Photoproduction of pions on pionsM. Gourdin ^{*)} and A. Martin

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SUMMARY

Photoproduction of pions on pions is computed by means of the Cini-Fubini version of the Mandelstam technique. The problem is reduced to the resolution of a Fredholm integral equation containing an arbitrary multiplicative constant which could be determined by extrapolation of experimental cross-sections. Under the assumption of a sharp pion pion resonance we derive approximate solutions which exhibit the same resonant behaviour. These results will be soon applied to Compton scattering on pions and nucleons and photoproduction on the nucleons.

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I. Introduction.

The assumption of a double dispersion relation ¹⁾ for a two body - two body matrix element leads to interesting consequences when it is combined with the unitarity condition. It is hoped that the approximation of neglecting three body states is valid in a limited range of energy. This is being applied to pion pion ^{2),3),4)}, pion nucleon ^{5),6)} and nucleon nucleon ^{5),7)} interaction.

Examination of the processes involving photons, such as pion production on nucleons or Compton scattering on nucleons shows that a necessary intermediate step to apply the Mandelstam technique to these reactions is the calculation of the process :

$$\gamma + \pi \rightarrow \pi + \pi^{*})$$

The interest of this reaction also lies in its simplicity, due to the limitation of possible states by conservation laws and symmetry properties. Assuming a known pion pion $J = 1$ $T = 1$ phase shift, one can reduce the determination of the matrix element to the solution of a single Mushkelishvili - Omnès integral equation which can be transformed into a Fredholm equation containing an arbitrary multiplicative constant in the homogeneous term. There is no way of determining this arbitrary factor in this approach because the electromagnetic coupling constant of pions never appears and since, in this process, the isoscalar part of the photon alone is acting, there is no possibility of using a low energy limit property.

In the first sections we give the rigorous treatment of this problem, following the simplified version of the Mandelstam technique proposed by Cini and Fubini ⁸⁾. The last section is concerned with approximate solutions of the equation and discussion of the results under the assumption of a pion pion resonance.

*) This problem is also studied by H.S. Wong, but we have no detailed information on his work which will be presented at the November 1959 American Physical Society meeting.

II. Structure of the matrix elements. Kinematics.

We define p_1, p_2, p_3 as the incoming four momenta of the pions ^{*}), $k = -p_1 - p_2 - p_3$ as the incident momentum of the photon. We define channel one by the reaction $1 + \gamma \rightarrow 2 + 3$, and so on [Fig.1].

Fig.1

We introduce, as usual, the quantities

$$\begin{aligned} s_1 &= -(p_2 + p_3)^2 \\ s_2 &= -(p_1 + p_3)^2 \\ s_3 &= -(p_1 + p_2)^2 ; \end{aligned} \quad (1)$$

on the energy shell $s_1 + s_2 + s_3 = 3\mu^2$.

From Lorentz invariance and gauge invariance the matrix element has the form ⁹⁾ :

$$\frac{1}{2i} \epsilon_{\lambda\mu\nu\rho} \frac{p_{1\lambda} p_{2\nu} p_{3\rho} e_\mu}{4 [p_{10} p_{20} p_{30} k_0]^{\frac{1}{2}}} F(s_1, s_2, s_3) ,$$

where e is the photon polarization.

^{*}) It is useless, as will become clear later, to indicate the charge indices of the pions.

Application of G-invariance shows that only the isoscalar part of the photon contributes. Since the $T=0$ three pion state is completely anti-symmetric in isospace, it follows from the boson character of the pions that the function $F(s_1 s_2 s_3)$ is completely symmetrical. In the reaction $\gamma + \pi \rightarrow \pi + \pi$, the two final pions are in an isospin state $T=1$; their relative orbital angular momentum is odd.

We now consider channel one (the crossed reactions are exactly the same due to the symmetries of F). Then the invariants can be written in the centre of mass system :

$$\begin{aligned} s_1 &= (\omega_p + p)^2 = 4\omega_q^2 \\ s_2 &= \mu^2 - 2p\omega_q + 2pq \cos\theta, \\ s_3 &= \mu^2 - 2p\omega_q - 2pq \cos\theta, \end{aligned} \quad (3)$$

where p is the photon momentum, q is the relative momentum of the final pions, and θ the scattering angle.

The matrix element becomes

$$A_{\underline{e}}(q, \Omega_q) = \frac{(\underline{e}, \underline{p}, \underline{q})}{4\sqrt{p\omega_p}} F(s_1, s_2, s_3) \quad (4)$$

III. Mandelstam representation.

From G-conjugation the intermediate states containing an odd number of pions are forbidden, as in the pion pion scattering. In particular the one pion state is missing so that there is no pole in the scattering amplitude.

