Equations of Motion for Renormalized Fields

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ABSTRACT

Solutions to conventional equations of motion are generally well-defined only in the "wrong" Hilbert space with all anti-particle states filled. The meaningful quantity which can be renormalized is a Wick product of some functional of the physical particle operators. When the field theory has infinite wave function renormalization, there is no direct connection between the two representations for the field operators. An example is given for which the two forms have different space-time developments.

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I. Introduction

Several problems are encountered in any attempt to determine equations of motion for renormalized field quantities. It has been noted that \((\psi^R)^2\) or \((\psi^*R)^2\) \((\psi^R = Z^{-\frac{1}{2}}\psi)\) must be of order \(Z^{-1}\) since \(\psi^*\psi\) is fixed by the commutation relations to be of order unity. Whenever \(Z^{-1}\) is infinite, the equations of motion will therefore contain undefined operator products if everything is expressed in terms of renormalized fields. Some authors have proposed that this effect may be treated by averaging over small time intervals\(^1,2\), and by using special limiting procedures to define the products\(^2,3\).

In this note we wish to point out another, quite distinct, difficulty which is usually present when \(Z\) vanishes. In these cases, the solutions to the non-renormalized equations of motion for spinor fields are undefined in the Hilbert space containing the true vacuum state. Well-defined bare particle fields must be Wick-ordered products of the physical particle operators. When \(Z = 0\), these ordered expressions do not necessarily obey the "renormalized" equations derived from the original equations of motion. This is indeed found to be true for the Thirring model where the ordered quantity satisfies a non-local integral equation. No such difficulties arise for the neutral scalar meson theory with fixed source, although they do appear in the Lee model and should be present for realistic theories.

II. The Ordering Problem

Consider a system of \(k\) spinor fields with an interaction which is hermitian, local and covariant. The invariant Lagrangian, \(L(\overline{\psi}_i, \psi_i, \lambda_i), \ i = 1 \ldots k\), leads to the equation of motion

\[
(\partial^\mu \gamma_\mu + m_i) \psi_i + \int_i (\overline{\psi}_j, \psi_j, \lambda_j) = 0,
\]

(1)
and its hermitian conjugate. The Lagrangian also admits the constants of motion \( P_\nu (\overline{\phi}_i, \psi_i, \lambda_i) \) and \( N_i = \int d^3x \, j_i^{(0)} (\overline{\phi}_i, \psi_i, \lambda_i) \). If any of the \( m_i \) are zero, the corresponding pseudo-currents are conserved, and symmetry properties of the Lagrangian under isotopic rotations may lead to additional constants of motion.

The problem is solved by defining physical particle fields, \( \phi_i, \overline{\phi}_i \), such that \( (i \gamma^\mu \partial_\mu + m_i) \phi_i = 0 \) \((m_i \neq m_i)\), and

\[
P_\nu (\overline{\phi}_i, \psi_i, \lambda_i) = P_\nu^{(0)} (\overline{\phi}_i, \psi_i) + \text{const.}
\]

where \( P^{(0)} \) means all \( \lambda_i = 0 \). A particular solution of Eq. (1) is then written as

\[
\psi_i = \phi_i \frac{\partial_i}{\partial^i} (\lambda_j, \overline{\phi}_j, \phi_j, \chi_i)
\]

so that \( \psi_i \rightarrow \psi_i^{\text{in}} = \phi_i \) as \( t \rightarrow -\infty \). Finally, free field anti-commutation relations are imposed on the \( \phi \) operators. The state vectors are eigenstates of \( P_\nu \) and any of the other constants of motion whose operator representatives commute with \( P_\nu \). The entire discussion may also be carried out with the modified Lagrangian

\[
L' = :L: = L + \text{const.}
\]

This leads to \( P_\nu' = :P_\nu: \), etc., but it yields the same equation of motion for \( \psi_i \), and the same functional solution, \( \phi_i \).

When \( \psi_i \) is decomposed as

\[
\phi_i = (2\pi)^{-3/2} \int d^3p \, (m_i / \omega_i)^{1/2} \int d(\rho) \, a_i(\rho) e^{i\rho x} + \nu(\rho) b_i^*(\rho) e^{-i\rho x}
\]

so that

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\]
a Hilbert space is defined by \( a^*_i |0\rangle = b^*_i |0\rangle = 0 \). Then, since the \( P_\varphi \) (or \( P'_\varphi \)) are polynomials in \( (\bar{\varphi} F \bar{\varphi})_i \), and since \( \bar{\varphi}^*_i |0\rangle = 0 \), possible eigenstates of \( P \) are \( |\alpha, \beta\rangle = (a^*_i)^\alpha (b^*_i)^\beta |0\rangle \). In this Hilbert space \( N_{\lambda} \) and Eq. (3) are well-defined for any function \( \bar{\varphi}_i \) which can be expanded in powers of \( F, \bar{\varphi} \). Thus, a complete solution may be obtained in this representation and the equations of motion have direct meaning with reference to the space-time development of \( \psi \).

