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Pion photon scattering

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ABSTRACT

Combining the assumption of a Mandelstam representation with approximate unitarity one can get the pion photon scattering amplitude in terms of the pion-pion scattering amplitude in even states and of the matrix element for photoproduction of pion on pions. In solving this problem attention must be paid to the fact that in the low energy limit this amplitude should become equal to the classical amplitude. With some simplifying assumptions we obtain an expression which contains four quantities : the two-pion S-phase shifts, the total cross-section for photoproduction of pions on pions, and the electromagnetic coupling constant. Numerical estimates will be given later.

The interest of studying this process lies in its possible influence on the nucleon-photon scattering.

I. Introduction

In a preceding paper¹⁾ the authors have studied the reaction $\gamma + \pi \rightarrow \pi + \pi$, using the Mandelstamm representation and assuming a pion-pion resonance in the $T = 1$ $J = 1$ pion-pion state. From this reaction amplitude and the pion-pion scattering amplitude in even angular momentum states one can get the amplitude for pion-photon scattering, using again the Mandelstamm representation, and neglecting, in the application of unitarity, all the processes other than two-body two-body.

The pion-photon scattering amplitude cannot be directly confronted with experiment, but it plays a role in the nucleon-photon scattering, when the photon is scattered by the meson cloud of the nucleon. More precisely, when one tries to apply the Mandelstamm technique to the process $\gamma + N \rightarrow \gamma + N$, one has to know the imaginary part of the reaction amplitude in the channel $\gamma + \gamma \rightarrow N + \bar{N}$; the lowest intermediate state is, apart from the π_0 giving rise to a pole, the two-pion state. This is the main motivation of the present work. On the other hand, the present problem constitutes a good illustration of the Mandelstamm technique where an explicit solution of the equations can be given.

In the body of the paper several simplifying assumptions have been made, but we show in the appendices how one can solve the problems in the general case. First we only retain the $J = 1$ part of the $\gamma + \pi \rightarrow \pi + \pi$ amplitude, neglecting contributions from $J \geq 3$; second we neglect the D waves and higher waves in pion-pion scattering, because this is essentially a low energy treatment, and, due to the range of the forces, these waves are expected to be very small. Then it is possible to express the $\gamma - \pi$ scattering amplitude in terms of three quantities :

2.

- the electromagnetic coupling constant which enters as a multiplying factor in the low energy limit of the charged scattering amplitude (this limit has to coincide with the Thompson amplitude);
- the total cross-section σ_T for the process $\gamma + \pi \rightarrow \pi + \pi$;
- the two S phases of pion-pion scattering in $T = 0$ and $T = 2$ isospin states.

In the present paper we make no attempt to replace these quantities by numbers. We feel that for the moment it is not really worthwhile. One could in principle use the figures proposed by Frazer and Fulco²⁾ in the $T = 1 \quad J = 1$ pion-pion state, which gives, up to a factor, σ_T , and try to estimate the $T = 0, 2 \quad J = 0$ pion-pion amplitudes by inserting the $T = 1 \quad J = 1$ data in the Chew Mandelstam³⁾ or Cini and Fubini⁴⁾ equations for the pion-pion scattering. This will be done later.

II. Structure of the Matrix elements

We define $p_1 \quad p_2$ as the incoming four momenta of the pions, $k_1 \quad k_2$ as the incoming momenta of the photons with polarizations e_1 and e_2 .

Fig. 1

The relevant channels are indicated in Fig. 1. We introduce the three invariants :

$$\begin{aligned} s_1 &= - (k_1 + p_1)^2 = - (k_2 + p_2)^2 \\ s_2 &= - (k_1 + p_2)^2 = - (k_2 + p_1)^2 \\ t &= - (k_1 + k_2)^2 = - (p_1 + p_2)^2 \end{aligned} \quad (1)$$

with, on the mass shell :

$$s_1 + s_2 + t = 2\mu^2 \quad (2)$$

where μ is the pion mass.

Due to energy momentum conservation all the quantities can be expressed in terms of three four vectors : k_1, k_2 and $\Delta = p_2 - p_1$.

The matrix element $e_{1\mu} e_{2\nu} M_{\mu\nu}(k_1, k_2, \Delta)$ must fulfill the gauge invariance condition which can be expressed as well known by the following necessary requirements :

$$k_{1\mu} M_{\mu\nu} = 0 \quad \text{and} \quad k_{2\nu} M_{\mu\nu} = 0$$

If we retain only the terms giving non vanishing contributions we can write the matrix element Φ as

$$\Phi(e_1, e_2; k_1, k_2, \Delta) \equiv D_\alpha(s_1, s_2, t) I_\alpha + D_\beta(s_1, s_2, t) I_\beta \quad (3)$$

where the two invariants I_α and I_β are given by :

$$\begin{aligned} I_\alpha &= (e_1 \cdot e_2) - \frac{(e_1 \cdot k_2)(e_2 \cdot k_1)}{k_1 \cdot k_2} \\ I_\beta &= (e_1 \cdot \Delta)(e_2 \cdot \Delta)(k_1 \cdot k_2) - (e_1 \cdot k_2)(e_2 \cdot \Delta)(k_1 \cdot \Delta) \\ &\quad - (e_1 \cdot \Delta)(e_2 \cdot k_1)(k_2 \cdot \Delta) + \frac{(e_1 \cdot k_2)(e_2 \cdot k_1)(k_1 \cdot \Delta)(k_2 \cdot \Delta)}{(k_1 \cdot k_2)} \end{aligned} \quad (4)$$

With the above notations the S matrix element has the form

$$\langle f | S | i \rangle = \frac{1}{4 (2\pi)^4} \frac{1}{[k_1^0 k_2^0 p_1^0 p_2^0]^{\frac{1}{2}}} \oint_4 (p_1 + p_2 + k_1 + k_2) \bar{\Phi} \quad (5)$$

and we obtain for the differential cross-section $\gamma + \pi \rightarrow \gamma + \pi$ in the centre of mass system :

$$\frac{d\sigma}{d\Omega} = \frac{1}{s_1} \left| \frac{\bar{\Phi}}{8\pi} \right|^2 \quad (6)$$

We notice that I_1 and I_2 are invariant in the exchange $p_1 \leftrightarrow p_2$, i.e. $\Delta \rightarrow -\Delta$. The matrix element $\bar{\Phi}$ must be even in Δ by crossing property. It follows that D_α and D_ρ are symmetrical functions of s_1 and s_2 . Then it is easy to see that in channel III only even angular momentum states of the pion-pion system are permissible; therefore the isotopic $T = 1$ state is forbidden. Hence the two photons currents appearing in the matrix element are either both isoscalars or both isovectors.

