Decay of the Charged Pion

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The amplitude for pion decay is related to quantities involved in the meson propagator under the strong π coupling alone. For pseudovector π coupling the Goldberger and Treiman result follows under certain assumptions, but without an expansion in intermediate states. For pseudoscalar π coupling, similar, but somewhat inconsistent, results are obtainable by means of an equivalence relation applied to the weak interaction. An expansion in intermediate states of the absorptive part of the decay amplitude is related to a similar expansion for the meson propagator and in an approximation yields the Goldberger and Treiman result when states containing one nucleon pair and some number of pions are included.

1. INTRODUCTION

THERE are different points of view on the question of whether the charged pion decay can in principle be calculated from ordinary beta decay and pion-nucleon interactions. In conventional renormalization theory, based on a perturbation expansion, a new coupling constant must be measured for each new type of primitively divergent process. In this case one would need both direct pion-decay and beta-decay terms in the Lagrangian, and the pion-decay rate may not be calculated in terms of g_A . This point of view has recently been emphasized by Weinberg.¹

Another approach is to try at once to circumvent the difficulties of strong couplings and of new divergences (or the difficulties of bare coupling constants) by directly comparing processes via dispersion relations. This will leave questions of universality (e.g., $g_V = g_A$) unanswerable but could allow comparison of pion decay with the axial vector contribution to beta decay. This is the method of Goldberger and Treiman² in their recent treatment of charged pion decay. They obtain a result which is both simple and in impressive agreement with experiment. However, their result rests on several questionable assumptions, some of which will be mentioned below.

There is a third viewpoint possible which avoids the multiplicity of measured constants in the renormalization theory and the abolition of bare coupling constants in the dispersion theory. This is the assumption of a real cutoff which is not allowed to approach infinity at the end of a calculation. As much as possible of this cutoff is to be absorbed in the measured constants of the theory; the remainder appears in the results. In this case it is not necessary to introduce a primal direct π -decay coupling into the theory. The penalty for not doing so is that a perturbation calculation of pion decay yields a logarithmic dependence on the cutoff.³ Nevertheless there is some hope that in a better approximation the amplitude for this process will be

convergent when expressed in terms of the physical pion-nucleon and beta-decay coupling constants.

In the present work a close relation between the pion-decay process and a meson self-energy process will be exploited to write the transition amplitude for the decay in terms of quantities involved in the meson propagator (under the strong couplings alone). The approach is similar to that recently taken by Symanzik⁴ and by Norton and Watson⁵ in considering the same problem.

2. PSEUDOVECTOR PION COUPLING

In the dispersion theory approach, of course, there is no distinction between pseudoscalar (PS) and pseudovector (PV) coupling in the π -nucleon interaction. We shall begin by considering PV since it is here that one finds the closest analogy between pion decay and the meson self-energy process. The fact that a perturbation expansion is not renormalizable for the PV theory is not particularly relevant here because perturbation theory will not be used and because our theory without counter-terms for various weak processes is nonrenormalizable in any event.

The weak interaction involved in π^+ decay will be taken as

$$\mathcal{L}_w = g_A^0 (\bar{\psi}_n i \gamma_5 \gamma_\mu \bar{\psi}_p) (\psi_p \gamma_\mu (1 + i \gamma_5) \psi_n) + \text{H.c.} \quad (1)$$

The S matrix element for the decay may be written as

$$S = (2\pi)^4 \delta^4(p^\pi - p^\mu - p^\nu) (m_\mu / p_0^\mu)^{\frac{1}{2}} (2p_0^\pi p_0^\nu)^{-\frac{1}{2}} \times F(-\mu^2) p_\alpha \bar{u}_\nu \gamma_\alpha (1 + i \gamma_5) u_\mu, \quad (2)$$

where $F(k^2)$ is defined from^{2,6}

$$i \delta_{lm} F(k^2) k_\mu = i Z_3^{-\frac{1}{2}} \int d^4x e^{-ik \cdot x} \theta(x_0) \times \left\langle 0 \left| \left[\frac{g_A^0}{\sqrt{2}} \bar{\psi}(x) i \gamma_\mu \gamma_5 \tau_l \psi(x), J_m(0) \right] \right| 0 \right\rangle. \quad (3)$$

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¹ S. Weinberg, Phys. Rev. **106**, 1301 (1957).

