. 1b

Fig.1

Graphs contributing to the direct and to the exchange parts of the scattering amplitude.