We now assume for the scalar function $F(s_1 s_2 s_3)$ a Mandelstam representation, containing a single weight function, due to the above symmetry properties :

$$F(s_1 s_2 s_3) = f(s_1 s_2) + f(s_2 s_3) + f(s_3 s_1) \quad (5)$$

4.

where

$$f(s_1, s_2) = \frac{1}{\pi} \int_{(2\mu)^2}^{\infty} \int_{(2\mu)^2}^{\infty} \frac{\rho(\alpha, \beta) d\alpha d\beta}{(\alpha - s_1)(\beta - s_2)}$$

The questions of convergence of the integrals are disregarded for the moment.

Actually, as was noticed quite generally by Cini and Fubini ⁸⁾, the two variables α and β cannot reach simultaneously their lower limit, corresponding to the two pion state. More precisely, if s_1 lies between $(2\mu)^2$ and $(4\mu)^2$, the only possible intermediate state in channel I is a two pion state -- the three pion state being forbidden [Fig.2].

Fig.2

Then the minimum value of s_2 can be obtained by looking at channel II. It is easily seen that the lowest intermediate state contains at least four pions [Fig.3]

Fig.3

Therefore

$$\begin{aligned}
 f(s_1, s_2) &= \frac{1}{\pi} \int_{(2\mu)^2}^{(4\mu)^2} d\alpha \int_{(4\mu)^2}^{\infty} d\beta \frac{\rho(\alpha, \beta)}{(\alpha - s_1)(\beta - s_2)} \\
 &+ \frac{1}{\pi} \int_{(4\mu)^2}^{\infty} d\alpha \int_{(2\mu)^2}^{(4\mu)^2} d\beta \frac{\rho(\alpha, \beta)}{(\alpha - s_1)(\beta - s_2)} \\
 &+ \frac{1}{\pi} \int_{(4\mu)^2}^{\infty} d\alpha \int_{(4\mu)^2}^{\infty} d\beta \frac{\rho(\alpha, \beta)}{(\alpha - s_1)(\beta - s_2)}
 \end{aligned}$$

The first term of the right hand side has a strong dependence on s_1 and a weak dependence on s_2 . The second term has a strong dependence on s_2 and a weak dependence on s_1 . The last term has a weak dependence on both variables. Combining the three functions f one can rewrite (5) as :

$$F(s_1, s_2, s_3) = g(s_1, Z_1) + g(s_2, Z_2) + g(s_3, Z_3)$$

where

$$Z_1 = s_2 - s_3; \quad Z_2 = s_3 - s_1; \quad Z_3 = s_1 - s_2.$$

$g(s, Z)$ has a strong dependence on s and a weak dependence on Z ; it is an even function of Z because F is completely symmetrical with respect to s_1, s_2, s_3 .

We now neglect the Z dependence; the error is of the order Z^2 *). Then (5) becomes :

$$F(s_1, s_2, s_3) = \frac{1}{\pi} \int_{(2\mu)^2}^{\infty} \frac{\rho(s') ds'}{s' - s_1} + \frac{1}{\pi} \int_{(2\mu)^2}^{\infty} \frac{\rho(s') ds'}{s' - s_2} + \frac{1}{\pi} \int_{(2\mu)^2}^{\infty} \frac{\rho(s') ds'}{s' - s_3} \quad (6)$$

*) If it turned out that the $J=3$ states are important this would not be permissible.

In the case of the pion pion interaction, this treatment has been shown by Cini and Fubini ³⁾ to be completely equivalent to the Chew Mandelstam approach ²⁾ in which the corresponding approximations are made later.

We provisionally assume that the integrals appearing in (6) are convergent; actually we shall see later that the final results are equivalent, with or without subtraction.

IV. Derivation of the basic integral equation.

In order to apply the unitarity condition to the lowest angular momentum amplitude, we have to extract the P wave part of $A_e(q, \Omega_q)$. In the frame given in Fig.4 angles θ and ϕ are defined. Then $(\underline{e}, \underline{p}, \underline{q}) = pq \sin\theta \cos\phi$ represents a P wave with axis $\underline{\exp}$. The P wave part of the matrix element has the same axis.

Fig.4

With the help of the form (6), we can write the scattering amplitude :

$$A_e(q, \Omega_q) = \frac{pq \sin\theta \cos\phi}{4 \sqrt{p\omega_p}} \left\{ \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{\rho(s) ds}{s - 4q^2 - 4\mu^2 - i\varepsilon} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{\rho(s) ds}{s - \mu^2 + 2p\omega_q - 2pq \cos\theta} \right. \\ \left. + \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{\rho(s) ds}{s - \mu^2 + 2p\omega_q + 2pq \cos\theta} \right\} \quad (7)$$