² M. L. Goldberger and S. B. Treiman, Phys. Rev. **110**, 1178 (1958). Hereafter referred to as TG.

³ S. B. Treiman and H. W. Wyld, Phys. Rev. **101**, 1552 (1956).

⁴ K. Symanzik, Nuovo cimento **11**, 269 (1959).

⁵ R. E. Norton and W. K. R. Watson, Phys. Rev. **110**, 996 (1958).

⁶ M. L. Goldberger and S. B. Treiman, Phys. Rev. **111**, 354 (1958).

The fields here are unrenormalized.⁷ $Z_3^{-1/2}$ is the wave function renormalization for the incoming pion. J_m is the pion current,

$$J_m = (\square - \mu^2) \varphi_m = \frac{g_0}{2M} \partial_\mu (\bar{\psi} i \gamma_\mu \gamma_5 \tau_m \psi) + (\mu_0^2 - \mu^2) \varphi_m. \quad (4)$$

It has been assumed that there is no pion-pion coupling term; there need not be in an unrenormalizable theory.

If a sufficiently strong cutoff has been introduced we may expect $F(k^2) \rightarrow 0$ as k^2 approaches infinity, there being no direct π -decay coupling.⁴ A further discussion of the possible behavior of $F(k^2)$ at infinity is given in Appendix A.

Assuming the above leads to a Lehmann representation for $F(k^2)$ of the form

$$F(-\mu^2) = -\frac{1}{\pi} \int_{(3\mu)^2}^{\infty} da^2 \frac{\text{Im} F(-a^2)}{a^2 - \mu^2}. \quad (5)$$

Now we define another function, $P(k^2)$, which is related to the meson propagator,

$$\delta_{lm} P(k^2) = i \int d^4x e^{-ik \cdot x} \theta(x_0) \langle 0 | [J_l(x), J_m(0)] | 0 \rangle. \quad (6)$$

The imaginary part of $P(k^2)$ is to be related to the imaginary part of $F(k^2)$,

$$\begin{aligned} \delta_{lm}(k^2 + \mu_0^2) \text{Im} P(k^2) \\ = (k^2 + \mu^2) \text{Im} \left\{ i \int d^4x e^{-ik \cdot x} \theta(x_0) \right. \\ \left. \times \langle 0 | [(\square - \mu_0^2) \varphi_l(x), J_m(0)] | 0 \rangle \right\}, \quad (7) \end{aligned}$$

where μ_0^2 is the bare mass of the pion. We have

$$(\square - \mu_0^2) \varphi(x) = \frac{g_0}{2M} \partial_\mu (\bar{\psi} i \gamma_\mu \gamma_5 \tau_m \psi). \quad (8)$$

Inserting this in (7) and comparing with (3) yields

$$\text{Im} F(k^2) = - \left(\frac{2}{Z_3} \right)^{1/2} \frac{M g_A^0}{g_0} \frac{k^2 + \mu_0^2}{k^2 + \mu^2} \frac{\text{Im} P(k^2)}{k^2}. \quad (9)$$

We note in addition that the imaginary part of $P(k^2)$ is related to the density, $\rho(k^2)$, in the Lehmann representation of the unrenormalized meson propagator $\Delta(k^2)$,

$$\text{Im} P(k^2) = (k^2 + \mu^2)^2 \text{Im} \Delta(k^2) = (k^2 + \mu^2)^2 \rho(k^2). \quad (10)$$

Substituting (9) in (5) gives

$$F(-\mu^2) = \left(\frac{2}{Z_3} \right)^{1/2} \frac{M g_A^0}{g_0} \frac{1}{\pi} \int_{(3\mu)^2}^{\infty} \frac{a^2 - \mu^2}{a^2} \rho(-a^2) da^2. \quad (11)$$

⁷ Although $[\dot{\varphi}(x), \psi(x')]_{t=t'} \neq 0$ in the PV pion-nucleon theory, there is no equal-time contribution to Eq. (3).