However, this Hilbert space is not the correct one since its "vacuum" has all anti-particle states filled and the energy is not positive definite. The correct Hilbert space has \( a |0\rangle = b |0\rangle = 0 \); the two are connected by \( |0\rangle = T |0\rangle = \prod_{\alpha_i} b_i (P_N |0\rangle) \), where we have tentatively assumed that a cut-off is present so that there are a finite number of degrees of freedom. In this Hilbert space the same operators \( P_\varphi \) or \( P'_\varphi \) are diagonal and well-defined, and their eigenstates are \( |\alpha, \beta\rangle = (a^*_i)^\alpha (b^*_i)^\beta |0\rangle \).

We must now investigate to what extent Eqs. (1), (3) are well-defined in the true Hilbert space. This question may easily be answered if a cut-off exists. Then \( T \) is unitary and well-defined, \( \psi |0\rangle = \tilde{\psi} T |0\rangle \), and \( \psi \) may be commuted with \( T \) to evaluate the expression. However, it is impossible to obtain an exact solution with a finite cut-off and one cannot determine how \( \psi |0\rangle \) behaves in the limit \( k_{max} \to \infty \). Relativistic theories always contain an infinite number of degrees of freedom so that \( T \) does not exist. Under these circumstances, one must define the function

\[
\tilde{\psi}_i = Z^\lambda \varphi_i : G \left( \bar{\varphi}_i, \varphi_i, \lambda_j, \lambda \right) ;
\]

by requiring that a) \( \tilde{\psi}_i^R = \tilde{\psi}_i Z^{-\lambda} \to \varphi_i \) as \( t \to -\infty \). b) \( \langle 0 | \tilde{\varphi}_i |0\rangle \approx \langle 0 | \varphi_i |0\rangle \), \( t \) finite. c) \( \tilde{\psi}_i \) satisfies the same commutation relations as does \( \psi_i \). Then \( \tilde{\psi}_i^R \) is well-defined in the true Hilbert space (i.e., it has finite matrix elements) and it is the renormalized Heisenberg operator for the bare particle field.
If $Z^Z$ turns out to be finite, $\tilde{\psi}_i$ is presumably obtained from $\psi_i$ by regrouping terms, adding constants, etc. In general, however, $Z^Z$ vanishes. For these cases $\tilde{\psi}_i$ does not necessarily have the same space-time dependence as does $\psi_i$. In particular, the renormalized, ordered operator satisfies

\[ (i \delta^+ \omega + M_i^+ \omega^+) \tilde{\psi}_i^- = \phi_i^- (i \delta^+ \omega + M_i^+ \omega^+) \tilde{\psi}_i^- \]  \tag{6}

while the "renormalized", unordered operator obeys

\[ (i \delta^+ \omega + m_i) \psi_i^R + Z^{-1} \psi_i = 0, \quad \psi_i^R = \psi_i Z^{-1}. \]

Since $\tilde{\psi}_i^R$ and $\psi_i^R$ are connected only by having the same commutation relations and asymptotic values, there is no reason to believe that these equations describe the same entity. In fact, there is no meaning to the operation of renormalizing $\psi_i$, and we assert that only Eq. (6) may be regarded as the equation of motion for the renormalized operator. Similar considerations generally apply when $L$ contains coupled fermion and boson fields.

### III. Some Specific Examples

Let us first examine the two-dimensional Thirring model for which the equation of motion is

\[ i \delta^+ \omega \psi + 2\lambda (\bar{\psi} \psi) \psi = 0. \]