We define the P wave amplitude as :

$$A(q^2) = \frac{3}{4\pi} \int A_e(q, \Omega_q) \sin \theta \cos \phi d\Omega \quad (8)$$

and obtain the following representation for $A(q^2)$:

$$A(q^2) = \frac{pq}{4\sqrt{p\omega_q}} B(q^2)$$

with $B(q^2)$ given by

$$B(q^2) = \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{\rho(s)ds}{s-4q^2-4\mu^2-i\varepsilon} + \frac{3}{4\pi} \int_{-1}^{+1} \sin^2 \theta d \cos \theta \quad (9)$$

$$\int_{4\mu^2}^{\infty} \rho(s)ds \left\{ \frac{1}{s-\mu^2+2p\omega_q-2pq \cos \theta} + \frac{1}{s-\mu^2+2p\omega_q+2pq \cos \theta} \right\}$$

The denominators of the second and third integrals in the r.h.s. can never vanish in the physical region for channel I. The only contribution to the imaginary part of $B(q^2)$ comes from the first term

$$\text{Im } B(q^2) = \rho(4q^2 + 4\mu^2)$$

We can then replace the weight function ρ by the imaginary part of B .

$$B(y) = \frac{1}{\pi} \int_0^{\infty} \frac{\text{Im } B(y') dy'}{y'-y-i\varepsilon} + \frac{3}{4\pi} \int_{-1}^{+1} \sin^2 \theta d \cos \theta \int_0^{\infty} \text{Im } B(y') dy' \times \quad (10)$$

$$\times \frac{2y' + y + \frac{9}{4}}{\left[y' + \frac{y+9}{2+8}\right]^2 - \frac{y}{y+1} \left(\frac{y+3}{2+8}\right)^2 \cos^2 \theta}$$

where the following change of variables has been made :

$$\frac{s}{4} - 1 = y' \quad q^2 = y ;$$

the pion mass μ has been taken to be unity.

We now have to apply the unitarity requirement. For this purpose we use the relation

$$\text{Im } T = T T^+$$

In terms of matrix elements this becomes :

$$\text{Im } \langle \gamma \pi | T | \pi \pi \rangle = \sum_{\substack{E_n = E \\ \pi \pi}} \langle \gamma \pi | T | n \rangle \langle n | T^+ | \pi \pi \rangle$$

When s_1 is less than $(4\nu)^2$, the only intermediate state contributing is a two pion state (the $\gamma \pi$ state is disregarded because it corresponds to higher order in the electromagnetic coupling constant); this state has the same quantum numbers as the final two pion state : $J=1$ $T=1$. The unitarity condition takes the simple form :

$$\text{Im } B(\nu) = B(\nu) h^*(\nu) \quad (11)$$

where $h(\nu) = e^{i\delta(\nu)} \sin \delta(\nu)$ is the amplitude for the pion pion scattering in the $T=1$ $J=1$ state. When $s_1 = 4\nu + 4$ is larger than 16, this relation is no longer rigorous but we maintain it because there is no way of taking into account properly the four pion states.

Equation (10) becomes a homogeneous Mushkelishvili - Omnès integral equation ¹⁰⁾ for the quantity $B(\nu)$:

$$B(\nu) = \frac{1}{\pi} \int_0^\infty \frac{B(\nu') h^*(\nu') d\nu'}{\nu' - \nu - i\varepsilon} + \frac{3}{4\pi} \int_{-1}^{+1} \sin^2 \Theta d \cos \Theta \int_0^\infty B(\nu') h^*(\nu') d\nu' \times$$

$$\times \frac{2\nu' + \nu + \frac{9}{4}}{\left(\nu' + \frac{\nu+9}{2}\right)^2 - \frac{\nu}{\nu+1} \left(\frac{\nu+3}{2}\right)^2 \cos^2 \Theta} \quad (12)$$

V. Reduction of the integral equation to a Fredholm type equation.

We want to put equation (12) in the form :

$$B(\nu) = \frac{1}{\pi} \int_0^{\infty} \frac{B(\nu') h^*(\nu') d\nu'}{\nu' - \nu - i\epsilon} + \frac{1}{\pi} \int_{-\infty}^{-a} \frac{c(z) dz}{\nu - z} \quad (13)$$

Then it is straightforward to transform equation (12) into a Fredholm equation.

The problem is to determine the left hand cut and the discontinuity across this cut of the function $B(\nu)$. The main difficulty of this transformation is due to kinematical complications coming from the presence of unequal masses.