Noting that

$$\frac{1}{\pi} \int_{(3\mu)^2}^{\infty} \rho(-a^2) da^2 = 1 - Z_3,$$

we have

$$F(-\mu^2) = 2^{1/2} \frac{g_A}{g} M \left(1 - Z_3 - \frac{1}{\pi} \int_{(3\mu)^2}^{\infty} da^2 \frac{\mu_0^2}{a^2} \rho(-a^2) \right). \quad (12)$$

Here we have set $Z_2(Z_3)^{1/2} Z_1^{-1} g_0 = g$, the pseudoscalar π - N coupling constant and we have put $Z_2 Z_1^{-1} g_A^0 = g_A$. That is, the axial vector coupling constant has been defined for an unphysical beta decay with a nucleon momentum transfer corresponding to a pion emission instead of with zero momentum transfer. Presumably this is still not far from the measured value of g_A .⁸

The first term in (12) is the numerically correct TG result²

$$F(-\mu^2) = 2^{1/2} M g_A / g. \quad (13)$$

We discard the second term since we expect $0 < Z_3 \ll 1$. The last term of (12) is difficult to discuss. In a theory in which $\langle 0 | [\dot{\varphi}(x), (\square - \mu_0^2) \varphi(x')] | 0 \rangle_{t=t'} \neq 0$ we may not relate the bare boson mass back to the density ρ .⁸ The above commutator is not zero in the PV theory, even without a meson-meson coupling. However, μ_0^2 may still be well-defined. The validity of neglecting the last term in (12) evidently depends on the bare meson mass being less than the physical masses giving the greatest contribution to the integral, $\int da^2 a^{-2} \rho(-a^2)$. This is not unreasonable, particularly since states containing nucleon pairs should be most important in this sum.

3. PSEUDOSCALAR PION COUPLING

In view of the equivalence relation one might expect a similar situation in the case of PS pion-nucleon coupling. We utilize instead an analogous equivalence relation for the weak coupling which is proved in Appendix B and reference 4. It enables Eq. (3) to be written, to the lowest order in the weak coupling,

$$\begin{aligned} \delta_{lm} F(k^2) = (2Z_3)^{-1/2} \frac{g_A^0}{k^2} i \int d^4x e^{-ik \cdot x} \theta(x_0) \\ \{ 2M_0 \langle 0 | [\bar{\psi}(x) \gamma_5 \tau_l \psi(x), J_m(0)] | 0 \rangle \\ + 2g_0 \langle 0 | [\bar{\psi}(x) \psi(x) \varphi_l(x), J_m(0) J | 0] \rangle \}, \quad (14) \end{aligned}$$

where M_0 is the bare mass of the nucleon. If the last term in (14) is ignored, and again there is no pion-pion interaction, one obtains Eq. (9) with M replaced by M_0 .

Actually we have neglected only the imaginary part of the second term on the right-hand side of (14). Symanzik⁴ has used an expression equivalent to the first term of Eq. (14), evaluated directly at $k^2 = -\mu^2$. This is not a result equivalent to our Eq. (11) because

⁸ H. Lehmann, Nuovo cimento 11, 342 (1954).

in writing the representation (5) we use the fact that $F(k^2)$ is regular at $k^2=0$, a condition not satisfied by the first term of (14) alone.

In proceeding from Eq. (11) to the result (12) we are no longer justified in taking the Z_1/Z_2 factor involved in the π -coupling renormalization as equal to that involved in the definition of g_A . However, one might take the experimental equality of beta-decay and μ -decay coupling constants as an indication that such $Z_2Z_1^{-1}$ factors tend to be equal to one.

An additional difficulty with (12) for the pseudoscalar case is that in the absence of a pion-pion term in the Lagrangian we may expect μ_0^2 to be given in terms of the density ρ by⁸

$$\mu_0^2 = Z_3\mu^2 + \int_{(3\mu)^2}^{\infty} da^2 \rho(-a^2)a^2. \quad (15)$$

In this case it is not consistent to assume that both Z_3 and the integral

$$\frac{1}{\pi} \int_{(3\mu)^2}^{\infty} \frac{\mu_0^2}{a^2} \rho(-a^2) da^2$$

are small compared to one. In fact it can be shown from (12) and (15) that $F(-\mu^2) < \frac{1}{4}(2^{\frac{1}{2}})Z_3^{-\frac{1}{2}}M_0g_A^0g_0^{-1}$ instead of $F(-\mu^2) \simeq 2^{\frac{1}{2}}Z_3^{-\frac{1}{2}}M_0g_A^0g_0^{-1}$ according to our earlier assumption. If the pion-pion term is present, perhaps (13) still has some validity.

