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Analytic Properties of $l \neq 0$ partial wave amplitudes
for a Given Class of Potentials

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ABSTRACT

A preceding work on the analytic properties of the S wave amplitude for a family of potentials including the Yukawa case is extended to higher waves. Apart from a transformation in order to eliminate the centrifugal term, the method is essentially the same as in the S wave case. It makes use of the Laplace transforms of two quantities related to independent solutions of the Schrödinger equation. The singularities in the upper half complex k plane are bound states on the imaginary axis plus a cut $i\frac{\nu}{2} \rightarrow i\infty$. In any direction (except perhaps the cut) $\lim_{|k| \rightarrow \infty} S(k) = 1$, so it is possible to write dispersion relations.

I. INTRODUCTION

In a preceding paper designated hereunder by A ¹⁾, we have studied the analytic properties of the S wave scattering amplitude for the family of potentials :

$$V(r) = e^{-\mu r} \int_0^{\infty} c(\alpha) e^{-\alpha r} d\alpha \quad (1)$$

with

$$|c(\alpha)| < M$$

It was shown that the scattering amplitude is holomorphic in the upper half complex k plane, except for poles due to bound states on the imaginary axis and a cut from $\frac{i\mu}{2}$ to $i\infty$; the analytic properties in the lower half plane follow by use of symmetry properties. Furthermore, it was shown that in any direction, the scattering amplitude goes to zero as $|k| \rightarrow \infty$. We shall now prove that these properties hold for any partial wave. Due to presence of the centrifugal term in the Schrödinger equation, the treatment presented in A must be seriously modified. Such a term cannot be merely included in the potential, since it is too singular at the origin and does not decrease fast enough at infinity.

II. Summary of the Method used in the S wave case

In A it was shown that the scattering matrix can be written :

$$S(k) = \frac{f(0,k)}{g(0,k)} \equiv \frac{f(0,k)}{f(0,-k)} \quad (2)$$

where

$$f(r,k)e^{-ikr}$$

is a solution of the Schrödinger equation, such that

$$\lim_{r \rightarrow \infty} f(r,k) = 1 \quad (3)$$

$f(r,k)$ was expressed in terms of its Laplace transform :

$$f(r,k) - 1 = \int_0^{\infty} e^{-\alpha r} \rho_k^S(\alpha) d\alpha \quad (4)$$

where ρ satisfies the equation :

$$\rho_k^S(\alpha) \alpha (\alpha + 2ik) = c(\alpha - \mu) + \int_0^{\alpha - \mu} c(\alpha - \mu - \beta) \rho_k^S(\beta) d\beta \quad (5)$$

(notice a slight change in notations), which means, in particular, that $\rho_k^S(\alpha) = 0$ for $0 < \alpha < \mu$. Equation (5) permits to compute by recursion $\rho_k^S(\alpha)$ for any arbitrarily large value of α , and gives the information needed to establish the analytic properties of $f(r, k)$ and $f(r, -k)$. Moreover, it was shown that the zeros of $g(0, k)$ in the upper half plane, correspond to bound state with the wave function :

$$u(r) = g(r, k)e^{ikr} \quad (6)$$

These poles cannot be outside the imaginary axis, because

$$\left| u' - iku \right|_{r_0}^2 \geq 4 \operatorname{Im} k (\operatorname{Re} k)^2 \int_0^{r_0} |u|^2 dr \quad (7)$$

cannot be satisfied for $r_0 \rightarrow \infty$, $\operatorname{Im} k \neq 0$, $\operatorname{Re} k \neq 0$.