Let us study the kernel $K(\nu, \nu')$ defined as the integral over $\cos\theta$ in the second term of the right hand side of (12) :

$$K(\nu, \nu') = \frac{6}{\pi} \int_0^{+1} (1-u^2) \frac{(\nu+1) (\nu+2\nu'+\frac{9}{4})}{(\nu+1) \left[\nu+2\nu'+\frac{9}{4} \right]^2 - \nu(\nu+\frac{3}{4})^2 u^2} du \quad (14)$$

It will be shown that the denominator $D(\nu, \nu', u^2)$ of the above integrand has three real negative roots with respect to ν . This enables us to define a left hand cut. These roots are functions of ν' and u^2 . The integrand will be decomposed in partial fractions with respect to ν and put in the form

$$\sum_{i=1,2,3} \frac{\beta_i(\nu', u^2)}{\nu - \alpha_i(\nu', u^2)} \quad (15)$$

Then α_i will be taken as a new variable of integration instead of u . It is not necessary to know the roots α_i of D but rather the expression of u in terms of α_i , and this turns out to be very simple, because D is linear in u^2 . The existence and location of the roots are quite trivially established by the following table :

$$D(-1, \nu', u^2) = \frac{u^2}{16} > 0$$

$$D(-\frac{9}{8}, \nu', u^2) = -\frac{\nu'}{2} (\nu' + \frac{9}{8}) - (\frac{3}{8})^3 (1-u^2) < 0$$

$$D(-2\nu' - \frac{9}{4}, \nu', u^2) = (3\nu' + 9) (\nu' + \frac{3}{4})^2 u^2 > 0$$

$$D(-\infty, \nu', u^2) < 0$$

$$-\infty \leq \alpha_3 \leq -2\gamma' - \frac{9}{4} \leq \alpha_2 \leq -\frac{9}{8} \leq \alpha_1 \leq -1 \quad (16)$$

For given γ' it can be shown that the three roots $\alpha_i(\gamma', u^2)$ are monotonous functions of u^2 . The limiting values are :

$$\begin{array}{ll} u = 0 & u = 1 \\ \alpha_1 = 1 & \alpha_m(\gamma') \geq -\frac{9}{8} \\ \alpha_2 = -2\gamma' - \frac{9}{4} & \alpha_M(\gamma') \leq -\frac{9}{8} \\ \alpha_3 = -2\gamma' - \frac{9}{4} & -\infty \end{array}$$

$\alpha_m(\gamma')$ and $\alpha_M(\gamma')$ are roots of $D(\gamma, \gamma', 1) = 0$, which in this special case becomes a second degree equation. In the limit $\gamma' = 0$

$$\alpha_m(0) = \alpha_M(0) = -\frac{9}{8}.$$

This means that when u and γ' vary in the whole physical domain one obtains a single continuous cut from $-\infty$ to -1 .

In each term of the decomposition (15) we take $\alpha_i = z$ as a new variable. Since for γ' fixed the ranges of variation of $\alpha_1, \alpha_2, \alpha_3$ do not overlap, the integrand is reduced to a single term :

$$\frac{1}{2} \frac{(z+1)}{(z+\frac{3}{4})} \frac{z(z+\frac{3}{4})^2 - (z+1)(z+2\gamma' + \frac{9}{4})^2}{(z+\frac{3}{4}) \left(z^2 + \frac{z}{2} - \gamma' - \frac{9}{8} \right) - z(z+1) \left(z+2\gamma' + \frac{9}{4} \right)}$$

We now study the Jacobian of the transformation

$$\frac{du}{dz} = \frac{1}{2\sqrt{u^2}} \frac{d(u^2)}{dz}$$

with

$$\sqrt{u^2} = \sqrt{\frac{z+1}{z}} \frac{z+2\gamma' + \frac{9}{4}}{z+\frac{3}{4}} \in$$

for $z > -2\gamma' - \frac{9}{4}$, i.e. $z = \alpha_1$ or α_2 , $\in = -1$; for $z = \alpha_3$ $\in = +1$.

Therefore

$$K(\nu, \nu') = \frac{1}{\pi} \left[\int_{-\infty}^{-2\nu' - \frac{9}{4}} + \int_{-2\nu' - \frac{9}{4}}^{\alpha_M(\nu')} - \int_{\alpha_m(\nu')}^{-1} \right] dz \frac{A(\nu', z)}{\nu' - z} \quad (17)$$

$A(\nu', z)$ turns out to be given by :

$$A(\nu', z) = 12(\nu' + 1) \frac{[z - \alpha_M(\nu')][z - \alpha_m(\nu')]}{z(z + \frac{3}{4})^3} \sqrt{\frac{z+1}{z}} \quad (18)$$

From this expression one can get equation (12) in the form (13) putting in evidence the discontinuity across the left hand cut :

$$C(z) = -\Xi(z + \frac{9}{8}) \int_0^{\nu'_M(z)} B(\nu') h^*(\nu') A(\nu', z) d\nu' \quad (19)$$

where $\Xi(x) = x/|x|$ and the upper limit of integration is related to ν' by :

$$\nu'_M(z) = -\frac{9}{8} - \frac{z}{2} - \frac{1}{2} \left(z + \frac{3}{4} \right) \sqrt{\frac{z}{z+1}}$$

Fig. 5 shows the domain of variation of z and ν' for physical values of ν .

The domain of integration over ν' and z .