Let us call $\bar{\Phi}_C$ and $\bar{\Phi}_N$ the matrix elements for the Compton effects on charged and neutral pions (by charge conjugation one sees that there is no difference between π^+ and π^-). The pions being in the $T = 0$ and $T = 2$ isotopic states, the corresponding reactions amplitudes are :

$$\bar{\Phi}_0 = \frac{1}{\sqrt{3}} (2\bar{\Phi}_C + \bar{\Phi}_N) = \frac{2}{\sqrt{3}} F_0$$

$$\bar{\Phi}_2 = \sqrt{\frac{2}{3}} (\bar{\Phi}_C - \bar{\Phi}_N) = \sqrt{\frac{2}{3}} F_2$$

$$\text{Conversely } \bar{\Phi}_N = \frac{2}{3} (F_0 - F_2) \text{ and } \bar{\Phi}_C = \frac{1}{3} (2F_0 + F_2).$$

Let us now compute the matrix element $\bar{\Phi}_C$ in the Born approximation. The three contributions indicated on Fig. 2 cannot be treated separately because of gauge invariance.

Fig. 2

We find for the two functions D_1 and D_2 the following contributions :

$$D_{\alpha}^B = 2 e^2 \quad D_{\beta}^B(s_1, s_2) = \frac{2 e^2}{(s_1 - \mu^2)(s_2 - \mu^2)} \quad (8)$$

where e is the rationalized electromagnetic coupling constant.

III. Kinematics in the various channels.

Because of the crossing properties one has only to consider two independent channels (See Fig.1).

III - 1 Channel I : if $\underline{k}$ and $\underline{k}'$ are momenta of the pions in the centre of mass system, the three invariants are given by :

$$\begin{aligned} s_1 &= (k + \omega_k)^2 \\ s_2 &= \mu^2 - 2 k \omega_k - 2 k^2 \cos \theta \\ t &= -2 k^2 (1 - \cos \theta) \end{aligned} \quad (9)$$

where θ is the scattering angle.

$$\begin{aligned} \text{Then} \quad k^2 &= \frac{(s_1 - \mu^2)^2}{4 s_1} \\ \cos \theta &= 1 + \frac{2 s_1 t}{(s_1 - \mu^2)^2} \end{aligned} \quad (10)$$

If we choose the gauge to eliminate longitudinal photons we get the following expressions for I_{α} and I_{β}

$$\begin{aligned} I_{\alpha} &= \underline{e}_1 \cdot \underline{e}_2 - \frac{2}{t} (\underline{e}_1 \cdot \underline{k}_1) (\underline{e}_2 \cdot \underline{k}) \\ I_{\beta} &= -\frac{2}{t} (s_1 - \mu^2)^2 (\underline{e}_1 \cdot \underline{k}') (\underline{e}_2 \cdot \underline{k}) \end{aligned} \quad (11)$$

6.

In spite of the presence of t in the denominators the invariants I_α and I_β are bounded in the limit $t \rightarrow 0$. In this limit $\underline{k} = \underline{k}'$; so $\underline{e}_1$ becomes orthogonal to $\underline{k}'$ and $\underline{e}_2$ to $\underline{k}$. In addition, it is easy to see that in the limit of zero energy, only I_α is non vanishing while I_β behaves like k^2 .

III - 2 Channel III : $\underline{p}$ and $-\underline{p}$ are the photon momenta, $\underline{q}$ and $-\underline{q}$ the pion momenta; then the invariants take the form :

$$\begin{aligned} s_1 &= - (q^2 + \omega_q^2 + 2q\omega_q \cos \varphi) \\ s_2 &= - (q^2 + \omega_q^2 - 2q\omega_q \cos \varphi) \\ t &= 4 p^2 = 4 \omega_q^2 \end{aligned} \quad (12)$$

The pion-photon angle, φ , is given by :

$$\cos \varphi = \frac{s_2 - s_1}{[t(t - 4\omega_q^2)]^{\frac{1}{2}}} \quad (13)$$

and I_α and I_β can be written for transverse photons :

$$\begin{aligned} I_\alpha &= \underline{e}_1 \cdot \underline{e}_2 \\ I_\beta &= 2 t (\underline{e}_1 \cdot \underline{q}) (\underline{e}_2 \cdot \underline{q}) \end{aligned} \quad (14)$$

IV. Mandelstam representation

We assume for $D_\alpha(s_1, s_2, t)$ and $D_\beta(s_1, s_2, t)$ a Mandelstam representation :

$$D_i(s_1, s_2, t) = D_i^B + \frac{1}{\pi} \iint \frac{\rho_i(s'_1, s'_2)}{(s'_1 - s_2)(s'_2 - s_2)} ds'_1 ds'_2 \quad (15)$$

$$i = \alpha, \beta$$

$$+ \frac{1}{\pi} \iint \frac{\sigma_i(s'_1, t')}{(s'_1 - s_1)(t' - t)} ds'_1 dt' + \frac{1}{\pi} \iint \frac{\sigma_i(s'_2, t')}{(s'_2 - s_2)(t' - t)} ds'_2 dt'$$

Let us first consider the case of two isoscalar photons. It can be easily seen that the variables of integration s'_1 and s'_2 or s' and t' cannot reach simultaneously their lower limit. More precisely if

$(2\mu)^2 \leq s'_1 \leq (4\mu)^2$ one has $s'_2 \geq (4\mu)^2$, and conversely; the same result holds for s' and t' . So the Cini and Fubini technique can be applied⁴⁾ and we introduce unidimensional representations.

The situation is not the same for isovector photons. One can just say that the integration domains are limited by

$$s'_1 \geq (3\mu)^2 \quad s'_2 \geq (3\mu)^2 \quad t' \geq (2\mu)^2$$

The weak dependence on s_1 and s_2 will be completely neglected. As will become clear later it is impossible to take this dependence into account as long as nothing is known on the reaction $\gamma + \pi \rightarrow \pi + \pi + \pi$.

The amplitude Φ_0 involves both isoscalar and isovector photons but Φ_2 corresponds only to isovector photons.

Under these assumptions one can write generally

$$D(s_1, s_2, t) = D^B + \sum_n (s_1 - \mu^2)^{2n} P_n \left[1 + \frac{2 s_1 t}{(s_1 - \mu^2)^2} \right] a^{(n)}(s_1) \quad (16)$$

$$+ \text{Sym} \left(s_1 \rightarrow s_2 \right) + \sum_n \left[t (t - 4\mu^2) \right]^n P_{2n} \left[\frac{s_2 - s_1}{[t(t - 4\mu^2)]^{\frac{1}{2}}} \right] b^{(n)}(t)$$

where P_n is the ordinary Legendre polynomial.