III. Necessary changes in view of an extension to higher waves

It is still possible to define two independent solutions of the Schrödinger equation

$$f(r, k)e^{-ikr}, \quad f(r, -k)e^{ikr}$$

satisfying the equation :

$$f'' - ikf' = \left[\frac{1(1+1)}{r^2} + V(r) \right] f \quad (8)$$

One still writes :

$$f(r,k) = 1 + \int_0^\infty \rho_k(\alpha) e^{-\alpha r} d\alpha \quad (9)$$

but equation (5) is replaced by :

$$\begin{aligned} \rho_k(\alpha) \propto (\alpha + 2ik) = 1(1+1) + C(\alpha - \mu) \\ + \int_0^\alpha [1(1+1)(\alpha - \beta) + C(\alpha - \mu - \beta)] \rho_k(\beta) d\beta \end{aligned} \quad (10)$$

It is no longer true that $\rho = 0$ for $\alpha < \mu$. One problem is to study the solution of equation (10). Another one is to define, from this solution the scattering matrix. In the general case, $f(r,k)$ and $f(r,-k)$ are singular at the origin and behave like r^{-1} . Therefore one has to define $S(k)$ by a limiting process :

$$S(k) \times S_0(k) = \lim_{r \rightarrow \infty} \frac{1 + \int_0^{\infty} e^{-\alpha r} \rho_k(\alpha) d\alpha}{1 + \int_0^{\infty} e^{-\alpha r} \rho_{-k}(\alpha) d\alpha} \quad (11)$$

$S_0(k)$ is a factor, independent of the potential, such that $S(k) = 1$ when no interaction is present. It will be shown in Section V that (11) can be replaced by a more convenient expression in terms of the asymptotic behaviour of $\rho_k(\alpha)$ and $\rho_{-k}(\alpha)$ for large α .

IV. Study of the Equations defining $\rho_k(\alpha)$

In equation (10), the centrifugal term is treated on the footing as the potential itself. This should be avoided, since for $V(r) = 0$ the solution of the Schrödinger equation can be expressed in terms of half integer order Bessel functions. Then $f(r, k)$ and $f(r, -k)$ are polynomials of degree 1 of the quantity $\frac{1}{ikr}$. From this one can deduce by Laplace transformation :

$$\rho_k^0(\alpha) = \frac{1}{(2ik)^l} \left(\frac{d}{dX} \right)^{l+1} [X(X+1)]^l \quad (12)$$

where $X = \frac{\alpha}{2ik}$; for $l = 0$ $\rho_k^0(\alpha) \equiv 0$.

Equation (10) can now be transformed in the following way: we introduce

$$y(\alpha) = 1 + \int_0^{\alpha} \rho(\alpha) d\alpha$$

and we take the derivative of equation (10). This gives :

$$\begin{aligned} \alpha(\alpha+2ik) y'' + (\alpha \Gamma \alpha + 2ik \Gamma)' y' - 1(1+1) y \\ = \frac{dC(\alpha-\mu)}{d\alpha} + \int_0^{\alpha-\mu} \frac{dC(\alpha-\beta-\mu)}{d\alpha} \rho_k(\beta) d\beta \end{aligned} \quad (13)$$

This equation can now be treated as a second order differential equation with "known" right hand side. The homogeneous equation has known solutions :

$$\begin{aligned} y_1(\alpha) &= \frac{1}{1!} \left(\frac{d}{d\alpha} \right)^1 \Gamma^{X(X+1)} \Gamma^1 \\ y_2(\alpha) &= y_1(\alpha) \int_{\alpha}^{\infty} \frac{d\beta}{y_1^2(\beta) \beta(\beta+2ik)} \end{aligned} \quad (14)$$

Then one can get the general solution of equation (13).