Fig.5

The solution of equation (13) reads ¹⁰⁾ :

$$B(\nu) = B_H(\nu) + \frac{e^{\rho(\nu)+i\delta(\nu)}}{\pi} \int_{-\infty}^{-1} \frac{C(z) e^{-\rho(z)} dz}{\nu-z} \quad (20)$$

where

$$\rho(\nu) = \frac{P}{\pi} \nu \int_0^{\infty} \frac{\delta(\nu') d\nu'}{\nu'(\nu'-\nu)} \quad (21)$$

and $B_H(\nu)$ obeys the equation :

$$B_H(\nu) = \frac{1}{\pi} \int_0^{\infty} \frac{B_H(\nu') h^*(\nu') d\nu'}{\nu' - \nu - i\varepsilon} \quad (22)$$

The possible solutions of (22) are

$$B_H(\nu) = \lambda \nu^m e^{\rho+i\delta} \quad (23)$$

where m is an algebraic integer ¹⁰⁾. If one assumes for the $\pi\pi$ $J=1$ $T=1$ phase shift the behaviour $\delta(0)=0$ $\delta(\infty)=\pi$, which corresponds to a single resonance, and furthermore that $\delta(\nu)-\pi$ goes to zero fast enough as ν goes to infinity, one sees that $e^{\rho(\nu)}$ behaves like $1/\nu$ at infinity. Since from (22) $B_H(\nu)$ behaves also like $1/\nu$ at infinity, the only acceptable solution is

$$B_H(\nu) = \lambda e^{\rho+i\delta} \quad (23')$$

Hence (12) is finally transformed into the following Fredholm equation

$$B(\nu) = e^{\rho(\nu)+i\delta(\nu)} \left[\lambda + \int_0^{\infty} N(\nu, \nu') h^*(\nu') B(\nu') d\nu' \right] \quad (24)$$

where

$$N(\nu, \nu') = \frac{1}{\lambda} \left[\int_{-\infty}^{\alpha_M(\nu')} - \int_{\alpha_m(\nu')}^{-1} \right] dz \frac{A(\nu', z) e^{-\rho(z)}}{\nu - z} \quad (25)$$

The constant λ is an arbitrary parameter which enters as a multiplying factor in the final solution. There is no way in this approach to relate this factor to other fundamental constants, in particular to the electromagnetic coupling constant which, in fact, certainly plays a fundamental role in the detailed mechanism of the interaction as can be seen by inspection of Feynman graphs. Therefore a direct confrontation of the present calculation with experiment cannot be made. One has either to compare two measurable processes where the $\gamma + \pi \rightarrow \pi + \pi$ reaction plays a role, for instance pion photoproduction on nucleons and Compton scattering on nucleons or to use the Chew-Low extrapolation method ¹¹⁾. This method, which would give an absolute measurement of the unknown parameter can be applied to two cases :

- a) photoproduction of two pions on a nucleon ¹²⁾ [Fig.6(a)];
- b) production of a pion by a pion in the Coulomb field of a heavy nucleus [Fig.6(b)].

Fig.6

A possible way of avoiding a difficult extrapolation in process b) consists in using various targets and putting in evidence a Z^2 effect in the cross section. This method, which is also suggested by B. Ferretti for the reaction $\gamma + \pi \rightarrow \pi + \pi + \pi$, is intended to be applied by O. Piccioni ¹³⁾.

Another equivalent way of introducing multiplicative parameter λ is to make one subtraction in the Mandelstam representation at an arbitrary reference point. The two methods give identical results.

VI. Approximate solutions.

In this section we present approximate solutions under the assumption that the $J = 1 \quad T = 1$ resonance is dominant in the $\pi\pi$ -scattering amplitude $h(\nu)$. Since nothing is really experimentally known about this scattering amplitude we do not think it is worthwhile to work out a rigorous solution of integral equation (24) which would probably necessitate machine computation. We only hope to reproduce general features of the matrix element and to see how much the pion pion resonance is felt in this reaction. Frazer and Fulco ⁴⁾ have estimated the effect of this resonance on the nucleon form factor. We use a resonance energy which is approximately the same, i.e. $\nu_R = 4$ *).

a) Approximation " $\cos\theta = 0$ ".

If we go back to equation (12) and look at the denominator of the second integral in the right hand side, we see that for $\nu = \nu_R = 4$ the relative coefficient of the $\cos^2\theta$ term is at most .45; on the other hand a weighing factor $\sin^2\theta$ in front of the whole integrand strongly favours $\cos^2\theta = 0$. It can be checked by analytic calculation of the kernel $K(\nu, \nu')$ that this is not a so bad approximation. Such an approximation has been used by Cini, Fubini and Stanghellini ¹⁴⁾ in the fixed angle nucleon nucleon dispersion relations problem and gives results in good agreement with the more rigorous calculations of Noyes ¹⁵⁾.

*) This corresponds to the quantity produced by Frazer and Fulco in their first paper. Calculations with the new value will be soon made.