To estimate the effect of the second term of the right-hand side of Eq. (14) and to obtain expressions in terms of the observed masses and coupling constants we now resort to insertion of intermediate physical states into the expressions for $\text{Im}F(k^2)$ and $\text{Im}P(k^2)$. From (3) and (6) are obtained

$$\delta_{lm}ik_\mu \text{Im}F(k^2) = Z_3^{-\frac{1}{2}}2^{-\frac{1}{2}}\pi \sum_I \delta^4(p_I - k) \times \langle 0 | g_A^0 \bar{\psi} i\gamma_\mu \gamma_5 \tau \psi | I \rangle \langle I | J_m(0) | 0 \rangle, \quad (16)$$

and,

$$\delta_{lm} \text{Im}P(k^2) = \pi \sum_I \delta^4(p_I - k) \times \langle 0 | J_l(0) | I \rangle \langle I | J_m(0) | 0 \rangle. \quad (17)$$

We consider first the nucleon pair state only. We tentatively make the approximation of replacing the matrix elements of the renormalized currents by the coupling constants alone, in spite of the huge nucleon momentum transfers involved,

$$Z_3^{-\frac{1}{2}} \langle 0 | J_l(0) | N\bar{N} \rangle = g \bar{u}_N \gamma_5 \tau_l u_N, \quad (18)$$

$$\langle 0 | g_A^0 \bar{\psi} i\gamma_\mu \gamma_5 \tau \psi | N\bar{N} \rangle = i g_A \bar{u}_N \gamma_\mu \gamma_5 \tau_l u_N. \quad (19)$$

When substituted in (16) and (17), these lead to the result

$$\frac{\text{Im}F(k^2)}{\text{Im}P(k^2)} = -\frac{2^{\frac{1}{2}}M}{k^2} \frac{g_A}{g} Z_3^{-1}. \quad (20)$$

The representation (5) now gives

$$F(-\mu^2) = 2^{\frac{1}{2}}M \frac{g_A}{g} Z_3^{-1} \int_{(3\mu)^2}^{\infty} da^2 \frac{\text{Im}P(-a^2)}{a^2(a^2 - \mu^2)} \\ = 2^{\frac{1}{2}}M \frac{g_A}{g} Z_3^{-1} (1 - Z_3). \quad (21)$$

In the last step we have used (10) and the fact that $\int \mu^2 a^{-2} \rho(-a^2) da^2 \ll 1$. Equation (21) differs from (13) by the presumably large factor Z_3^{-1} .

In TG an additional term in $\langle 0 | \bar{\psi} i\gamma_\mu \gamma_5 \tau \psi | N\bar{N} \rangle$ is obtained by writing a new dispersion relation for this quantity. The expression (19) is the subtraction term in this relation and the extra term is obtained from an intermediate one-pion state. This extra term then manages almost to cancel the coupling-constant term when inserted into the dispersion relation (5). This exactly removes what in our treatment is the Z_3^{-1} factor [which is TG's quantity $1 + \frac{1}{2}(G/\pi)^2 J$]. We also may find such a damping effect with an elementary approximation as in (19) if we take better account of polarization processes on the external pion line. For the real pion decay these contribute only a renormalization constant, which has already been taken into account. But for the unphysical momenta involved in the dispersion relation there are additional effects.

