Investigation of the Invariance Group in the
Three Fundamental Fields Model

by

J. E. Wess

C E R N Geneva

A B S T R A C T

For a theory in which elementary particles are represented by products of fundamental fields, consequences of **an** invariance group are investigated. The invariance group is the three-by-three unitary group. Tensor-calculus is used to investigate the representations, quantum numbers are defined and certain families of elementary particles are identified with irreducible representations.

Introduction.

The problem as to how many fields are necessary to get all conservation laws which hold in strong interaction, has been studied by W. Thirring¹⁾. Through the commutation relations of the conserved quantities and the invariance properties of the Lagrangian-formulism, Thirring found that at least three massless Weyl fields (or Mayorana fields) or six fields with mass are necessary.

In this paper, consideration will be given only to the three-fields case, where the invariance group is the three-by-three unitary group. The group theoretical consequences of this invariance will be investigated.

In chapter 1 the tensor representations, the only finite dimensional representations of this group, will be discussed.

Then the coordinate system in the representation space is adapted to transformations, which, in our theory, are connected with the isotopic spin and the displacement of the centre of charge. We find that we can label the tensor components in a unique way with numbers which we can associate with the displacement of the centre of charge (u), the isospin (τ), and the third component of isospin (τ_3).

In chapter 3 we construct the tensors by means of "fundamental fields". If we take the fields at the same space-time-point, we learn from the Pauli-principle that only a few representations are possible. These representations will be listed.

Then we try to identify certain families of elementary particles (as n p $\wedge$ or π - k - mesons) with these representations. Doing this, we demand an agreement between the quantum numbers of isospin, strangeness and baryon number.

CHAPTER I.

We consider the quantities $\vec{T}$ (isotopic spin), N (Baryon number) and U (displacement of the centre of charge), which obey the commutation relations

$$[T_\ell, T_m] = i\epsilon_{\ell mk} T_k, \quad [N, \vec{T}] = [N, U] = [U, \vec{T}] = 0 \quad (1,1)$$

They form an algebra and it is very easy to be seen that the lowest dimensional faithful representation of this algebra is three-dimensional and that the elements of this algebra can be represented by the following matrices

$$\vec{t} = \begin{vmatrix} \vec{\sigma} & \\ & 0 \end{vmatrix} \quad n = \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} \quad u = \begin{vmatrix} 1 & & \\ & 1 & \\ & & 0 \end{vmatrix} \quad (1,2)$$

($\vec{\sigma}$ are the Pauli matrices)

These matrices are contained in the Lie-algebra of the unitary three-by-three group (but not in the Lie-algebra of the orthogonal three-by-three group).

Now we take the full three-by-three unitary group (u_3) although only the algebra (2), corresponding to a sub-group of u_3 , has been connected with physical quantities up to now. Starting, however, like Yamaguchi ²⁾ with three fields ($p \ n \ \wedge$) and an invariance against unitary transformations between p and n as well as a symmetry against permutations of the three fields, we are again concerned with the full unitary group.

Let us assume now that in physics there is an invariance against our group. Thus, physical quantities have well-defined transformation properties against this group, i.e. they belong to certain representations. We assume that they belong to irreducible representations. Therefore, we have to investigate at first the representations of the group and the quantities belonging to the representations (these are the vectors in the representation space). In order to construct the representations, it is useful to introduce covariant vectors (x_i) and contravariant vectors (x^i); $x_i x^i$ being an invariant.

$$x'_i = u_i^{\quad k} x_k \quad x^{i'} = u^i_{\quad k} x^k \quad (i, k = 1, 2, 3) \quad (1, 3)$$

where: $\left\| u_i^{\quad k} \right\|$ is an element of u_3 , and $u^i_{\quad k} = u_i^{\quad k*}$.

With the help of these vectors we can build up tensors.

$$\begin{array}{c} \ell_1 \cdots \ell_s \\ A \\ k_1 \cdots k_r \end{array} \quad (1, 4)$$

should be a tensor component with s contravariant and r covariant indices.