This corresponds to a partial wave expansion in the various channels (see formulas 10 and 13).

We assume for $a^{(n)}$ and $b^{(n)}$ a spectral representation

$$a^{(n)}(s) = \frac{1}{\pi} \int_{(2\mu)^2}^{\infty} \frac{\sigma^{(n)}(x) dx}{x-s} \quad \text{for the } T=0 \text{ amplitude}$$

$$a^{(n)}(s) = 0 \quad \text{for the } T=2 \text{ amplitude} \quad (16')$$

$$b_T^{(n)}(t) = \frac{1}{\pi} \int_{(2\mu)^2}^{\infty} \frac{\tau_T^{(n)}(y) dy}{y-t}$$

The convergence of the integrals will be studied later. The low energy limits of D_α and D_β must, in addition satisfy certain requirements⁵⁾ which will be taken into account later.

V. Unitarity in Channel I

In this section we shall only be concerned with the F_0 amplitude. In channel I, F_0 is given by :

$$F_0 = (e_1 e_2) D_{\alpha 0} + \frac{1}{k^2 (1-\cos \theta)} (e_1 k') (e_2 k) \left[D_{\alpha 0} + 4k^2 (k+\omega_k)^2 D_{\beta 0} \right] \quad (17)$$

Application of the unitarity condition enables us to determine the imaginary part of this amplitude :

$$\text{Im} \langle \gamma \pi | T | \gamma \pi \rangle = \sum_n \langle \gamma \pi | T | n \rangle \langle n | T^* | \gamma \pi \rangle \quad (18)$$

The only intermediate state we retain is the two-pion state, because we are interested into an energy domain where s_1 is less than $(3\mu)^2$. The $\gamma + \pi \rightarrow \gamma + \pi$ amplitude has been calculated in a preceding paper¹⁾. The structure of this matrix element is

$$M_{fi} = \frac{1}{(2\pi)^6} \frac{1}{4\omega_q (k\omega_k)^{\frac{1}{2}}} (\underline{e}, \underline{k}, \underline{q}) H \left[\underline{k}^2, (\underline{k}, \underline{q})^2 \right] \quad (19)$$

where $\underline{k}$ is the photon momentum and $\underline{q}$ the outgoing pion momentum in the centre of mass system. If one assumes that the reaction mainly occurs in the $T = 1 \quad J = 1$ state, H does not depend on the angles (this approximation was done in¹⁾). Under this assumption the imaginary part of the $\gamma\pi \rightarrow \gamma\pi$ amplitude can be related to the total cross-section σ_T for $\gamma\pi \rightarrow \pi\pi$:

$$\text{Im } F_0 = \frac{2(k + \omega_k)}{k} (\underline{e}_1 \times \underline{k}) \cdot (\underline{e}_2 \times \underline{k}') \sigma_T, \quad (20)$$

$$\text{with } \sigma_T = \frac{1}{48\pi} \frac{k q^3}{(k + \omega_k)^2} \left| H(k^2) \right|^2,$$

where $2\omega_q = k + \omega_k$. By comparison of formulae (17) and (20) one gets :

$$\text{Im } D_{\alpha 0} = 2k(k + \omega_k) \cos \theta \sigma_T$$

$$\text{Im } D_{\beta 0} = - \frac{\sigma_T}{2k(k + \omega_k)}$$

These formulae could have been obtained by use of the "optical" theorem.

The more general case where H possesses an angular dependence is considered in Appendix I.

If we return to formula (16) for functions D , the knowledge of (21) gives the values of the weight functions $\sigma^{(n)}$. Then one can write D_α and D_β as :

$$\begin{aligned}
D_i(s_1 s_2 t) &= D_i^B(s_1 s_2) + C_i(s_1 t) + C_i(s_2 t) \\
i = \alpha, \beta &+ \sum_n [t(t-4\mu^2)]^n P_{2n} \left[\frac{s_2 - s_1}{\sqrt{t(t-4\mu^2)}} \right] b^{(n)}(t) \\
C_{\alpha 0}(s, t) &= \left[2st + (s - \mu^2)^2 \right] \frac{1}{\pi} \int_{(2\mu)^2}^{\infty} \frac{\sigma_T(x) dx}{(x - \mu^2)(x - s)} \\
C_{\beta 0}(s) &= - \frac{1}{\pi} \int_{(2\mu)^2}^{\infty} \frac{\sigma_T(x) dx}{(x - \mu^2)(x - s)}
\end{aligned} \tag{22}$$

VI. Unitarity in Channel III

In this channel the unitarity condition connects the reaction amplitude $\gamma + \gamma \rightarrow \pi + \pi$ to pion-pion scattering. Only even angular momenta are present, and we shall provisionally assume that the D-waves (and higher waves) of the pion-pion scattering can be neglected (The general case will be considered in Appendix II). The first invariant I_1 is isotropic but I_2 contains both S and D parts. Therefore, if $\text{Im } D_2$ is non vanishing in Channel III the $J \neq 0$ amplitudes have a non vanishing imaginary part in contradiction with the above assumption. It follows that all the

functions appearing in D_β are zero and that in D_α the only non vanishing weight function, $\tau^{(0)}$ is proportional to the $J = 0$ pion-pion scattering amplitude. We now extract the S part of F both in $T = 0$ and $T = 2$ states :

$$(\underline{e}_1 \underline{e}_2) F_{0,2}^S(q^2) = \frac{1}{4\pi} \int F_{0,2}(\underline{e}_1; \underline{e}_2; \underline{q}) d\Omega_{\underline{q}} \tag{23}$$

Then, for $T = 0$ or 2 :

$$\text{Im } F_T^S(q^2) = \tau_T^{(0)}(4\omega_q^2) = F_T^S(q^2) h_T^{S*}(q^2). \tag{24}$$

$h_T^S(q^2) = e^{i\delta_T^S} \sin \delta_T^S$ is the pion-pion scattering amplitude for $J = 0$ and $T = 0$ or 2 .