All our various expectation values of Heisenberg field operators may be expressed in terms of appropriate S matrix elements, $\langle \alpha_{\text{out}} | \beta_{\text{in}} \rangle$. These in turn may be calculated by perturbation theory in the interaction representation. We shall define modified matrix elements with a subscript P (for proper) to indicate the contribution from Feynman diagrams not containing a single intermediate pion line. Then we may write (3) as

$$k_\mu F(k^2) = Z_3^{-\frac{1}{2}}(k^2 + \mu^2) \Delta(k^2) k_\mu M(k^2), \quad (22)$$

where

$$i\delta_{lm} k_\mu M(k^2) = i \int d^4x e^{-ik \cdot x} \theta(x_0) \times \langle 0 | [2^{-\frac{1}{2}} g_A^0 \bar{\psi}(x) i\gamma_\mu \gamma_5 \tau \psi(x), J_m(0)] | 0 \rangle_P. \quad (23)$$

$\Delta(k^2)$ is written in terms of proper contributions as

$$\Delta(k^2) = [k^2 + \mu^2 - \Pi(k^2)]^{-1}, \quad (24)$$

and we have

$$\delta_{lm} \text{Im}\Pi(k^2) = \text{Im} \left\{ i \int d^4x e^{-ik \cdot x} \theta(x_0) \langle 0 | [J_l(x), J_m(0)] | 0 \rangle_P \right\}. \quad (25)$$

From (22), (24) and the renormalization requirement,

$$\Pi(k^2) = (1 - Z_3^{-1})(k^2 + \mu^2) + Z_3^{-1}(k^2 + \mu^2)^2 R(k^2), \quad (26)$$

we obtain

$$F(k^2) = Z_3^{\frac{1}{2}} \frac{M(k^2)}{1 - (k^2 + \mu^2)R(k^2)}, \quad (27)$$

where

$$\delta_{im} \text{Im} R(k^2) = Z_3 \frac{1}{(k^2 + \mu^2)^2} \text{Im} \left\{ i \int d^4x e^{-ik \cdot x} \theta(x_0) \right. \\ \left. \times \langle 0 | [J_i(x), J_m(0)] | 0 \rangle_P \right\}. \quad (28)$$

We are going to find approximations in which

$$\text{Im} M(k^2) \simeq C k^2 \text{Im} R(k^2), \quad (29)$$

where C is a constant. Before discussing these approximations we note that the dispersion relation (5) for $F(-\mu^2)$ may now be written, using (27), as

$$F(-\mu^2) = Z_3^{\frac{1}{2}} \int_{(3\mu)^2}^{\infty} \frac{da^2}{a^2 - \mu^2} \frac{1}{|1 + (a^2 - \mu^2)R(-a^2)|^2} \\ \times \{ \text{Im} M(-a^2) + (a^2 - \mu^2) [\text{Re} R(-a^2) \\ \times \text{Im} M(-a^2) - \text{Im} R(-a^2) \text{Re} M(-a^2)] \}. \quad (30)$$

With the aid of (27), (29), and the relations

$$\text{Re} R(k^2) = -\frac{P}{\pi} \int_{(3\mu)^2}^{\infty} \frac{da^2}{a^2 + k^2} \frac{\text{Im} R(-a^2)}{a^2 + k^2}, \quad (31)$$

and

$$\text{Re} M(k^2) = M(-\mu^2)$$

$$-\frac{1}{\pi} (k^2 + \mu^2) P \int_{(3\mu)^2}^{\infty} \frac{da^2}{(a^2 + k^2)(a^2 - \mu^2)} \frac{\text{Im} M(-a^2)}{(a^2 + k^2)(a^2 - \mu^2)}, \quad (32)$$

one obtains

$$F(-\mu^2) = Z_3^{\frac{1}{2}} \int_{(3\mu)^2}^{\infty} \frac{da^2}{|1 + (a^2 - \mu^2)R(-a^2)|^2} \\ \times [-C - Z_3^{-\frac{1}{2}} F(-\mu^2)]. \quad (33)$$

In obtaining this expression μ^2 has been neglected compared to a^2 where necessary. Equation (33) may be written

$$F(-\mu^2) = Z_3^{-\frac{1}{2}} \int_{(3\mu)^2}^{\infty} \frac{da^2}{\pi} \\ \times \text{Im} \Delta(-a^2) [-C - Z_3^{-\frac{1}{2}} F(-\mu^2)]. \quad (34)$$