From group theory ³⁾ we know that all finite representations are capable of tensor representations and that u_3 induces irreducible representations in the tensor space of a certain rank and symmetry class. Thus, we have to construct tensors of highest symmetry in order to get the irreducible representations.

When forming the determinant of three vectors

$$\text{Det} (xyz) = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \quad (1,5)$$

we see that $\text{Det} (x' \ x' \ z') = \text{Det} \left\| u_i^k \right\| \text{Det} (xyz)$. $\text{Det} (xyz)$, therefore is an invariant against the uni-modular unitary group:

$$\epsilon^{ijk} x_i^{(1)} x_j^{(2)} x_k^{(3)} \sim \text{invariant} \quad (1,6a)$$

$$\epsilon^{ijk} x_j^{(1)} x_k^{(2)} \sim x^i \quad (1,6b)$$

If we restrict ourselves to the uni-modular group, (1,6b) indicates that a pair of covariant (contravariant) indices, in which a tensor is antisymmetric, can be replaced by one contravariant (covariant) index. Furthermore, three indices form an invariant, if a tensor is antisymmetric in each pair of these three indices (1,6a). Obviously there exists no tensor that is antisymmetric in each pair of more than three indices. Thus if we use covariant and contra-

variant indices, we have to construct only tensors whose components

$$\begin{array}{c} \ell_1 \cdots \ell_s \\ A \\ k_1 \cdots k_r \end{array} \quad (1,7)$$

are symmetrical in the r covariant and the s contravariant indices. The space spanned by such a tensor, however, is not irreducible, for it contains an invariant sub-space formed by the linear combinations

$$\sum_i \begin{array}{c} i \ell_2 \cdots \ell_s \\ A \\ i k_2 \cdots k_r \end{array} \quad (1,8)$$

The tensor components therefore should satisfy the following condition :

$$\sum_i \begin{array}{c} i \ell_2 \cdots \ell_s \\ a \\ i k_2 \cdots k_r \end{array} = 0 \quad (1,9)$$

(we write small a to indicate that the components obey the condition (1,9)). In such a tensor space the uni-modular unitary group induces an irreducible representation. We call it $a(rs)$. Then $(\text{Det } u_3)^e a(rs)$, (e integer) are representations of u_3 . We call it ${}^e a(rs)$.

Our quantities are the tensor components

$$\begin{array}{c} \ell_1 \cdots \ell_s \\ {}^e a \\ k_1 \cdots k_l \end{array} \quad (1,10)$$

From this we learn that all the irreducible representations of u_3 can be labelled with the three numbers e, r, s ; (e integer, $r, s \geq 0$).

We have, for instance

$$\varepsilon^{ik\ell} x_k y_\ell \sim 1_a^i, \quad \varepsilon^{ik\ell} x_i y_k z_\ell \sim 1_a \quad (1,11)$$

The tensor space formed by

$$x_j^{e_a} \begin{matrix} \ell_1 \cdots \ell_s \\ k_1 \cdots k_r \end{matrix} \quad (1,12)$$

contains the invariant sub-spaces spanned by the tensor components

$$e_a \begin{matrix} \ell_1 \cdots \ell_s \\ k_1 \cdots k_r \end{matrix} j; \quad \varepsilon^{ik_1 j} x_j \begin{matrix} \ell_1 \cdots \ell_s \\ k_1 \cdots k_r \end{matrix}; \quad x_j^{e_a} \begin{matrix} j \ell_2 \cdots \ell_s \\ k_1 \cdots k_r \end{matrix} \quad (1,13)$$

In these sub-spaces, the representations

$$e_{a(r+1,s)}; \quad e_{a(r-1,s+1)}^{+1}; \quad e_{a(r,s-1)} \quad (1,14)$$

are induced.

If we want to know the degree of the representations, we have to count the linear independent tensor components. This will be done in appendix I. We find that $\frac{1}{2}(r+1)(s+1)(r+s+2)$ is the degree of the representation $e_{a(r,s)}$.