By inspection of equation (10) one sees that the right boundary condition is that $\rho(\alpha)$ and $\rho_0(\alpha)$ coincide for $\alpha < \mu$. This gives the solution :

$$y = y_1(\alpha) + \int_0^\alpha [y_1(\alpha) y_2(\beta) - y_1(\beta) y_2(\alpha)] \frac{dF(\beta)}{d\beta} d\beta$$

where

$$F(\alpha) = C(\alpha - \mu) + \int_0^{\alpha - \mu} C(\alpha - \beta - \mu) \rho_k(\beta) d\beta$$

This expression can be transformed by integration by parts, differentiation, and inversion of order of integration to get :

$$\begin{aligned} \rho_k(\alpha) = & \rho_k^0(\alpha) + \frac{C(\alpha - \mu)}{\alpha(\alpha + 2ik)} + \int_\mu^\alpha K(\alpha, \beta) C(\beta - \mu) d\beta \\ & + \int_0^{\alpha - \mu} d\gamma \rho_k(\gamma) \left[\frac{C(\alpha - \gamma - \mu)}{\alpha(\alpha + 2ik)} + \int_{\gamma + \mu}^\alpha K(\alpha, \beta) C(\beta - \gamma - \mu) d\beta \right] \end{aligned} \quad (15)$$

where

$$K(\alpha, \beta) = y_1'(\alpha) y_2'(\beta) - y_1'(\beta) y_2'(\alpha)$$

$$(y_1' \equiv \rho^0)$$

This new integral equation has the same structure as the S wave equation (5) : when $\rho(\alpha)$ is known for $\alpha < \alpha_0$ one can get

$\rho(\alpha)$ for $\alpha < \alpha_0 + \mu$ by inserting the known function in the right hand side. The derivation we give assumes that $C(\alpha)$ is continuous and derivable, but the final result is independent of this assumption; the derivative of C is not present in equation (14). As a check we see that $\rho = \rho^0$ when $C = 0$, and the case $l = 0$ gives equation (5). Equation (15) has a well defined solution when $\alpha + 2ik$ can never vanish, α being in the range $\mu - + \infty$. This means that when k is outside the cut $\frac{i\mu}{2} - i\infty$, the solution exists.

Let us now study the behaviour of $\rho_k(\alpha)$ for large α . It is perhaps more convenient to use equation (15) in the following form :

$$R_K(\alpha) = 1 + F_K(\alpha) + \int_0^{\alpha-\mu} d\gamma R_K(\gamma) G_K(\gamma, \alpha) \quad (15')$$

with

$$R_K(\alpha) = \frac{\rho_K(\alpha)}{\rho_K^0(\alpha)}$$

$$F_K(\alpha) = \frac{C(\alpha-\mu)}{y_1'(\alpha)\alpha(\alpha+2ik)} + \int_{\mu}^{\alpha} \left[y_2'(\beta) - \frac{y_2'(\alpha)}{y_1'(\alpha)} y_1'(\beta) \right] \times C(\beta-\mu) d\beta$$

$$G_K(\gamma, \alpha) = \frac{y_1'(\gamma)C(\alpha-\gamma-\mu)}{y_1'(\alpha)\alpha(\alpha+2ik)} + y_1'(\gamma) \int_{\gamma+\mu}^{\alpha} \left[y_2'(\beta) - y_1'(\beta) \frac{y_2'(\alpha)}{y_1'(\alpha)} \right] C(\beta-\gamma-\mu) d\beta$$

Here we just state the result which will be proven in Appendix I: in the union of the domains,

$$|\operatorname{Arg} - ik| \geq \epsilon, \quad |k| > \frac{\mu}{2} - \epsilon$$

(this excludes the cut $i\frac{\mu}{2} - i\infty$).

$$\lim_{\alpha \rightarrow \infty} R_k(\alpha)$$

is a finite holomorphic quantity, and

$$\lim_{|k| \rightarrow \infty} \left[\lim_{\alpha \rightarrow \infty} R_k(\alpha) \right] = 1 \quad (16)$$

The proof of this result is based on upper and lower bounds of y_1, y'_1, y_2, y'_2 giving upper bounds of $F_k(\alpha)$ and $G_k(\gamma, \alpha)$.