In the standard representation one of the components satisfies $\partial \psi / \partial u = i \lambda \psi \bar{\psi} \psi$, $u = t+ x$, $v = t-x$; we have also let $\psi_1^{-} \rightarrow \psi_2^{+} \exp i u \delta(0)$ and the solution is $\psi_1^{+}(x, t) = \phi_1(v) \exp i \lambda \int_x^u \psi_2^{+} \phi_2 \bar{\phi}_2 \, du'$. Glaser has shown that the commutation relations $\bar{\gamma}_2(u') \psi_1^{+}(u, v) = \psi_1^{+}(u, v) \bar{\phi}_2(u') \exp [i \lambda \bar{\phi}(u-u')]$ can be used to recast $\tilde{\psi}_1^{-}$ in the well-defined ordered form, $\tilde{\psi}_1^{-} = Z^{1/2} \exp Q_1$, or

\[ \tilde{\psi}_1^- = Z^{1/2} \phi(v) \exp \left\{ \frac{(2\lambda - 1)}{2 \hbar c} \int d \rho \, d \bar{\rho} \left( \frac{\rho_1^+}{2} \right) \frac{\Theta^{(2-\rho)} u}{\rho_1 - \frac{\bar{\rho}}{2}} c_\rho^+ (\rho) c_2 (\bar{\rho}) \right\} \] \tag{7}
where \( C_2(q) = \omega(-q) a(q) + \Theta(q) b^*(q) \) and \( \lambda' = \lambda + 2\pi N \) so that \( |\lambda'/2\pi| < 1 \); \( \alpha \) is real for \( p \) real and positive and it has a branch cut from 0 to \(+\infty\). In Glaser's paper \( Z^{1/2} \) is evaluated by requiring that

\[
\frac{\mathcal{D} \langle \Psi_1 | \Psi_1 \rangle}{\mathcal{D} \lambda} = i \int_{-\infty}^{\infty} du' \langle \Psi_1 | \phi_2^* \phi_2 | \Psi_1 \rangle
\]

be valid for \( \Psi_1 \rightarrow \tilde{\Psi}_1 \). Eq. (8) gives the renormalization constant to within a factor independent of \( \lambda \); since \( Z^{1/2} \) must be unity for \( \lambda = 0 \), this factor is unity.

To see that \( \tilde{\Psi}_1^R = \tilde{\Psi}_1 Z^{1/2} \) differs from \( \Psi_1 \), consider the equation \( \frac{\partial \tilde{\Psi}_1^R}{\partial u} = i\lambda \phi_2^* \phi_2 \tilde{\Psi}_1^R \). The matrix element \( \langle n | \tilde{\Psi}_1^R | 0 \rangle \) is zero since \( \langle n | \tilde{\Psi}_1^R | 0 \rangle = \langle n | \phi_1 | 0 \rangle \) but the other side gives \( i\lambda \sin(\lambda/2) \times \int_0^\infty dk \times 1 \int_{-1}^{1} = \langle n | \phi_2 \phi_2 \rangle \) has infinite norm. One might try to redefine \( \tilde{\Psi}_1^R \rightarrow \tilde{\Psi}_1^R \exp(i u x \infty) \) to eliminate the infinite term but this procedure does not help; the matrix elements between \( \langle n | \) and \( | m \rangle \) still give different values for the two sides of the equation. In other words \( \tilde{\Psi}_1^R \) has a space-time dependence which differs from that of \( \Psi_1 \), and the difference cannot be eliminated by a c-number adjustment. If a cut-off exists in k-space, \( \langle n | \phi_2^* \phi_2 | \tilde{\Psi}_1^R | 0 \rangle \) is finite. One may then write \( i\lambda \phi_2^* \phi_2 \tilde{\Psi}_1^R = R(\phi_2^* \phi_2 + Z_1) \Psi_1 + Z \Psi_1 \) and \( \tilde{\Psi}_1^R \) can be derived from the equation for \( \Psi_1 \) (it should be noted that the ordering operation is generally non-local, even if there is a cut-off). In this sense, \( \tilde{\Psi}_1^R \), which obeys

\[
\frac{\mathcal{D} \tilde{\Psi}_1^R}{\mathcal{D} u} = \mathcal{D} \frac{\mathcal{D} \tilde{\Psi}_1}{\mathcal{D} u} \tilde{\Psi}_1^R
\]

also "satisfies" the original equations of motion, but with a cut-off \( Z^{1/2} \) depends on \( u \). In the limit of no cut-off the \( \Psi \rightarrow \tilde{\Psi}_1^R \) connection becomes ill-defined. The quantity \( Z^{1/2} \) formally depends on \( u \), but it is usually regarded as a constant. Each function has meaning in its own Hilbert space, but the original equations of motion are meaningless in the representation with \( \langle 10 | 10 \rangle = \langle 00 | 00 \rangle = 0 \).
This result has several immediate consequences. The customary derivation of the Tamm-Dancoff equations \(^{(6)}\) for many particle amplitudes involves the original equations of motion. For instance, one would have for the Thirring model

\[
\frac{\partial}{\partial \lambda} \langle \tilde{\Psi}^R(a) \tilde{\Psi}_\sigma^R(b) | 0 \rangle = i \lambda \langle \tilde{\Psi}^R(a) \tilde{\Phi}^R(0) | \tilde{\Psi}_\sigma^R(b) | 0 \rangle
\]