Equation (12) becomes

$$B_a(\nu) = \frac{1}{\pi} \int_0^\infty \frac{B_a(\nu') h^*(\nu') d\nu'}{\nu' - \nu - i\varepsilon} + \frac{4}{\pi} \int_0^\infty \frac{B_a(\nu') h^*(\nu') d\nu'}{2\nu' + \nu + \frac{9}{4}} \quad (26)$$

It is readily transformed into the Fredholm equation

$$B_a(\nu) = e^{\rho(\nu) + i\delta(\nu)} \left[\lambda + \frac{4}{\pi} \int_0^\infty \frac{B_a(\nu') h^*(\nu') e^{-\rho(-2\nu' - \frac{9}{4})} d\nu'}{2\nu' + \nu + \frac{9}{4}} \right] \quad (27)$$

It is convenient to make the following change of function

$$B_a(\nu) = e^{\rho(\nu) + i\delta(\nu)} \phi(\nu);$$

then

$$\phi(\nu) = \lambda + \frac{4}{\pi} \int_0^\infty \frac{\phi(\nu') \sin \delta(\nu') e^{[\rho(\nu') - \rho(-2\nu' - \frac{9}{4})]} d\nu'}{2\nu' + \nu + \frac{9}{4}} \quad (28)$$

This equation is easier to solve, because of more rapid convergence of the integral, in subtracted form :

$$\frac{\phi(\nu)}{\nu + \frac{3}{4}} \equiv F(\nu) = \frac{\phi(-\frac{3}{4})}{\nu + \frac{3}{4}} - \frac{2}{\pi} \int_0^\infty \frac{\sin \delta(\nu') e^{[\rho(\nu') - \rho(-2\nu' - \frac{9}{4})]} F(\nu') d\nu'}{2\nu' + \nu + \frac{9}{4}} \quad (29)$$

($\nu = -\frac{3}{4}$ corresponds to $s_1 = s_2 = s_3 = \mu^2$).

If we first make the approximation that the width of the resonance is extremely small, equation (29) can be put in a form which has an exact solution :

$$F(\nu) = \frac{\phi(-\frac{3}{4})}{\nu + \frac{3}{4}} - \frac{2}{\pi} \frac{e^{-\rho(-2\nu_R - \frac{9}{4})}}{\nu + 2\nu_R + \frac{9}{4}} \int_0^\infty \sin \delta(\nu') e^{\rho(\nu')} F(\nu') d\nu' \quad (30)$$

making use of the relation $\frac{1}{\pi} \int_0^{\infty} \frac{\sin \delta(\nu) e^{\rho(\nu)} d\nu}{\nu + \nu_0} = e^{\rho(-\nu_0)}$, derived in the Appendix, we readily get the solution

$$F(\nu) = \phi(-\frac{3}{4}) \left[\frac{1}{\nu + \frac{3}{4}} - \frac{2}{3} \frac{e^{\rho(-\frac{3}{4})} - e^{\rho(-2\nu_R - \frac{9}{4})}}{\nu + 2\nu_R + \frac{9}{4}} \right] \quad (31)$$

In the case of zero width, i.e. $\delta(\nu) = \pi \theta(\nu - \nu_R)$ [where $\theta(x)$ is the step function] $e^{-\rho(z)}$, for negative values of z is given by

$$e^{-\rho(z)} = \frac{\nu_R - z}{\nu_R} \quad (32)$$

Then $\phi(\nu)$ becomes :

$$\phi(\nu) = \phi(-\frac{3}{4}) \frac{3 + 8\nu_R + 4\nu}{9 + 8\nu_R + 4\nu} \quad (33)$$

and the P wave amplitude becomes

$$A(\nu) = \frac{1}{8} \sqrt{\frac{\nu(4\nu+3)}{\nu+1}} \exp[\rho(\nu) + i\delta(\nu)] \phi(-\frac{3}{4}) \frac{3 + 8\nu_R - 4\nu}{9 + 8\nu_R + 4\nu} \quad (33')$$

This amplitude is dominated by the $\pi\pi$ resonance since the factor $\exp \rho(\nu)$ has a very sharp maximum at $\nu = \nu_R$, but, due to the properties of $\phi(\nu)$, it vanishes at an energy $\nu = \frac{3}{4} + 2\nu_R$, i.e. about 8.75 for $\nu_R \sim 4$; this is a fairly large energy where the Mandelstam treatment might break down and one should not consider this feature of the solution too seriously, but we believe that an exact treatment of the integral equation will qualitatively give the same result because the approximation " $\cos\theta = 0$ " slightly underestimates the kernel of the Fredholm equation.