V. Connection between the Solutions of Equations (15') and the scattering matrix.

We want to prove here that definitions (11) of the scattering matrix is equivalent to the following :

$$S(k) = \frac{\lim_{\alpha \rightarrow \infty} R_k(\alpha)}{\lim_{\alpha \rightarrow \infty} R_{-k}(\alpha)} \quad (17)$$

If we assume first that these two limits are different from zero, this means that

$$\rho_k(\alpha) \simeq c_k \alpha^{1-1}$$

$$\rho_{-k}(\alpha) \simeq c_{-k} \alpha^{1-1}$$

then it is not difficult to prove that

$$\lim_{r \rightarrow 0} r^1 \frac{1 + \int_0^\infty e^{-\alpha r} \rho_k(\alpha) d\alpha}{(1-1)! c_k} = 1$$

hence, from (11) :

$$S(k)S_0(k) = \frac{c_k}{c_{-k}} = \frac{\alpha \xrightarrow{\lim} \infty R_k(\alpha)}{\alpha \xrightarrow{\lim} \infty R_{-k}(\alpha)} \quad \alpha \rightarrow \infty \frac{\rho_k^0(\alpha)}{\rho_{-k}^0(\alpha)}$$

The last factor in this equation is easily seen to be $(-1)^1$, from equation (12), so that :

$$S_0(k) = (-1)^1$$

$$S(k) = \frac{\alpha \xrightarrow{\lim} \infty R_k(\alpha)}{\alpha \xrightarrow{\lim} \infty R_{-k}(\alpha)}$$

It remains to prove that whenever $S(k)$, as defined by equation (11) has a pole, this pole appears also in the new definition (17) ; it is shown in Appendix II that whenever

$$\lim_{\alpha \rightarrow \infty} R_{-k}(\alpha) = 0 ,$$

i.e. when

$$\lim_{\alpha \rightarrow \infty} \alpha^{-l+1} \rho_{-k}(\alpha) = 0 ,$$

$$\lim_{r \rightarrow 0} \left[1 + \int_0^{\infty} e^{-\alpha r} \rho_{-k}(\alpha) d\alpha \right] \equiv \lim_{r \rightarrow 0} f(r, -k) = 0$$

VI. Final Results

From sections IV and V, we obtain that $S(k)$ is an analytic function in the upper half plane with a cut going from $\frac{i\epsilon}{2}$ to $i\infty$, and poles corresponding to

$$\lim_{r \rightarrow 0} f(r, -k) = 0 .$$

The reasoning used in the S wave case to show that these poles correspond to bound states is still valid. At infinity, except

perhaps in the direction $\text{Re} k = 0$, $S(k)$ becomes equal to unity, so that it is possible to write a dispersion relation for the quantity $\frac{S(k)-1}{2i^2}$ by following the contour indicated in figure I in the k^2 complex plane (the contribution from the large circle vanishes).

Figure 1.

It is perhaps worthwhile to add one remark: the quantity $\lim_{\alpha \rightarrow \infty} R_{-k}(\alpha)$ has the phase of $e^{-i\delta(k)}$ for real k , and is regular in the whole upper half plane; its zeros correspond to bound states. Application of the Cauchy formula to the function

$$\frac{d}{dk} \left[\lim_{\alpha \rightarrow \infty} R_{-k}(\alpha) \right] / \lim_{\alpha \rightarrow \infty} R_{-k}(\alpha)$$

leads to the Levinson theorem

$$\delta(0) - \delta(\infty) = \pi \times \text{number of bound states.}$$

The properties derived here are essentially the same as those obtained in the S wave case. Of course these results could be derived from the Mandelstam representation of the total scattering amplitude ³⁾ by taking the average of this amplitude over the angles, conveniently weighted by a Legendre polynomial.