VII. Low energy requirements

From the above equations we finally get the following representations :

$$\begin{aligned}
 D_{\alpha 0}(s_1, s_2; t) = & D_{\alpha}^B(s_1, s_2) \\
 & + \left[2 s_1 t + (s_1 - \mu^2)^2 \right] \frac{1}{\pi} \int_{(2\mu)^2}^{\infty} \frac{\sigma_T(x) dx}{(x - \mu^2)(x - s_1)} \\
 & + \left[2 s_2 t + (s_2 - \mu^2)^2 \right] \frac{1}{\pi} \int_{(2\mu)^2}^{\infty} \frac{\sigma_T(x) dx}{(x - \mu^2)(x - s_2)} \\
 & + \frac{4}{\pi} \int_0^{\infty} \frac{F_0^S(q^2) h_0^{s*}(q^2) dq^2}{4(\mu^2 + q^2) - t}
 \end{aligned} \tag{25}$$

$$\begin{aligned}
 D_{\beta 0}(s_1, s_2; t) = & D_{\beta}^B(s_1, s_2) \\
 & - \frac{1}{\pi} \int_{(2\mu)^2}^{\infty} \frac{\sigma_T(x) dx}{(x - \mu^2)(x - s_1)} \\
 & - \frac{1}{\pi} \int_{(2\mu)^2}^{\infty} \frac{\sigma_T(x) dx}{(x - \mu^2)(x - s_2)}
 \end{aligned} \tag{26}$$

$$D_{\alpha 2}(s_1, s_2; t) = D_{\alpha}^B(s_1, s_2) + \frac{4}{\pi} \int_0^{\infty} \frac{F_2^S(q^2) h_2^{S*}(q^2) dq^2}{4(\mu^2 + q^2) - t} \quad (27)$$

$$D_{\beta 2}(s_1, s_2; t) = D_{\beta}^B(s_1, s_2) \quad (28)$$

It can be easily seen that in channel I (compton scattering)

$$\lim_{\substack{s_1 \rightarrow \mu^2 \\ \cos \theta \text{ fixed}}} I_{\alpha} D_{\alpha}^B + I_{\beta} D_{\beta}^B \text{ is a finite quantity.}$$

It is known that in the zero energy limit, for any charged system scattering photons⁵⁾, the cross-section becomes equal to the Thomson cross-section. Moreover the derivative with respect to k , at zero energy, is proportional to the magnetic moment of the system; here the magnetic moment is zero since we are dealing with spin zero particles; for the neutral case, the cross-section and its derivative should vanish. In fact the explicit proof was only given for spin 1/2 particles but everybody believes that this result is independent of the spin of the particle. We shall not give here a justification of this statement but we shall take it for granted. Then one gets the following condition :

$$I_{\alpha} D_{\alpha} + I_{\beta} D_{\beta} - (I_{\alpha} D_{\alpha}^B + I_{\beta} D_{\beta}^B) \sim k^2 (s_1 - \mu^2)^2 \sim (s_2 - \mu^2)^2 \sim t$$

$$\text{for } s_1 - \mu^2 \rightarrow 0, \cos \theta \text{ fixed.}$$

Since I_{β} behaves like $(s_1 - \mu^2)^2$, the expressions (26) and (28) for $D_{\beta 0}$ and $D_{\beta 2}$ are unaffected by this condition, but in $D_{\alpha 0}$ and $D_{\alpha 2}$ a subtraction is needed :

$$D_{\alpha 0} = D_{\alpha}^B + \left[2 s_1 t + (s_1 - \mu^2)^2 \right] \frac{1}{\pi} \int_{(2\mu)^2}^{\infty} \frac{\sigma_T(x) dx}{(x - \mu^2)(x - s_1)} + \left[2 s_1 t + (s_2 - \mu^2)^2 \right] \frac{1}{\pi} \int_{(2\mu)^2}^{\infty} \frac{\sigma_T(x) dx}{(x - \mu^2)(x - s_2)} \quad (29)$$

$$+ \frac{t}{\pi} \int_0^{\infty} \frac{F_0^S(q^2) h_0^{*S}(q^2) dq^2}{(q^2 + \mu^2) [4(q^2 + \mu^2) - t]} \\ D_{\beta 2} = D_{\beta}^B + \frac{t}{\pi} \int_0^{\infty} \frac{F_2^S(q^2) h_2^{*S}(q^2) dq^2}{(q^2 + \mu^2) [4(q^2 + \mu^2) - t]} \quad (30)$$

These expressions still satisfy the unitarity requirements in channel III.

The very fact that subtractions are feasible in equations (25), (27) naturally leads to ask the question : why other subtractions should not be made ?

Here, we have done the minimum number of necessary subtractions. It would look quite unnatural that the pion-pion interaction does not influence the k^2 dependence of the $\gamma\pi$ scattering cross-section, as viewed, for instance, in a perturbation approach.

VII. Equations for F_0^S and F_2^S and their solutions.

We take the average of the F_s' over angle φ in channel III as was done in Section VI (see formula 23). It is more convenient to write equations for

$$f_0^S = \frac{F_0^S}{\omega_q^2} \quad \text{and} \quad f_2^S = \frac{F_2^S}{\omega_q^2} :$$

$$f_2^S(q^2) = \frac{1}{\pi} \int_0^\infty \frac{f_2^S(q'^2) h_2^{S*}(q'^2) dq'^2}{q'^2 - q^2 - i\epsilon} + \frac{e^2 \mu^2}{\omega_q^2} \int_{-1}^{+1} \frac{d \cos \varphi}{\omega_q^2 - q^2 \cos^2 \varphi} \quad (31)$$

$$f_0^S(q^2) = \frac{1}{\pi} \int_0^\infty \frac{f_0^S(q'^2) h_0^{S*}(q'^2) dq'^2}{q'^2 - q^2 - i\epsilon} + \frac{e^2 \mu^2}{\omega_q^2} \int_{-1}^{+1} \frac{d \cos \varphi}{\omega_q^2 - q^2 \cos^2 \varphi} \quad (32)$$

$$+ \frac{2}{\pi} \int_{-1}^{+1} d \cos \varphi \left[s_1 \int_{(2\mu)^2}^\infty \frac{\sigma_T(x) dx}{(x - \mu^2)(x - s_1)} + \text{sym} (1 \rightleftharpoons 2) \right]$$

These equations are obtained after some manipulations using the expressions of q^2 and $\sin^2 \varphi$ in terms of s_1 , s_2 , and t . These equations are of Omnès-Mushkelishvili type⁶⁾; they could be solved directly but it is perhaps better to exhibit first the analytical structure of the inhomogeneous terms. By the change of variable $Z = -1/\sin^2 \varphi$, one shows :

$$\int_{-1}^{+1} \frac{d \cos \varphi}{\omega_q^2 - q^2 \cos^2 \varphi} = \int_{-\infty}^{\mu^2} \frac{dz}{(q^2 - z) \sqrt{z(z + \mu^2)}} \quad (33)$$

One makes also use of the relation :

$$\int_{-1}^{+1} \frac{d \cos \varphi}{x - s_1} = \int_{-1}^{+1} \frac{d \cos \varphi}{x - s_2} = \int_{-\infty}^{-\frac{(x+\mu^2)^2}{4x}} \frac{dz}{2(q^2-z) \sqrt{z(z+\mu^2)}} \quad (34)$$