Keeping the same analytical form (31) with undetermined coefficients we have tried to solve equation (29) for a non-zero width. We have constructed a P wave amplitude, characterized by a pole at $\nu = 3.75 - 2i$ in the second Riemann sheet corresponding to a resonance (see Appendix). It turns out that the trial form of the solution is very suitable. The numerical result is :

$$\phi(\nu) = \phi(-\frac{3}{4}) \frac{56.4 - 4\nu}{65.4 + 8\nu}$$

This should be compared to :

- (i) the case of zero width which gives a factor

$$\frac{35 - 4\nu}{41 + 4\nu}$$

- (ii) application of formula (31), taking the $\exp(\phi(z))$ corresponding to the actual width :

$$\frac{51 - 4\nu}{58 + 5.7\nu}$$

All these solutions have the same qualitative behaviour. As could be expected formula (31) gives an intermediate result between the exact solution and the limiting case of zero width.

b) Possibility of more refined calculations.

One of the difficulties of the present calculation is that when the previous approximation is abandoned, the kernel $N(\nu, \nu')$ is given as an integral over z [Eq. (25)]. For special forms of the factor $\exp[-\rho(z)]$ the integral can be performed.

We have already given in (32) the very simple form of this factor for the case of zero width. More generally we have checked on the case of non zero width considered in VI.a) and in the Appendix that a linear form

$$\exp[-\rho(z)] = 1 - z/\alpha$$

can be used for $-20 < z < 0$.

Let us compare the integrands of $K(\gamma, \gamma')$ and $N(\gamma, \gamma')$ which are then respectively

$$\frac{A(\gamma', z)}{\gamma - z} \quad \text{and} \quad \frac{A(\gamma', z)}{\gamma - z} \times \frac{\alpha - z}{\alpha}$$

the last one can be written

$$\lim_{\gamma \rightarrow \infty} \left[\frac{\gamma A(\gamma', z)}{\alpha(\gamma - z)} \right] + \frac{\alpha - \gamma}{\alpha} \frac{A(\gamma', z)}{\gamma - z},$$

so that the connection between K and N is just

$$\alpha N(\gamma, \gamma') = \lim_{\gamma \rightarrow \infty} [\gamma K(\gamma, \gamma')] + (\alpha - \gamma) K(\gamma, \gamma') \quad (34)$$

K can be directly computed in the form (14) :

$$K(\gamma, \gamma') = \frac{3}{\pi} \left[2 \frac{(\gamma+1)(\gamma+2\gamma'+\frac{9}{4})}{\gamma(\gamma+\frac{3}{4})^2} - \sqrt{\frac{\gamma+1}{\gamma}} \frac{(\gamma+1)(\gamma+2\gamma'+\frac{9}{4})^2 - \gamma(\gamma+\frac{3}{4})^2}{\gamma(\gamma+\frac{3}{4})^3} \log \frac{\sqrt{\gamma+1}(\gamma+2\gamma'+\frac{9}{4}) + \sqrt{\gamma}(\gamma+\frac{3}{4})}{\sqrt{\gamma+1}(\gamma+2\gamma'+\frac{9}{4}) - \sqrt{\gamma}(\gamma+\frac{3}{4})} \right] \quad (35)$$

then since $\lim_{\gamma \rightarrow \infty} (\gamma K(\gamma, \gamma'))$ is independent of γ' , one can write the integral equation as :

$$B(\gamma) = B(\alpha) + \frac{\alpha - \gamma}{\alpha} \int_0^{\infty} K(\gamma, \gamma') h^*(\gamma') B(\gamma') d\gamma' . \quad (36)$$

VII. Concluding remarks.

Starting from the Mandelstam representation of the matrix element for photoproduction of pions on pions, we have derived an integral equation for the $J = 1$ reaction amplitude. Under the assumption of a sharp resonance in the $J = 1 \quad T = 1 \quad \pi - \pi$ scattering, we have given explicit approximate forms of the solution which exhibit the same resonant behaviour. By inserting in the Mandelstam representation this result one could estimate the $J = 3$ state contribution and see if the approximation made in going from a two dimensional representation to a sum of unidimensional representations is justified.

We have already noticed that the solution contains an arbitrary multiplicative constant which cannot be determined in a theoretical way in the present approach. On the other hand, the validity of this treatment certainly breaks down for total c.m. energies appreciably larger than 4 .

The knowledge of this amplitude and of the pion pion amplitude permits to determine explicitly, within the same framework, the Compton effect on pions ^{*)}. This amplitude is also needed in order to apply the Mandelstam treatment to photopion production on nucleons and, then, to Compton scattering on nucleons.

Our calculation also indicates a possible enhancement of the three pion contribution ¹²⁾ to the isoscalar part of the nucleon form factors; this is far from certain since here we are dealing with a real photon.

We have indicated a way of determining the unknown factor in the solution by extrapolation of the cross-section for double photoproduction; this experiment is certainly very difficult since all the kinematics must be obtained from the final products of the reaction, the energy of the photon being unknown. Another way would be to extrapolate data on single pion production by a pion in the Coulomb field of a heavy nucleus; but then the recoil is difficult to measure; it is better ¹³⁾ to use both a Z^2 effect and angular correlation effects to isolate this phenomenon.