(10)

and this can be expanded to obtain an approximate equation for

\[
\langle \tilde{\Psi}^R(a) \tilde{\Psi}_\sigma^R(b) | 0 \rangle.
\]

However, we find that these amplitudes have no meaning in the true representation. The meaningful amplitude satisfies

\[
\frac{\partial}{\partial \lambda} \langle \tilde{\Psi}^R(a) \tilde{\Psi}_\sigma^R(b) | 0 \rangle = \langle \tilde{\Psi}^R(a) \tilde{\Psi}_\sigma^R(b) | 0 \rangle
\]

(11)

which leads to considerably different results.

There are no difficulties in the neutral scalar meson theory with fixed source \(^{(7)}\). The content of this model is best explored by examining the interaction representation operator \(\mathcal{U}(t, - \infty)\),

\[
i \frac{\partial \mathcal{U}}{\partial t} = \mathcal{H} \mathcal{U} = \int d^3x \left( \mathcal{B} - \bar{\delta} \mathcal{m} \right) N \mathcal{U}
\]

(12)

where \(N = \psi^\dagger \psi = \phi^\dagger \phi\), \(\mathcal{B}\) is the physical boson field operator and \(\bar{\delta} \mathcal{m}\) is the mass renormalization constant. For fixed nucleons, \(\mathcal{J}_N = 0\), \(N = N^2 = N\) and \(\int N \mathcal{B} = 0\). Thus, \(N\) may be treated as a c-number in Eq. (12). The general solution is \(\mathcal{U} = \exp \left[ - i \int_{-\infty}^{t} d\tau' \mathcal{H} \right] \) (time-ordered), which is well defined in the representation with \(\mathcal{B}(+) | 0 \rangle = \mathcal{B}(-) | 0 \rangle = 0\). This solution is not meaningful in the true Hilbert space, \(\mathcal{B}(+) | 0 \rangle = \mathcal{B}(-) | 0 \rangle = 0\). However, \(\mathcal{H}\) of Eq. (12) obeys \(\int_{-\infty}^{t} (\mathcal{H}(+) N \mathcal{J}) = - N \mathcal{C}(\mathcal{H}(+) N - 1\), so that \(^{(8)}\)

\[
i \frac{\partial \mathcal{U}}{\partial t} = \mathcal{H} \mathcal{U} + f(c) \mathcal{U}.
\]

\(^{(8)}\) L. van Hove, Physica, 16, 145 (1952)
The simple substitution $U = \hat{\psi} R \exp \left[ -i \int_{-\infty}^{t} f(c) dt' \right]$ leads to the ordered form, $\hat{\psi} R$, so that $U$ differs from $\hat{\psi} R$ merely by a vanishing c-number factor, and $\hat{\psi} R$, $\hat{\beta} R$ obey the original equations of motion after this factor is absorbed.

For the Thirring model $H = \lambda \int d^3 x \phi_1^* \phi_1 \phi_2^* \phi_2$, and $\int_{H(-)}, (H(+))^d \int$ is not simply related to $H$. Thus, $\hat{\psi} R$ obeys an equation which is essentially different from the one satisfied by $U$, even after c-number renormalization. Generally, when $H$ is not a linear function of the field operators $H U = \alpha; H U + \sum \beta_i \phi_i U; + \ldots$, where $\phi_i$, $\beta_i$ etc., are non-local functions, and $\hat{\psi} R$ does not satisfy the renormalized equation for $U$.

IV. Conclusions

It has been conjectured that renormalized field quantities obey local equations which can be obtained from equations of motion by regrouping terms (to account for mass renormalization, etc.) and factors (to account for coupling constant and wave function renormalization). Since products of renormalized operators are undefined at a sharp time if $Z = 0$, various averaging and limiting processes must accompany the manipulations.
We assert that the equations obtained by these techniques have solutions which are generally ill-defined in the true Hilbert space. For these problems the meaningful renormalized operator is a Wick-ordered product. If $Z = 0$, the latter is not obtained directly from the original equation of motion or from its "renormalized" form (the neutral scalar meson theory is an exception). It is still necessary to use the averaging and limiting techniques if all quantities in the new equation \( \langle \tilde{e} \rangle \), Eq. (2)\( \tilde{e} \) are to be expressed in terms of renormalized bare particle operators.

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REFERENCES

1) W. Heisenberg, Nuclear Phys. 4, 532 (1957).