CHAPTER 2.

Under the invariance group, u_3 physical quantities should transform like vectors in the representations space of this group. Thus, we shall discuss the representation space rather than the representation itself. In this chapter we shall characterise the vectors in this space; for this purpose we have to specify the coordinate system. This can be achieved by giving a special form to matrices representing certain group elements, so that there is no freedom for equivalence-transformations.

The most obvious way is to make diagonal the transformations generated by N, U, T_3 . These are the only transformations that are diagonal in the same coordinate system, as we can find no other element in the Lie-algebra of u_3 commuting with N, U, T_3 . Then we try to characterise the vectors by labelling them with the eigenvalues of N, U, T_3 .

In the three-dimensional vector space the transformations:

$$x'_1 = e^{i\varphi_1} x_1, \quad x'_2 = e^{i\varphi_2} x_2, \quad x'_3 = e^{i\varphi_3} x_3 \quad (2,1)$$

are generated by N, U, T_3 . It follows that the tensor component (v, μ, v^*, μ^*) (see appendix I) are eigenvectors with the eigenvalue (= weight of the component)

$$\exp i \left\{ (v-v^*+e) \varphi_1 + (\mu-\mu^*+e) \varphi_2 + (r-s-v+v^*-\mu+\mu^*+e) \varphi_3 \right\} \quad (2,2)$$

According to (1,2) the eigenvalues of N, U, T_3 are

$$\begin{aligned} n &= r - s + 3e \\ u &= v - v^* + \mu - \mu^* + 2e \\ T_3 &= \frac{1}{2} (v - v^* - \mu + \mu^*) \end{aligned} \quad (2,3)$$

We see that the weight depends only on

$$\begin{aligned} v - v^* &= \alpha \\ \mu - \mu^* &= \beta \end{aligned} \quad (2,4)$$

so that components with the same value for α and β have the same weight and are not distinguished through labelling with n, u, T_3 .

In appendix II, it will be shown that there are only

$$\frac{1}{2} \int (r+1)(s+1)(r+s+2) - rs(r+s)$$

different pairs $\alpha \beta$, i.e. for $r \neq 0, s \neq 0$ less than the degree of the representation. Therefore, we have linear independent tensor components with ^{the} same weight. If the transformations (2,1) are required to be diagonal, there is still a possibility for equivalence transformations, affecting only components with the same weight. Thus, we cannot characterise the vectors by the numbers n, u, T_3 only, but we have to distinguish between linear independent vectors with the same n, u, T_3 .

Now we demand that the representations of the sub-group generated by $\vec{T}$ are contained in $^e a(rs)$ in fully reduced form.

The sub-group is a two-by-two unitary uni-modular group (u_2), the irreducible representations of which are well known. They can be characterised by the eigenvalues of $T_1^2 + T_2^2 + T_3^2$, which have the form $\hat{\tau}(\hat{\tau}+1)$, or, as usual, with the isospin $\hat{\tau}$ itself. Furthermore, we know that for a representation of u_2 characterised by $\hat{\tau}$ there is exactly one linear independent eigenvector of T_3 to every eigenvalue: $-\hat{\tau}, -\hat{\tau}+1, -\hat{\tau}+2, \dots, +\hat{\tau}$. These eigenvectors span the whole representation space.

As U commutes with all $\vec{T}$ matrices, it follows from the Lemma of Schur that U has the same eigenvalue u for all vectors belonging to the same irreducible representation of u_2 . From (2,3), we see that the eigenvalue of N depends only on r, s and e and that it has therefore the same value for vectors of the same irreducible representation $^e a(rs)$.

