CM-P00057215

Some remarks about the spectral representation of the
 $K^+ \rightarrow 3\pi$ decay amplitude

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The effect of pion-pion interaction on the $K^+ \rightarrow 3\pi$ spectrum has been recently investigated^{1),2)}.

The following representation has been proposed for the decay amplitude $A^d(s_1, s_2, s_3)$:

$$A^d(s_1, s_2, s_3) = \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{g(x)dx}{x-s_1-i\epsilon} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{g(x)dx}{x-s_2-i\epsilon} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{g(x)dx}{x-s_3-i\epsilon} \quad (1)$$

where

$$s_i = (K - \pi_i)^2$$

(K and π_i denote the momenta of the K and of the pions).

From the representation (1) an integral equation has been derived which can be approximately solved by taking into account only the effect of the lowest singularities. This has enabled some information to be obtained from the K spectrum about the s-wave pion-pion interaction.

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Representation (1) can be simply understood in the following manner. Let us consider the unphysical case in which the mass of the K-meson is of the same order of magnitude as the pion mass; in such a case we have to deal with the scattering process

$$K + \pi \longrightarrow \pi + \pi \quad (2)$$

Perturbation theory then suggests the validity of the Mandelstam representation for the scattering amplitude A^S of the process (2)

$$A^S(s_1, s_2, s_3) = \frac{1}{\pi^2} \iint \frac{\rho(x, y) dx dy}{(x-s_1-i\epsilon)(y-s_2-i\epsilon)} + \frac{1}{\pi^2} \iint \frac{\rho(x, y) dx dy}{(x-s_2-i\epsilon)(y-s_3-i\epsilon)} + \frac{1}{\pi^2} \iint \frac{\rho(x, y) dx dy}{(x-s_3-i\epsilon)(y-s_1-i\epsilon)} \quad (3)$$

It has been shown by M. Cini and one of us⁴⁾ that under very general arguments on the location of the singularities, the Mandelstam representation can be transformed into the approximate form, valid for $|s_1| < 16\mu^2$

$$A^S(s_1, s_2, s_3) = \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{g(x) dx}{x-s_1-i\epsilon} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{g(x) dx}{x-s_2-i\epsilon} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{g(x) dx}{x-s_3-i\epsilon} \quad (4)$$

the spectral function $g(x)$ can be obtained by computing the imaginary part of A^S for $s_1 > 4\mu^2$. Since only the denominator of the first term can vanish, we get

$$\text{Im } A^S(s_1, s_2, s_3) = g(s_1) \quad (4')$$

On the other hand, the theorem of final state interaction⁵⁾ as applied to the process (2) gives

$$g(s) = A_0(s) e^{-i\delta(s)} \sin \delta(s) \quad (5)$$

where $\delta(s)$ is the s-wave pion-pion phase shift and $A_0(s)$ is the s-wave part of the amplitude A . Equation (5) follows from the fact that the K-meson is here connected to pions only through a weak interaction.

Now it is easy to observe that the representation for the decay amplitude can be written in the form (1) by assuming that Eq. (4), together with the expression (5) of $g(x)$ valid for the scattering process, can be extended also to the actual value of M_K corresponding to the decay process.

The possibility of justifying such continuation has been studied by G. Bonnevey who has investigated in detail the analytical properties of the decay amplitude in the mass of the K-meson.

We point out here that in the representation (1) for the decay process a new feature arises. All three denominators in Eq. (1) can vanish simultaneously for values of s_1, s_2, s_3 in the physical region.