(see Appendix III)

Then the equations become :

$$f_2^S(q^2) = \frac{1}{\pi} \int_0^\infty \frac{f_2^S(q'^2) dq'^2}{q'^2 - q^2 - i\varepsilon} + \frac{g^2 \mu^2}{\omega_q^2} \int_{-\infty}^{-\mu^2} \frac{dz}{(q^2-z) \sqrt{z(z+\mu^2)}} \quad (35)$$

$$f_0^S(q^2) = \frac{1}{\pi} \int_0^\infty \frac{f_0^S(q'^2) dq'^2}{q'^2 - q^2 - i\varepsilon} + \frac{g^2 \mu^2}{\omega_q^2} \int_{-\infty}^{-\mu^2} \frac{dz}{(q^2-z) \sqrt{z(z+\mu^2)}}$$

$$- \frac{1}{\pi} \int_{(2\mu)^2}^\infty \frac{8\sigma_T(x) dx}{x - \mu^2} + \frac{1}{\pi} \int_{(2\mu)^2}^\infty \frac{2x \sigma_T(x) dx}{x - \mu^2} \times$$

$$\times \int_{-\infty}^{-\frac{(\mu^2+x)^2}{4x}} \frac{dz}{(q^2-z) \sqrt{z(z+\mu^2)}} \quad (36)$$

Actually the two last terms in the last equation could be combined to show that it is still meaningful if $\sigma_T(x)$ increases like $x^{1-\varepsilon}$ at infinity.

Under the following assumptions :

$$\begin{aligned} \oint_T^S (0) &= \oint_T^S (\infty) = 0 \\ \oint_T^S (q'^2) &\sim q' \text{ as } q' \rightarrow 0 \quad \oint_T^S (q'^2) \sim \frac{A_T}{q'} + \frac{B_T}{q'^2} + \dots \text{ as } q' \rightarrow \infty \end{aligned} \quad (37)$$

One gets the solutions of equations (35), (36) (see Appendix III)

$$F_2^S(q^2) = e^{\rho_2^S(q^2) + i \oint_2^S(q^2)} e^{2\mu^2} \int_{-\infty}^{-\mu^2} \frac{dz}{\sqrt{z(z+\mu^2)}} \left[\frac{e^{-\rho_2^S(z)}}{q^2 - z} + \frac{e^{-\rho_2^S(z)} - e^{-\rho_2^S(-\mu^2)}}{z + \mu^2} \right]$$

where $\rho_2^S(q^2) = \frac{P}{\pi} \int_0^\infty \frac{\oint_2^S(q'^2) dq'^2}{q'^2 - q^2}$

and

$$\begin{aligned} F_0^S(q^2) &= e^{\rho_0^S(q^2) + i \oint_0^S(q^2)} \times \\ &\left[e^{2\mu^2} \int_{-\infty}^{-\mu^2} \frac{dz}{\sqrt{z(z+\mu^2)}} \left(\frac{e^{-\rho_0^S(z)}}{q^2 - z} + \frac{e^{-\rho_0^S(z)} - e^{-\rho_0^S(-\mu^2)}}{z + \mu^2} \right) \right. \\ &\left. + \frac{2\omega_q^2}{\pi} \int_{(2\mu)^2}^\infty \frac{\sigma_T(x) dx}{x - \mu^2} \left(x \int_{-\infty}^{\frac{-(x+\mu^2)^2}{4x}} \frac{e^{-\rho_0^S(z)}}{(q^2 - z) \sqrt{z(z+\mu^2)}} dz - 4 \right) \right] \end{aligned} \quad (38)$$

with a similar definition of ρ_0^S .

The advantage of putting the solution in this form, is that, apart $\rho(q^2)$, there is no principal value integral to evaluate, which is a great advantage from a computational point of view. It can be seen that F_0^S exists even if $\sigma_T(x)$ behaves like $x^{1-\epsilon}$ at infinity : the assumptions on δ show that $\rho(z)$ behaves like $\frac{\log z}{z}$ at infinity;

hence

$$\text{for } x \rightarrow \infty \left| \int_{-\infty}^{\frac{(x+\mu^2)^2}{4x}} \frac{e^{-\rho_0^S(z)} dz}{(q^2-z)\sqrt{z(z+\mu^2)}} - \frac{4}{x} \right| < C \frac{\log x}{x^2}$$

The assumptions on the phase shift behaviour at zero energy and at infinity permit to guarantee the uniqueness of the solutions of equations (35), (36) (See appendix III).

By inserting F_0^S and F_2^S in the subtracted $D_{\pi 0}$ and $D_{\pi 2}$ (equations 29, 30) and substituting these quantities in equation (17) one gets the compton scattering amplitude.

IX. Concluding remarks

We have given explicit expressions of F_0^S and F_2^S in terms of σ_T , δ_0^S , δ_2^S which permit to obtain the γ - π scattering amplitude or the $\pi+\pi \rightarrow \gamma+\gamma$ amplitude. However one cannot consider this problem as completely solved since we have not made any attempt of numerical evaluation of these amplitude, because of the lack of information on pion-pion scattering. Nevertheless we felt it might be useful to give this algebraic solution. Of course it would be very interesting to give numerical estimates of the deviations from Born approximation as the energy of the photon increases. This will be done in the future.

It is important to notice that the equations we solve have a unique solution if the S-phases of pion-pion scattering are such that $\delta(0) = \delta(\infty) = 0$, but that the equations are not really unique, because an arbitrary number of subtractions can be made before solving them.

Here, we have done one subtraction, which is the minimum number of subtraction needed to ensure that the low energy cross-section for γ - π scattering is given by the Thomson formula. On the other hand one cannot see any mathematical argument to forbid a larger number of subtractions at $t = 0$; nevertheless this would give the result that the derivative of the amplitude F_2 , related to the difference between the charged and neutral amplitude, with respect to k^2 , at zero energy, would not be influenced by the pion-pion interaction, which looks rather unsatisfactory. So, on the basis of this argument and also for simplicity reasons we felt that one should not make more than one subtraction at zero energy.