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APPENDIX I.

a) upper and lower bounds for y_1, y_1', y_2, y_2'

From equation (14)

$$y_1(\alpha) = \frac{1}{!} \left(\frac{d}{dX} \right)^! \left[X(X+1) \right]^1$$

where

$$X = \frac{\alpha}{2ik}$$

one sees that the roots of $y_1(\alpha) = 0$ lie on a straight line between $\alpha = 0$ and $\alpha = -2ik$. This is sufficient to show

$$|y_1(\alpha)| < \text{const.} \left[|X| + 1 \right]^1$$

and, in the same way

$$|y_1'(\alpha)| < \text{const} \frac{\left[|X| + 1 \right]^{1-1}}{2 \quad k}$$

Now, provided $\alpha > \mu$, it can be shown that if k lies inside the union of the domains $|\text{Arg } -ik| > \epsilon$, $|k| < \frac{\mu}{2} - \epsilon$,

which excludes the cut,

$$C_1 < \frac{\left| \frac{\alpha}{2ik} + X_0 \right|}{\left| \frac{\alpha}{2ik} \right| + 1} < C_2$$

if $0 < \epsilon < X_0 < 1 - \epsilon$.

So it is also possible to get lower bounds of $|y_1(\alpha)|$ and $|y_1'(\alpha)|$:

$$\text{Const} < \frac{|y_1|}{\sqrt{|X| + 1}} < \text{Const}$$

(I.2)

$$\text{Const} < \frac{2 |y_1| |k|}{\sqrt{|X| + 1}^{l-1}} < \text{Const},$$

in the above defined domain. By substitution in

$$y_2(\alpha) = y_1(\alpha) \int_{\alpha}^{\infty} \frac{d\beta}{\beta \sqrt{\beta + 2ik} y_1^2(\beta)}$$

and in $y_2'(\alpha)$, of these upper and lower bounds, one can prove

in the same way, in the same domain :

$$|y_2(\alpha)| < \frac{\text{Const.}}{\alpha \sqrt{|x|} + 1} \quad (I.3)$$

$$|y_2'(\alpha)| < \frac{\text{Const.}}{|2k|^2 \sqrt{|x|} + 1} \sqrt{|x|}$$

b) Upper bounds for the quantities entering in the integral equation (15').

Keeping in mind $|c(\alpha)| < M$ and using the bounds obtained in a), it is possible to get upper bounds for $F_k(\alpha)$ and $G_k(\gamma, \alpha)$. It must be noticed that $F_k(\alpha)$ is non-vanishing for $\alpha > \mu$ and that $G_k(\gamma, \alpha)$ is non-vanishing for $\alpha - \mu > \gamma > 0$. Then, by straightforward, but tedious calculations, one gets

$$|F_k(\alpha)| < \frac{\text{Const.}}{\sqrt{\frac{\mu}{2|k|}} + 1} \quad (I.4)$$

This is not always sufficient; a more complicated bound can be obtained

$$F_k(\alpha) < \frac{\text{Const.}}{|k| \sqrt{\frac{\mu}{2|k|}} + 1} + \frac{\text{Const.}}{\sqrt{\frac{\alpha}{2|k|}} + 1} \quad (I.5)$$

which means that

$$\lim_{\substack{|k| \rightarrow \infty \\ \alpha \rightarrow \infty}} F_k(\alpha) = 0$$

inside the domain excluding the cut.

It can also be shown :

$$|G_k(\gamma, \alpha)| < \frac{\text{Const.}}{(\gamma + \mu)(\gamma + \mu + 2|k|)} \quad (\text{I.6})$$

c) Existence and analyticity of $\lim_{\alpha \rightarrow \infty} [R_k(\alpha)]$ outside the cut.

The bounds obtained in b) will be used first to prove that $R_k(\alpha)$ is uniformly bounded, for any α , in the domain already defined for k . From b), one sees that

$$|R_k(\alpha)| < C_1 + \int_0^{\alpha - \mu} \frac{R_k(\gamma) C_2}{(\gamma + \mu)^2} d\gamma$$

where C_1 is a constant larger than unity.