From (2,3), (Appendix II, 1) and (Appendix II, 2) follows that the values which can be taken on by u are

$$2e - s, 2e - s + 1, \dots; r + 2e - 1, r + 2e \quad (2,5)$$

When forming the sub-space of all the vectors with the same eigenvalue u , we know already that this is an invariant sub-space under the transformations induced by u_2 . We are interested in the irreducible invariant sub-spaces contained in this sub-space. For this end we take the vectors belonging to the greatest eigenvalue of $T_3 = \hat{\tau}_3 \max$. The number of the linear independent vector belonging to $\hat{\tau}_3 \max$ gives us the number of the representations

of u_2 with $\gamma = \gamma_{3\max}$, which are induced in the above mentioned sub-space. Then, we take away the eigenvectors of T_3 belonging to these irreducible representations and start again with the greatest remaining eigenvalue of T_3 .

With the help of this procedure we find all the irreducible representations of U_2 induced in the given sub-space and we can adapt the coordinate system to these representations. In other words, we label the vectors with γ , γ_3 and u .

The corresponding calculations are also given in appendix II. The final results are

For u between	there is exactly one irreducible representation of u_2 for each value of γ .
$r-s+2e \leq u \leq +2e$	$\gamma = \frac{u}{2} - e, \frac{u}{2} - e + 1, \dots, r - \frac{u}{2} + e$
$2e \leq u \leq r-s+2e$	$\gamma = \frac{u}{2} - e, \frac{u}{2} - e + 1, \dots, s + \frac{u}{2} - e$
$-s+2e \leq u \leq 2e$	$\gamma = -\frac{u}{2} + e, -\frac{u}{2} + e + 1, \dots, s + \frac{u}{2} - e$

Table 1.

(we have assumed $r \geq s$)

Thus we find that the quantities can be characterised by the eigenvalues of U , T_3 and T^2 in a unique way; the possible eigenvalues are listed in Table 1. The fact that we need the three numbers indicates that the value of u does not determine the value of τ in a certain representation.

The irreducible representation can be characterised by n , r and s .

CHAPTER 3.

Up to now we have only been concerned with symmetry properties arising from an invariance against the three-by-three unitary group and no other properties of the quantities have been involved. Now we shall consider the case that the vectors in the three-dimensional unitary space (internal space) are fields. We can build up all our tensors with these three "fundamental fields" just as products. As we have taken fields - they are quantities with a known space-time dependence - our quantities have also a certain space-time dependence.

In other words, we shall take fields belonging to the two-dimensional representation of the Lorentz group (Weyl fields) and we shall make products, which then transform according to certain representations of the Lorentz group. We are interested in the spin to which they belong.

If we take the fields in the products at the same space-time point, as will be done throughout this paper, the products are just spin tensors. From spinor-calculus we know the spin representations induced in such a tensor space.

As is well known, an antisymmetric pair of spin-indices forms an invariant under the uni-modular unitary two-by-two group (determinant). In our consideration we need therefore only spin tensors with covariant vector indices, which are symmetric also. Such a tensor can now be written

$$b_{\xi_1} \dots \xi_t \quad (3,1)$$

and it belongs to a spin representation with spin $t/2$. The tensor space

$$\psi_{\sigma} b_{\xi_1} \dots \xi_t \quad (3,2)$$

contains the invariant sub-spaces

$$b_{\xi_1} \dots \xi_{t\sigma}, \quad \epsilon^{\xi_1 \sigma} \psi_{\sigma} b_{\xi_1} \dots \xi_t \quad (3,3)$$

belonging to the spin $\frac{1}{2}(t+1)$ and $\frac{1}{2}(t-1)$

W. Thirring has shown that the Lagrangian formulism is invariant against a unitary group only in case the mass is zero.

Thus we take massless Weyl fields ⁴⁾ :

$$\psi_{k\sigma}(x) \quad (3,4)$$

$k = 1, 2, 3$ is a vector index in the internal space. $\sigma = 1, 2$ is a spin index.