A C K N O W L E D G M E N T

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APPENDIX I

Unitarity in Channel I

The matrix element for the process $\gamma + \pi \rightarrow \gamma + \pi$ has been defined in the centre of mass system as :

$$\langle \underline{k}', \underline{e}_2 | T | \underline{k}, \underline{e}_1 \rangle = \frac{1}{(2\pi)^6} \frac{1}{4k\omega_k} \Phi(\underline{e}_1, \underline{e}_2, \underline{k}, \underline{k}') \quad (\text{I},1)$$

and for the process $\gamma + \pi \rightarrow \pi + \pi$

$$\langle \underline{q} | T | \underline{k}, \underline{e} \rangle = \frac{1}{(2\pi)^6} \frac{1}{4(k\omega_k)^{\frac{1}{2}}\omega_q} \Psi(\underline{e}, \underline{k}, \underline{q}) \quad (\text{I},2)$$

The unitarity condition has the simple form :

$$\text{Im} \Phi(\underline{e}_1, \underline{e}_2; \underline{k}, \underline{k}') = \frac{1}{(2\pi)^2} \frac{1}{8\omega_q^2} \int d_3\underline{q} \int \delta(2\omega_q - \omega_k - k) \Psi(\underline{e}_1, \underline{k}, \underline{q}) \Psi^*(\underline{e}_2, \underline{k}', \underline{q}) \quad (\text{I},3)$$

and after integration over the modulus of q

$$\text{Im} \Phi(\underline{e}_1, \underline{e}_2; \underline{k}, \underline{k}') = \frac{1}{(8\pi)^2} \frac{q}{\omega_q} \int \Psi(\underline{e}_1, \underline{k}, \underline{q}) \Psi^*(\underline{e}_2, \underline{k}', \underline{q}) d\Omega_q$$

$$\text{with } 2\omega_q = k + \omega_k \quad (\text{I},4)$$

From gauge and Lorentz invariance conditions, we know that $\Psi(\underline{e}, \underline{k}, \underline{q})$, has the form

$$\Psi(\underline{e}, \underline{k}, \underline{q}) = (\underline{e}, \underline{k}, \underline{q}) H [k^2, (k \cdot q)^2]$$

It follows that the integral over the $\underline{q}$ angle can be written as a linear combination of two invariants directly related to I_1 and I_2 . By definition of U_1 and U_2

$$\int \bar{\Psi}(\underline{e}_1, \underline{k}, \underline{q}) \bar{\Psi}^*(\underline{e}_2, \underline{k}', \underline{q}) d\Omega_q = \frac{4\pi}{3} q^2 \left\{ (\underline{e}_1 \cdot \underline{e}_2) (\underline{k} \cdot \underline{k}') U_1(k^2, \cos^2 \theta) - (\underline{e}_1 \cdot \underline{k}') (\underline{e}_2 \cdot \underline{k}) U_2(k^2, \cos^2 \theta) \right\} \quad (I,5)$$

By comparing this expression of $\text{Im } \Phi$ and the definition

$$\text{Im } \Phi = I_1 \text{Im } D_1 + I_2 \text{Im } D_2$$

we easily deduce by identification :

$$\begin{aligned} \text{Im } D_1 &= \frac{1}{24\pi} \frac{q^3 k^2}{k + \omega_k} \cos \theta U_1(k^2, \cos \theta) \\ \text{Im } D_2 &= \frac{1}{96\pi} \frac{q^3}{(k + \omega_k)^3} \left[\cos \theta (U_2 - U_1) - U_2 \right] \end{aligned} \quad (I,6)$$

At this stage, it is interesting to relate U_1 to the total cross-section for the reaction $\gamma + \pi \rightarrow \pi + \pi$. In order to obtain the latter we must calculate the integral

$$\int \left| \bar{\Psi}(\underline{e}, \underline{k}, \underline{q}) \right|^2 d\Omega_q = \frac{4\pi}{3} q^2 k^2 U_1(k^2, 1)$$

and the total cross-section is given by

$$\sigma_T = \frac{1}{48\pi} \frac{k q^3}{(k + \omega_k)^2} U_1(k^2, 1) \quad (I,7)$$

In the particular case, considered in this paper where H does not depend of the angle between $\underline{k}$ and $\underline{q}$, we simply have

$$U_1(k^2, \cos^2 \theta) = U_2(k^2, \cos^2 \theta) = \left| H(k^2) \right|^2 \quad (I,8)$$

From the general form given for D_1 and D_2 we can expand their imaginary parts in Legendre polynomials of $\cos\theta$

$$\text{Im } D_i = \sum_n \left[2 k (k + \omega_k) \right]^{2n} P_n(\cos\theta) \sigma_i^{(n)}(s_1)$$

and the unitarity condition permits to know the weight functions $\sigma_i^{(n)}(s_1)$:

$$\begin{aligned} \sigma_{\alpha}^{(n)}(s_1) &= \frac{2n+1}{48\pi} \frac{q^3 k^2}{k + \omega_k} \frac{1}{[2k(k + \omega_k)]^{2n}} \int_{-1}^{+1} U_1(k^2, u^2) u P_n(u) du \\ \sigma_{\beta}^{(n)}(s_1) &= \frac{2n+1}{192\pi} \frac{q^3}{(k + \omega_k)^3} \frac{1}{[2k(k + \omega_k)]^{2n}} \int_{-1}^{+1} \left[u (U_1 - U_2) - U_2 \right] P_n(u) du \end{aligned} \quad (\text{I}, 9)$$

For the channel III, ϕ is given by :

$$\phi = (\underline{e}_1 \cdot \underline{e}_2) D_1 - 8 \omega_q^2 (\underline{e}_1 \cdot \underline{q}) (\underline{e}_2 \cdot \underline{q}) D_2 \quad (\text{II-1})$$

Taking the incident momentum $\underline{p}$ as z axis and the bissector of $\underline{e}_1$ and $\underline{e}_2$ as x axis, we have :

$$(\underline{e}_1 \cdot \underline{q}) (\underline{e}_2 \cdot \underline{q}) = \frac{q^2}{2} \sin^2 \varphi [\underline{e}_1 \cdot \underline{e}_2 + \cos 2\omega]$$

where φ and ω are the polar angles of the final momentum $\underline{q}$.

Consequently the amplitude ϕ takes the form :

$$\phi = (\underline{e}_1 \cdot \underline{e}_2) \left\{ D_1 - \frac{8}{3} (q \omega_q)^2 [1 - P_2(\cos \varphi)] D_2 \right\} - \frac{4}{3} (q \omega_q)^2 P_2^2(\cos \varphi) \cos 2\omega D_2$$

where P_2 and P_2^2 are respectively the Legendre polynomial and the Legendre associated function of order 2.