^{*)} This result will be published later.

Acknowledgements.

We wish to thank Professors M. Cini and S. Fubini for many useful suggestions during the course of this work and Professor B. Ferretti for a fruitful discussion. Mr. W. Klein is acknowledged for high speed hand computations.

One of us (M.G.) would like to thank Professors C.J. Bakker and M. Fierz for hospitality at the CERN Theoretical Studies Division and French C.N.R.S. for financial support.

* * * * *

APPENDIX

We assume a $J = 1$ $T = 1$ monotonic phase shift, with a single resonance, such that

$$\begin{aligned} \delta(\nu) &\sim \text{ct. } \nu^{3/2} \text{ for } \nu \rightarrow 0 \\ |\pi - \delta(\nu)| &< \text{ct. } \nu^{-1/2} \text{ for } \nu \rightarrow +\infty \end{aligned}$$

$$\text{then } \rho(\nu) = \nu \frac{P}{\pi} \int_0^\infty \frac{\delta(\nu') d\nu'}{\nu'(\nu' - \nu)}$$

is such that

$$\rho(\nu) \simeq \nu \frac{1}{\pi} \int_0^\infty \frac{\delta(\nu') d\nu'}{\nu'} \text{ for } \nu \rightarrow 0$$

$$\rho(\nu) \simeq \text{ct. } -\log \nu \text{ for } \nu \rightarrow \infty$$

The quantity

$$\int_0^\infty \frac{\rho(\nu) \sin \delta(\nu) d\nu}{\nu + \nu_0} = \frac{1}{2i} \int_0^\infty \frac{\exp[\rho(\nu) + i\delta(\nu)] - \exp[\rho(\nu) - i\delta(\nu)]}{\nu + \nu_0} d\nu$$

can be evaluated by contour integration (fig. A1)

of the function $\frac{1}{z+z_0} \exp \left[\frac{i}{\pi} \int_0^\infty \frac{\delta(\nu')}{(\nu' - z)} d\nu' \right]$

Fig. A1

In the special case of zero width resonance at $\nu = \nu_R$
 $\exp \rho(\nu) = \left| \frac{\nu_R}{\nu_R - \nu} \right|$, and then

$$\int_0^{\infty} \frac{\rho(\nu) \sin \delta(\nu)}{\nu + \nu_0} d\nu \rightarrow \pi \frac{\nu_R}{\nu_R + \nu_0}$$

and

$$\int_0^{\infty} \rho(\nu) \sin \delta(\nu) d\nu \rightarrow \pi \nu_R$$

We have built a model of scattering amplitude with non-zero width of the resonance such that $\exp(\rho)$ could be evaluated easily. It turns out that the minimum number of poles needed in the S matrix is four if one wants to reproduce the features indicated at the beginning of the appendix. One gets

$$e^{i\delta} \sin \delta = \frac{[\Gamma^2 + K_1 - K_0][\nu]}{(K_0 + i\sqrt{\nu})(K_1 - i\sqrt{\nu})[(\Gamma - i\sqrt{\nu})^2 + \nu_R]}^{3/2}$$

With the conditions

$$2\Gamma K_0 K_1 = (K_1 - K_0)(\Gamma^2 + \nu_R),$$

$$K_1 > K_0 > 0.$$

The only pole of $e^{i\delta} \sin \delta$ in the first Riemann sheet is $\nu = -K_0^2$ it has no zeros except at $\nu = 0$. Therefore the quantity

$(z)^{3/2} \exp \frac{z}{\pi} \int_0^{\infty} \frac{\delta(\nu) d\nu}{\nu(\nu - z)} \Big/ [z + K_0^2] e^{i\delta(z)} \delta(z)$ is regular in the whole z plane, real along the cut $0 - \infty$, and has a finite non zero limit as $|z| \rightarrow \infty$. It is therefore a real constant.

Finally

$$e^{\rho(\nu)} = \frac{K_1}{K_0} (\Gamma^2 + \nu_R) \sqrt{\frac{K_0^2 + \nu}{(K_1^2 + \nu)[(\Gamma^2 + \nu_R - \nu)^2 + 4\nu\Gamma^2]}} \quad \text{for } \nu > 0$$

$$e^{\rho(\nu)} = \frac{K_1}{K_0} (\Gamma^2 + \nu_R) \frac{K_0 + \sqrt{-\nu}}{(K_1 + \sqrt{-\nu})(\Gamma^2 + \nu_R - \nu + 2\Gamma\sqrt{-\nu})} \quad \text{for } \nu < 0$$

The parameters used in VI a) are:

$$\nu_R = 4 \quad \Gamma = 0.5 \quad K_0 = 1.755 \quad K_1 = 2.995$$

This corresponds approximately to the case

$$\Gamma = 0.3 \quad \nu_R = 3.5 \text{ in Frazer and Fulco's notations } ^4).$$

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