Naturally, $\psi_{k\sigma}^+(x)$ transforms contravariant. The fields are supposed to obey the commutation relations

$$\begin{aligned} [\psi_{k\sigma}(x), \psi_{k'\sigma'}(x)]_+ &= 0 \\ [\psi_{k\sigma}^+(x), \psi_{k'\sigma'}(x)]_+ &= 0 \end{aligned} \quad (3,5)$$

It follows from (3,5)

$$\dots \psi_{k\sigma} \dots \psi_{k'\sigma'} \dots = - \dots \psi_{k'\sigma'} \dots \psi_{k\sigma} \dots \quad (3,6)$$

The products of ψ fields or ψ^+ fields only must be antisymmetric in the index pairs $(k\sigma)$, i.e. they have to be

symmetric in k and antisymmetric in the corresponding σ
or

antisymmetric in k and symmetric in the corresponding σ

Thus, forming the irreducible tensor space, we have only the freedom to choose the symmetry class for either the spin or the isospin tensor. This restricts the possible representations to a large extent, for spin tensors are antisymmetric in not more than two indices, isospin tensors in not more than three indices.

In the construction of the irreducible invariant tensor spaces that are spanned by our field products, we follow the way indicated by (1,12), (1,13), (3,2) and (3,3). Taking into account the rules

outlined above, it is very easy to find the possible representations. A list of the representations will be given in Table 2.

field products	spin	$^e a(r,s)$	degree of $^e a(rs)$
$\psi_{k_1 \sigma_1}$	1/2	$^0 a(10)$	3
$\psi_{k_1 \sigma_1} \psi_{k_2 \sigma_2}$	0 1	$^0 a(20)$ $^1 a(01)$	6 3
$\psi_{k_1 \sigma_1} \psi_{k_2 \sigma_2} \psi_{k_3 \sigma_3}$	1/2 3/2	$^1 a(11)$ $^1 a(00)$	8 1
$\psi_{k_1 \sigma_1} \dots \psi_{k_4 \sigma_4}$	0 1	$^2 a(02)$ $^1 a(10)$	6 3
$\psi_{k_1 \sigma_1} \dots \psi_{k_5 \sigma_5}$	1/2	$^2 a(01)$	3
$\psi_{k_1 \sigma_1} \dots \psi_{k_6 \sigma_6}$	0	$^2 a(00)$	1

Table 2.

The same table holds for products of the form

$$\psi_{k_1 \sigma_1}^+ (x) \dots \psi_{k_e \sigma_e}^+ (x)$$

when we interchange r and s and put $-e$ for e .

In order to get the irreducible representations according to which products of the form

$$\psi_{k_1 \sigma_1}^+ \dots \psi_{k_e \sigma_e}^+ \psi_{k'_1 \sigma'_1} \dots \psi_{k'_e \sigma'_e} \quad (3,7)$$

transform, we have to reduce the direct product of the corresponding representations as listed in Table 2. This can be done by considering all the possibilities, in which the indices available are fully symmetrised.

The tensor space

$$a_{\ell_2}^{\ell_1} \times a_{k_1 k_2} \quad (3,8)$$

for instance, contains the

irreducible subspaces	belonging to the representations.
$a_{\ell_2}^{\ell_1} a_{k_1 k_2}$	${}^0_a(31)$
$\sum_{\ell_1} a_{\ell_2}^{\ell_1} a_{\ell_1 k_2}$	${}^0_a(20)$
$\sum_{\ell_2 k_1} \epsilon^{\ell_3 \ell_2 k_1} a_{\ell_2}^{\ell_1} a_{k_1 k_2}$	${}^1_a(12)$
$\sum_{\ell_1 \ell_2 k_1} \epsilon^{\ell_3 \ell_2 k_1} a_{\ell_2}^{\ell_1} a_{k_1 \ell_1}$	${}^1_a(10)$

The spin, to which the products belong, follows from vector addition; there is no restriction as is in the first case (3,6).

For a certain class of product, i.e. products belonging to a Baryon number 0 and 1, the representations will be listed in Table 3 and Table 4.