So $R_k(\alpha)$ is bounded by

$$B(\alpha) = C_1 + C_2 \int_0^{\alpha-\mu} \frac{B(\gamma) d\gamma}{(\gamma+\mu)^2}$$

which is an increasing function of α . Then it can be shown as in A that

$$B(\alpha) = C_1 + C_2 \int_0^{\infty} \frac{B(\alpha) d\alpha}{(\alpha+\mu)^2} < \frac{C_1 + C_2 \int_0^{\alpha_1} \frac{B(\alpha) d\alpha}{(\alpha+\mu)^2}}{1 - C_2 \int_{\alpha_1}^{\infty} \frac{d\alpha}{(\alpha+\mu)^2}}$$

with $\alpha_1 + \mu > C_2$. This establishes that $R_k(\alpha)$ is uniformly bounded.

Now it is not difficult to establish that the right hand side of equation (15') has a finite limit which is given by :

$$\lim_{\alpha \rightarrow \infty} R_k(\alpha) = 1 + \int_{\mu}^{\infty} y_2'(\beta) c(\beta) d\beta + \int_0^{\infty} R_k(\gamma) y_1'(\gamma) d\gamma \int_{\gamma+\mu}^{\infty} y_2'(\beta) c(\beta-\gamma) d\beta \quad (I.7)$$

The integrals appearing in (I.7) converge uniformly with respect

to k in the domain excluding the cut; since $R_k(\alpha)$, for any finite α can be obtained by a finite number of integrations over holomorphic functions, $\lim_{\alpha \rightarrow \infty} R_k(\alpha)$ is holomorphic in this domain. Using again the bounds (I.2), (I.3), one can show by substitution in (I.7) that

$$\lim_{|k| \rightarrow \infty} \left[\lim_{\alpha \rightarrow \infty} R_k(\alpha) \right] = 1$$

outside the cut. Further, one can see that this limit is reached faster than $\log |k| / |k|$.

APPENDIX II.

We want to prove that whenever $\lim_{\alpha \rightarrow \infty} R_k(\alpha) = 0$

$$\lim_{r \rightarrow 0} 1 + \int_0^{\infty} e^{-\alpha r} \rho_k(\alpha) d\alpha = \lim_{r \rightarrow 0} f(r, k) = 0$$

One starts from the original equation (10), rather than from the transformed equation (15). Assume for ρ the asymptotic behaviour

$$c_k \alpha^p$$

Then, inserting in equation (1), one sees that if $p > -1$ the right hand side is equivalent to

$$c_k \frac{1(1+1)}{(p+1)(p+2)} \alpha^{p+2}$$

while the left hand side is equivalent to

$$c_k \alpha^{p+2}$$

Hence, either $p = -1$ or $p < -1$. So if $\lim_{\alpha \rightarrow \infty} R_k(\alpha) = 0$, $\rho_k(\alpha)$ goes to zero as $\alpha \rightarrow \infty$ at least as fast as α^{-1} .

The case $p = -1$ can be easily excluded by inserting the asymptotic $\rho_k(\alpha)$ in equation (10). If $p < -1$, the left hand side behaves like $\alpha^{1-\epsilon}$ while the right hand side is dominated by the term

$$\alpha^{1(1+1)} \left[1 + \int_0^\infty \rho_k(\beta) d\beta \right].$$

This means that

$$1 + \int_0^\infty \rho_k(\beta) d\beta = 0,$$

and so

$$\lim_{r \rightarrow 0} f(r, k) = 0.$$

R E F E R E N C E S

- 1) A. Martin, Preprint CERN/6992/Th/37, to be published in Nuovo Cimento.
- 2) Magnus and Oberettringer, "Formulae and Theorems of the functions of Mathematical Physics", Chelsea Publ. Co. (1954), pp. 19, 22, 122, 124.
- 3) T. Regge, Preprint,
R. Blankenbeckler, M.L. Goldberger, N.N. Khuri,
S.B. Treiman, Preprint.