This has the solution

$$F(\mu^2) = -Z_3^{\frac{1}{2}} (1 - Z_3) C. \quad (35)$$

It remains to establish (29) and determine the constant C . From (23) and (28), we have

$$\delta_{im} \text{Im} R(-\mu^2) = Z_3 (k^2 + \mu^2)^{-2} \pi^{-1} \sum_I \delta^4(k - p_I) \\ \times \langle 0 | J_i(0) | I \rangle_P \langle I | J_m(0) | 0 \rangle_P, \quad (36)$$

and

$$i \delta_{im} k_\mu \text{Im} M(k^2) \\ = \pi^{-1} \sum_I \delta^4(k - p_I) \langle 0 | 2^{-\frac{1}{2}} g_A \bar{\psi}(0) i \gamma_\mu \gamma_5 \tau_i \psi(0) | I \rangle_P \\ \times \langle I | J_m(0) | 0 \rangle_P. \quad (37)$$

It is clear here that in a perturbation evaluation of the matrix elements, e.g., in (36), $\langle 0 | J(0) | I \rangle$, any nonproper contribution corresponds to a nonproper contribution to the left-hand side; hence the subscript P on the intermediate matrix elements.

Again we consider first the nucleon pair state and make a coupling-constant estimate. For the pion current term an extra factor of Z_3^{-1} must be included to compensate for the lack of processes on the external pion line,

$$\langle 0 | J_i(0) | N\bar{N} \rangle_P \simeq Z_3^{-\frac{1}{2}} \langle 0 | Z_3^{-\frac{1}{2}} J_i(0) | N\bar{N} \rangle \\ = Z_3^{-\frac{1}{2}} g \bar{u} \gamma_5 \tau_i u. \quad (38)$$

There are no one pion processes involved in the coupling constant for beta decay, so we write,

$$\langle 0 | g_A \bar{\psi} i \gamma_\mu \gamma_5 \tau_i \psi | N\bar{N} \rangle_P = g_A \bar{u} \gamma_5 \tau_i u. \quad (39)$$

When substituted in (36) and (37), these expressions yield the result

$$\frac{\text{Im} M(k^2)}{\text{Im} R(k^2)} \simeq - \left(\frac{2}{Z_3} \right)^{\frac{1}{2}} \frac{g_A M}{g} k^2. \quad (40)$$

From (35), we finally obtain

$$F(-\mu^2) = (1 - Z_3) 2^{\frac{1}{2}} g_A M / g, \quad (41)$$

which is again the TG result [with $Z_3^{-1} = 1 + \frac{1}{2}(G/\pi)^2 J]$, derived by an equivalent method.

It is hoped to establish (40) and hence (41) for a more inclusive class of intermediate states. Indeed, the equivalence (14) suggests that this is possible. We have considered intermediate states which contain one pair and some number of pions. A perturbation treatment similar to (38) and (39) has been followed. That is, the lowest order result with renormalized couplings has been used for the quantities,

$$\langle N\bar{N} + \text{pions} | \bar{\psi} i \gamma_\mu \gamma_5 \tau_i \psi | 0 \rangle_P$$

and

$$\langle N\bar{N} + \text{pions} | J_i(0) | 0 \rangle_P.$$

As explained above, an extra Z_3^{-1} must be included in the second of these expressions. The result of this perturbation treatment is the identical ratio (40) for the partial contribution to M and R of every state with nonrelativistic nucleon momentum and with pion momentum $|k| \ll M$. This is independent of the form of the common factor in the definition of M and R , $\langle 0 | J(0) | N\bar{N} + \text{pions} \rangle$.⁹

⁹ It is not unreasonable that the two factors in the expansion in intermediate states should be treated differently, since only one vertex is to be corrected in a boson self-energy part.

Now a feature of the TG treatment is that in the sum $\sum \langle 0 | \bar{\psi} i \gamma_\mu \gamma_5 \tau \psi | N \bar{N} \rangle \langle N \bar{N} | J | 0 \rangle$ the second factor is damped at high $N \bar{N}$ relative momentum by a unitarity condition. We may hope that a similar damping for high particle momenta will occur in the more complicated states. If so, the ratio (40) and the result (41) may be substantially unaltered when the complete contributions of the states containing one pair plus pions are included.

Thus the Goldberger and Treiman assumption of the dominance of the $N \bar{N}$ state may be unnecessary. Their second assumption, essentially that one can use a kind of perturbation theory for one of the factors in an expansion in states, we have retained. Their additional assumption that an integral J depending on the nucleon-antinucleon phase shift is large has here been replaced by the assumption that Z_3 is small.