In Table 1 we find the possible eigenstates of T^2 , T_3 and U for the representations. Thus, we know the possible eigenvalues T , T_3 and u of the field products under consideration, as we know the representations to which they belong.

Field products with baryon number 0			
Field products	Spin	$e_a(rs)$	degree of $e_a(rs)$
$\psi_{k_1\sigma_1}^+ \psi_{k'_1\sigma'_1}$	0,1	$^0_a(00)$	1
		$^0_a(11)$	8
$\psi_{k_1\sigma_1}^+ \psi_{k_2\sigma_2}^+ \psi_{k'_1\sigma'_1} \psi_{k'_2\sigma'_2}$	0	$^0_a(00)$	1
		$^0_a(11)$	8
		$^0_a(22)$	27
	1	$^1_a(03)$	10
		$^0_a(11)$	8
		$^{-1}_a(30)$	10

0, 1, 2

 $^0_a(00)$

1

 $^0_a(11)$

8

$\psi_{k_1\sigma_1}^+ \dots \psi_{k_2\sigma_2}^+ \psi_{k_1'\sigma_1'} \dots \psi_{k_3'\sigma_3'}$	0,1	$^0_a(22)$	27
		$^1_a(03)$	10
		$^{-1}_a(30)$	10
		$^0_a(11)$	8
		$^0_a(00)$	1
	1,2	$^0_a(11)$	8
	0,1,2,3	$^0_a(00)$	1
$\psi_{k_1\sigma_1}^+ \dots \psi_{k_4\sigma_4}^+ \psi_{k_1'\sigma_1'} \dots \psi_{k_4'\sigma_4'}$	0	$^0_a(00)$	1
		$^0_a(11)$	8
		$^0_a(22)$	27
	1	$^0_a(11)$	8
		$^{-1}_a(30)$	10
		$^1_a(03)$	10
	0,1,2	$^0_a(00)$	1
		$^0_a(11)$	8
$\psi_{k_1\sigma_1}^+ \dots \psi_{k_5\sigma_5}^+ \psi_{k_1'\sigma_1'} \dots \psi_{k_5'\sigma_5'}$	0,1	$^0_a(00)$	1
		$^0_a(11)$	8
$\psi_{k_1\sigma_1}^+ \dots \psi_{k_6\sigma_6}^+ \psi_{k_1'\sigma_1'} \dots \psi_{k_6'\sigma_6'}$	0	$^0_a(00)$	1

Table 3.

Field products with baryon number 1.

Field products	Spin	$e_a(rs)$	degree of $e_a(rs)$
$\psi_{k'_1 \sigma'_1}$	1/2	$0_{a(10)}$	3
$\psi_{k_1 \sigma_1}^+ \psi_{k'_1 \sigma'_1} \psi_{k'_2 \sigma'_2}$	1/2	$0_{a(10)}$	3
		$0_{a(21)}$	15
	1/2, 3/2	$1_{a(02)}$	6
		$0_{a(10)}$	3
$\psi_{k_1 \sigma_1}^+ \psi_{k_2 \sigma_2}^+ \psi_{k'_1 \sigma'_1} \psi_{k'_3 \sigma'_3}$	1/2	$1_{a(13)}$	24
		$0_{a(21)}$	15
		$1_{a(02)}$	6
		$0_{a(10)}$	3
	3/2	$1_{a(02)}$	6
	1/2, 3/2	$0_{a(21)}$	15
		$1_{a(02)}$	6
	1/2, 3/2, 5/2	$0_{a(10)}$	3