The first part of ϕ contains only cylindrical symmetrical waves and the second part non cylindrical symmetrical waves. For one given value of the orbital angular momentum ℓ we can define two partial waves (except for $\ell = 0$) :

$$\left\{ \begin{aligned} F_{2\ell}(q^2) &= \frac{4\ell+1}{2} \int_{-1}^{+1} \left[D_1 - \frac{8}{3} (q \omega_q)^2 D_2 (1 - P_2(\cos \varphi)) \right] P_{2\ell}(\cos \varphi) d\cos \varphi \\ f_{2\ell}(q^2) &= \int_{-1}^{+1} D_2 P_2^2(\cos \varphi) P_{2\ell}^2(\cos \varphi) d\cos \varphi \end{aligned} \right. \quad (\text{II-2})$$

corresponding to the following expansion for ϕ :

$$\phi = (\underline{e}_1 \cdot \underline{e}_2) \sum_{\ell=0}^{\infty} F_{2\ell}(q^2) P_2(\cos \varphi) - \frac{4}{3} (q \omega_q)^2 \sum_{\ell=1}^{\infty} f_{2\ell}(q^2) P_2^2(\cos \varphi) \cos 2\omega$$

The imaginary parts of $F_{2\ell}(q^2)$ and $f_{2\ell}(q^2)$ are easily related to the imaginary parts of D_1 and D_2 . With the definitions given in the text

$$\text{Im } D_{1,2} = \sum_{\ell} (4q \omega_q)^2 P_2(\cos \varphi) \rho_{1,2}^{(\ell)}(4\omega_q^2)$$

It follows that angular integrations over φ can be performed and we define for the cylindrical part :

$$\left\{ \begin{aligned} \alpha_n &= \int_{-1}^{+1} (1-u^2) P_{2n}(u) P_{2n+2}(u) du \\ \beta_n &= \int_{-1}^{+1} (1-u^2) [P_{2n}(u)]^2 du \end{aligned} \right.$$

and the imaginary part of $F_{2\ell}(q^2)$ appears as a linear combination of the unknown weight functions ρ :

$$\begin{aligned} \text{Im } F_{2\ell}(q^2) &= (4q\omega_q)^2 \left\{ \rho_1^{(\ell)}(4\omega_q^2) - \right. \\ &\quad \left. - \frac{4\ell+1}{8} \left[\alpha_{\ell-1} \rho_2^{(\ell-1)}(4\omega_q^2) + (4q\omega_q)^2 \beta_{\ell} \rho_2^{(\ell)}(4\omega_q^2) + (4q\omega_q)^4 \alpha_{\ell} \rho_2^{(\ell+1)}(4\omega_q^2) \right] \right\} \end{aligned} \quad (\text{II-3})$$

In the same way, we obtain for the non cylindrical functions :

$$\text{Im } f_{2\ell}(q^2) = (4q\omega_q)^{2\ell-2} \left[\gamma_{\ell\ell-1}^{(\ell-1)} \rho_2^{(\ell-1)}(4\omega_q^2) + (4q\omega_q)^2 \gamma_{\ell\ell}^{(\ell)} \rho_2^{(\ell)}(4\omega_q^2) + (4q\omega_q)^4 \gamma_{\ell\ell+1}^{(\ell+1)} \rho_2^{(\ell+1)}(4\omega_q^2) \right] \quad (\text{II-4})$$

where ;

$$\gamma_{\ell m} = \int_{-1}^{+1} (1-u^2) P_{2\ell}^2(u) P_{2m}(u) du$$

It is easy to apply the unitary condition for each partial wave. For a given isospin, we have :

$$\left\{ \begin{aligned} \text{Im } F_{2\ell}(q^2) &= F_{2\ell}(q^2) h_{2\ell}^*(q^2) \\ \text{Im } f_{2\ell}(q^2) &= f_{2\ell}(q^2) h_{2\ell}^*(q^2) \end{aligned} \right.$$

where $h_{2\ell}(q^2)$ is the pion-pion scattering amplitude for the angular momentum 2ℓ .

By inspection of equations (II-3) and (II-4) we see that $\rho_1^{(\ell)}$ appears in $\text{Im } F_{2\ell}(q^2)$ and $\rho_2^{(\ell)}$ in $\text{Im } F_{2\ell+2}$, $\text{Im } F_{2\ell}$, $\text{Im } F_{2\ell-2}$, $\text{Im } f_{2\ell+2}$, $\text{Im } f_{2\ell}$, $\text{Im } f_{2\ell-2}$. If we assume the pion-pion scattering amplitude to be zero for the angular momenta higher than $2L$, we can conclude: $\rho_1^{(\ell)} = 0$ for $\ell > L$ and $\rho_2^{(\ell)} = 0$ for $\ell > L-1$. The non vanishing weight functions are solutions of a linear system of $(2L+1)$ equations (corresponding to $(2L+1)$ unitarity conditions) for $(2L+1)$ functions ($\rho_1^{(\ell)}$ for $\ell = 0, L$ and $\rho_2^{(\ell)}$ for $\ell = 0, L-1$). In general we have a unique solution.

The particular case considered in the text corresponds to $L = 0$ and we have only one weight function $\rho_1^{(0)}$ given by the unitarity condition corresponding to the S-wave.

A P P E N D I X I I I

We first transform the integral (34). A general technique has been given by the authors in ref.¹⁾, but here a trick due to S. Fubini gives the answer in a straightforward way :

$$\int_{-1}^{+1} \frac{d \cos \varphi}{x - s_1} = \int_x^\infty dy \int_{-1}^{+1} \frac{d \cos \varphi}{[y - s_1]^2}$$

Since s_1 is a linear function of $\cos \varphi$ the last integral can be performed and gives a rational expression :

$$\int_x^\infty \frac{dy}{2y \left[q^2 + \frac{(y+\mu^2)^2}{4y} \right]}$$

which, by a trivial change of variable gives :

$$\int_{-\infty}^{-\frac{(x+\mu^2)^2}{4x}} \frac{dz}{2(q^2 - z) \sqrt{z(z+\mu^2)}} \quad (\text{III},1)$$

Then we try to solve equations (35) and (36). Let us first solve the general equation

$$f(q^2) = \frac{1}{\pi} \int_0^\infty \frac{f(q'^2) h^*(q'^2) dq'^2}{q'^2 - q^2 - i\epsilon} + \int_{-\infty}^{-M < 0} \frac{p(z) dz}{z - q^2} + c \quad (\text{III},2)$$

where $h(q^2) = e^{i\delta(q^2)} \sin \delta(q^2)$ and

where $p(z)$ may include a finite number of δ -functions and behaves at most like $\frac{1}{z}$ at infinity.