4. INCLUSION OF HYPERONS

Hyperon interactions can hardly be irrelevant to the pion decay problem. Even if there were no weak hyperon-lepton coupling, π -hyperon interactions would contribute to the meson self-energy process and hence to our result. Here we note only that if one has a globally symmetric π coupling^{10,11} and if one generalizes the β and μ decay couplings to hyperon pairs in complete analogy with the π coupling,¹¹ then one obtains exactly our above results. This is because in all ratios such as that of M/R in Eq. (40) the particle multiplicity increases the numerator by the same factor as the denominator.

APPENDIX A

From the definition (3) we may establish for pseudo-vector pion coupling the relation,

$$k^2 F(k^2) = 2^{\frac{1}{2}} M Z_3^{-1} g_0^A g_0^{-1} \times [(k^2 + \mu^2)(k^2 + \mu_0^2) \Delta_R(k^2) - (k^2 + \mu^2)], \quad (\text{A1})$$

where $\Delta_R(k^2)$ is the retarded meson propagator. For PS pion coupling the same result is obtained as an approximation by neglecting the second term in Eq.

(14). In either case we expect⁸

$$1 = Z_3 + Z_3 \int_{(3\mu)^2}^{\infty} \rho_c(-a^2) da^2, \quad (\text{A2})$$

where

$$\Delta(k^2) = Z_3 \left(\frac{1}{k^2 + \mu^2} + \int_{(3\mu)^2}^{\infty} da^2 \frac{\rho_c(-a^2)}{k^2 + a^2 - i\epsilon} \right),$$

provided the integral in (A2) exists. If in addition the integral $\int a^2 \rho(-a^2) da^2$ exists, we may conclude from (A1) that

$$\lim_{k^2 \rightarrow \infty} k^2 F(k^2) = \text{const},$$

which is sufficient for the existence of the representation (5).

APPENDIX B

Consider a weak derivative coupling to a fictitious scalar meson field χ_i ,

$$\mathcal{L}_w = g_w \bar{\psi} i \gamma_\mu \gamma_5 \tau_i \psi \partial_\mu \chi_i. \quad (\text{B1})$$

To first order in g_w we have

$$\begin{aligned} \langle \chi_i | \pi_j \rangle &= (4\omega_\chi \omega_\pi)^{-\frac{1}{2}} (2\pi)^4 i k_\mu \delta^4(p_\chi - k) \\ &\times \int d^4x e^{-ik \cdot x} \theta(x_0) \\ &\times \langle 0 | [g_w \bar{\psi}(x) i \gamma_\mu \gamma_5 \tau_i \psi(x), J_j(0)] | 0 \rangle. \end{aligned} \quad (\text{B2})$$

We may make a canonical transformation by adding to $\mathcal{L}$ the divergence $-\partial_\mu [g_w \bar{\psi} i \gamma_\mu \gamma_5 \tau_i \psi \chi_i]$. The new weak interaction may be written, after using the field equation for ψ (in the presence of the π field) as

$$\mathcal{L}_w' = 2M_0 g_w \bar{\psi} \gamma_5 \tau_i \psi \chi_i + 2g_w g_0 \bar{\psi} \psi \chi_i \varphi_i. \quad (\text{B3})$$

An alternative form for (B2) is thus

$$\begin{aligned} \langle \chi_i | \pi_j \rangle &= (4\omega_\chi \omega_\pi)^{-\frac{1}{2}} (2\pi)^4 \delta^4(p_\chi - k) \\ &\times \int d^4x e^{-ik \cdot x} \theta(x_0) \\ &\times \langle 0 | [2M_0 g_w \bar{\psi}(x) \gamma_5 \tau_i \psi(x), J_j(0)] \\ &+ [2g_w g_0 \bar{\psi}(x) \psi(x) \varphi_i(x), J_j(0)] | 0 \rangle. \end{aligned} \quad (\text{B4})$$

Comparison of (B2) and (B4) yields the result (14).

¹⁰ M. Gell-Mann, Phys. Rev. **106**, 1296 (1957).

¹¹ J. Schwinger, Ann. Phys. (N. Y.) **2**, 407 (1957).