If we assume the validity of (1) and (5) for the decay process, then the spectral function $g(s)$ must be a complex quantity⁶⁾. This can be easily seen in the following manner. Let us assume $g(s)$ to be real, then Eq. (5) tells us that

$$\text{Im } A_0(s) = g(s)$$

This result is in contrast with what one would obtain for g real by taking the imaginary part of Eq. (1)

$$\text{Im } A^d(s_1, s_2, s_3) = g(s_1) + g(s_2) + g(s_3)$$

We see therefore that the representation (1) for decay process has a completely different mathematical structure than the similar equation for scattering. In order to understand the physical meaning of the imaginary part of g let us write

$$\rho(s) = \frac{g(s) - g^*(s)}{2i} \quad (6)$$

Let us take the imaginary part of A^d

$$\text{Im } A^d = g(s_1) + g(s_2) + g(s_3) + \frac{1}{\pi} \left[\int_{\mu^2}^{\infty} \frac{\rho(x) dx}{x-s_1+i\epsilon} + \int_{\mu^2}^{\infty} \frac{\rho(x) dx}{x-s_2+i\epsilon} + \int_{\mu^2}^{\infty} \frac{\rho(x) dx}{x-s_3+i\epsilon} \right] \quad (7)$$

We note that the term containing $\rho(x)$ must vanish for $M_K \rightarrow 3\mu$.

Now, using the unitarity of the S matrix, and invariance under time reversal and under the Lorentz transformations⁵⁾, one obtains the following general expression for the imaginary part of the decay amplitudes:

$$\begin{aligned} \text{Im } A^d(\pi_1, \pi_2, \pi_3) &= \int T^+(\pi_1, \pi_2, \pi_3; \pi'_1, \pi'_2, \pi'_3) A^d(\pi'_1, \pi'_2, \pi'_3) \cdot \\ &\quad \delta(\pi_1^2 + \mu^2) \delta(\pi_2^2 + \mu^2) \delta(\pi_3^2 + \mu^2) \theta(\pi'_1) \theta(\pi'_2) \theta(\pi'_3) \cdot d^4\pi'_1 d^4\pi'_2 d^4\pi'_3 \end{aligned} \quad (8)$$

where T is the transition matrix element from the state π_1, π_2, π_3 to the state π'_1, π'_2, π'_3 . By comparing Eq. (7) with Eq. (8) it is possible to recognize the contributions to Eq. (7) as due to the effect of correlations between two pions or three pions.

We thus split the matrix element T into a connected and a disconnected part

$$T = T_c + T_{nc} \quad (9)$$

where

$$\begin{aligned} T_{nc} &= 2\pi_1^0 \delta^3(\vec{\pi}_1 - \vec{\pi}'_1) t(\pi_2, \pi_3; \pi'_2, \pi'_3) + 2\pi_2^0 \delta^3(\vec{\pi}_2 - \vec{\pi}'_2) t(\pi_1, \pi_3; \pi'_1, \pi'_3) + \\ &\quad + 2\pi_3^0 \delta^3(\vec{\pi}_3 - \vec{\pi}'_3) t(\pi_1, \pi_2; \pi'_1, \pi'_2) \end{aligned} \quad (10)$$

$t(\pi_i, \pi_j; \pi'_i, \pi'_j)$ is the pion-pion scattering amplitude, and $T_c(\pi_1, \pi_2, \pi_3; \pi'_1, \pi'_2, \pi'_3)$ is the matrix element representing the sum of all those graphs in which all three pions are involved.

Let us substitute Eqs. (9) and (10) into Eq. (8). It is easy to verify that the contribution of the disconnected part T_{nc} (if one keeps only the pion-pion s-waves) reproduces exactly the term $\sum_i g(s_i)$ in the r.h.s. of Eq. (7) where $g(s)$ is given by Eq. (5).

The contribution of the connected part T_c can therefore be identified with the term $\sum_i \int \frac{\rho(x) dx}{x-s_i+i\epsilon}$, such a term, as previously stated, vanishes where M_K decreases to 3μ .

We see therefore that the imaginary part $\rho(x)$ of the spectral function $g(x)$ is directly connected with the 3 pion transition amplitude. Thus Eq. (1) might be a tool for constructing the 3 pion amplitude in terms of the pion-pion phaseshifts.

We wish to thank Dr. D. Amati, G. Bonnevey, A. Stanghellini and B. Vitale for very useful comments and criticism. We are particularly grateful to Dr. G. Bonnevey for having shown us his unpublished work; many questions have become clearer from comparison with his results.

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- 6) This result has also been obtained independently by G. Bonnevey and A. Stanghellini.