Equation (III,2) defines an analytic function $f(Z)$ with the following properties :

- it is such that $\lim_{|Z| \rightarrow \infty} f(Z) = C$, if an oscillatory behaviour of $f(q^2)h^*(q^2)$ at infinity is excluded;
- it is holomorphic everywhere except along two cuts from $-\infty$ to $-M$ and from 0 to $+\infty$;
- the discontinuity across the left hand cut is $p(z)$;
- the discontinuity across the right hand cut is such that
$$f(q^2 + i\varepsilon) / f(q^2 - i\varepsilon) = e^{2i\delta(q^2)}$$

$$\begin{aligned} \text{If } \delta(0) = \delta(\infty) = 0 \text{ and} \\ \delta(q^2) \sim q \text{ as } q \rightarrow 0 \\ \delta(q^2) \sim \frac{1}{q} \text{ as } q \rightarrow \infty \end{aligned} \quad (\text{III},3)$$

one can define

$$U(Z) = \frac{1}{\pi} \int_0^\infty \frac{\delta(q'^2) dq'^2}{q'^2 - Z}, \quad U(q^2 + i\varepsilon) = \rho(q^2) + i\delta(q^2)$$

Then the quantity

$$e^{U(Z)} \left[C + \int_{-\infty}^{-M} \frac{e^{-\rho(z)} p(z)}{z - Z} dz \right] = e^{U(Z)} g(Z) \quad (\text{III},4)$$

fulfils all the above requirements. To check that this is a solution one can insert in (III,2). Then one has to compute the integral

$$\frac{1}{\pi} \int_0^\infty \frac{e^{U(x+i\varepsilon)} g(x) h^*(x) dx}{x - Z_0} = \frac{1}{2i\pi} \int_0^\infty \frac{[e^{U(x+i\varepsilon)} - e^{U(x-i\varepsilon)}] g(x) dx}{x - Z_0}$$

This integral can be computed by following C_+ (fig.3)
 (the contribution from the circle surrounding the origin vanishes).
 Then $\int_{C_+} = \text{Residue}(Z_0) \times 2\pi i - \int_{C_-} - \int_{\Gamma_-} - \int_{\Gamma_+}$.

Fig. 3

$$\frac{1}{\pi} \int_0^{\infty} \frac{e^{u(x+i\varepsilon)} h^*(x) g(x) dx}{x - Z_0} = e^{u(Z_0)} g(Z_0) - \frac{1}{2\pi i} \int_{C_-} \frac{e^{u(Z)} g(Z) dZ}{Z - Z_0} - C$$

Now, if $g(Z)$ is regular at $Z = -M$ (which is the case here)

$$\begin{aligned} \int_{C_-} \frac{e^{u(Z)} g(Z) dZ}{Z - Z_0} &= \int_{-\infty}^{-M} \frac{e^{p(z)} [g(z+i\varepsilon) - g(z-i\varepsilon)]}{z - Z_0} dz \\ &= 2\pi i \int_{-\infty}^{-M} \frac{p(z) dz}{z - Z_0} \end{aligned} \quad (\text{III}, 5)$$

Then it can be seen that the equation is satisfied. The multiplicity of the solutions has already been discussed elsewhere (7). The homogeneous equation has no solution when the hypothesis (III,3) are made on the phase shift. This is shown by inserting in the homogeneous equation the possible solutions, i.e. $Z^m e^{U(Z)}$, m integer (6), and using contour integration.

The equations we have to solve in the present paper can be formally written in the form (III, 2), but the singularities of the weight functions make the integral divergent : one writes the term arising from Born approximation as

$$\frac{e^2 \mu^2}{q^2 + \mu^2} \int_{-\infty}^{\mu^2} \frac{dz}{(q^2 - z) \sqrt{z(z + \mu^2)}} = e^2 \mu^2 \left[-\frac{1}{q^2 + \mu^2} \int_{-\infty}^{\mu^2} \frac{dz}{(\mu^2 + z) \sqrt{z(z + \mu^2)}} + \int_{-\infty}^{\mu^2} \frac{dz}{(q^2 - z) (z + \mu^2) \sqrt{z(z + \mu^2)}} \right] \quad (\text{III}, 6)$$

If we disregard the divergences and apply formula (III, 4) we finally get for the solutions convergent expressions. Though this procedure is not well justified we can see by contour integration that the result is correct. As we said the only trouble comes from the term (III, 6) which appears both in the equation for f_0^S and in the equation for f_2^S . The contribution of (III, 6) to the solution is found to be

$$\begin{aligned} f_B(q^2) &= e^{u(q^2 + i\epsilon)} \frac{e^2 \mu^2}{q^2 + \mu^2} \int_{-\infty}^{\mu^2} \left[\frac{e^{-\rho(z)}}{q^2 - z} + \frac{e^{-\rho(z)} - e^{-\rho(-\mu^2)}}{z + \mu^2} \right] \frac{dz}{\sqrt{z(z + \mu^2)}} \\ \text{Now } \frac{1}{\pi} \int_0^\infty \frac{f_B(q'^2) h^*(q'^2) dq'^2}{q'^2 - Z_0} &= f_B(Z_0) \\ &- \frac{e^2 \mu^2}{2\pi i} \int_{C_-} \frac{e^{u(Z)}}{Z - Z_0} \frac{dZ}{(Z + \mu^2)} \int_{-\infty}^{\mu^2} \left[\frac{e^{-\rho(z)}}{Z - z} + \frac{e^{-\rho(z)} - e^{-\rho(-\mu^2)}}{z + \mu^2} \right] \frac{dz}{\sqrt{z(z + \mu^2)}} \\ &= f_B(Z_0) - \frac{e^2 \mu^2}{2\pi i} \int_{C_-} \frac{dZ}{(Z - Z_0) (Z + \mu^2)} \int_{-\infty}^{\mu^2} \frac{dz}{(Z - z) \sqrt{z(z + \mu^2)}} \\ &- \frac{e^2 \mu^2}{2\pi i} \int_{C_-} \frac{dZ}{(Z - Z_0) (Z + \mu^2)} \int_{-\infty}^{\mu^2} \left[\frac{e^{-\rho(z)} - e^{-u(Z)}}{Z - z} + \frac{e^{-\rho(z)} - e^{-\rho(-\mu^2)}}{z + \mu^2} \right] \frac{dz}{\sqrt{z(z + \mu^2)}} \end{aligned}$$

In the first part of the integral the path C_- can be obviously replaced by a contour surrounding Z_0 . In the second part of the integral the integrand has no discontinuity across the left hand cut and the only possible singularity inside C_- lies at $Z = -\mu^2$, but the residue is zero, so, finally

$$\frac{1}{\pi} \int_0^{\infty} \frac{f_B(q'^2) h^*(q'^2) dq'^2}{q'^2 - Z_0} = f_B(Z_0) - \frac{e^2 \mu^2}{(Z_0 + \mu^2)} \int_{-\infty}^{-\mu^2} \frac{dz}{(Z_0 - z) \sqrt{z(z + \mu^2)}}$$

q.e.d.

R E F E R E N C E S

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