$\psi_{k_1 \sigma_1}^+ \dots \psi_{k_3 \sigma_3}^+ \psi_{k'_1 \sigma'_1} \dots \psi_{k'_4 \sigma'_4}$	1/2	$1_a(13)$	24
		$1_a(02)$	6
		$0_a(21)$	15
		$0_a(10)$	3
	1/2, 3/2	$0_a(10)$	3
		$0_a(21)$	15
		$1_a(02)$	6
	3/2	$1_a(02)$	6
	1/2, 3/2, 5/2	$0_a(10)$	3
$\psi_{k_1 \sigma_1}^+ \dots \psi_{k_4 \sigma_4}^+ \psi_{k'_1 \sigma'_1} \dots \psi_{k'_5 \sigma'_5}$	1/2	$0_a(21)$	15
		$0_a(10)$	3
	1/2, 3/2	$1_a(02)$	6
		$0_a(10)$	3
$\psi_{k_1 \sigma_1}^+ \dots \psi_{k_5 \sigma_5}^+ \psi_{k'_1 \sigma'_1} \dots \psi_{k'_6 \sigma'_6}$	1/2	$0_a(10)$	3

To products with baryon number -1 belong
contravariant representations.

Table 4.

To the representations of Table 3 and Table 4 belong the states with the following quantum numbers

Representation	u	τ
${}^0a(00)$	-0	0
${}^0a(11)$	-1 0 1	$1/2$ 0,1 $1/2$
${}^0a(22)$	-2 -1 0 1 2	1 $1/2, 3/2$ 0, 1, 2 $1/2, 3/2$ 1
${}^{\dagger 1}a(03)$	-1 0 1 2	$3/2$ 1 $1/2$ 0
${}^{\omega 1}a(30)$	-2 -1 0 1	0 $1/2$ 1 $3/2$
${}^0a(10)$	0 1	0 $1/2$
${}^0a(21)$	-1 0 1 2	$1/2$ 0,1 $1/2, 3/2$ 1

$^1_a(02)$	0	1
	1	1/2
	2	0
$^1_a(13)$	-1	3/2
	0	1, 2
	1	1/2, 3/2
	2	0, 1
	3	1/2

Table 5.

In a composite model of elementary particles one would try to identify the field products with elementary particles. According to the quantum numbers n , u , τ , τ_3 there are among others the following possibilities:

	τ	τ_3	u
τ_1	1/2	1/2	1
τ_2	1/2	-1/2	1
τ_3	0	0	0

Table 6.

These fundamental fields have the quantum numbers of p , n and $\wedge$ as a consequence of $(1,2)$, but they belong to the mass zero. Therefore, we would have to identify these particles with products of at least three fields.

From Table 4 we see that the 27 different states formed by the product of three fields belong to different irreducible representations. They have the degrees 3, 15 and spin $\frac{1}{2}$ and degrees 3, 6 and spin $\frac{1}{2}$ or $\frac{3}{2}$.

As we believe that there is no degeneration among states belonging to different irreducible representations, we have to identify the particles with a family of states connected by the same irreducible representation, and we have also some reason to exclude the other states. According to Table 5, there are states which we can identify with p , n , $\wedge$, $\lesseqgtr$ and $\equiv$ particles, but there are several possibilities among which group theory predicts no unique way.

In the case of two-fields product, we can form the following states.

Products	τ	τ_3	u	Particles
$\psi_1^+ \psi_2$	1	-1	0	π^-
$\psi_1^+ \psi_1 - \psi_2^+ \psi_2$	1	0	0	π^0
$\psi_2^+ \psi_1$	1	1	0	π^+
$\psi_1^+ \psi_3$	1/2	-1/2	-1	$\overline{K^+}$
$\psi_2^+ \psi_3$	1/2	1/2	-1	$\overline{K^0}$
$\psi_3^+ \psi_2$	1/2	-1/2	1	K^0
$\psi_3^+ \psi_1$	1/2	1/2	1	K^+
$\psi_3^+ \psi_3 - \psi_i^+ \psi_i$	0	0	0	$\pi^{00} \quad *)$

Table 7.

$\sum_i \psi_i^+ \psi_i$ again belongs to a different representation and can be excluded.

*) π^{00} is a state degenerated with the π^- and K^- mesons.

FINAL COMMENTS.

Invariance under a group of transformations yields a great deal of information. We can identify families of physical states with representations of the group, through this we get transformation laws and orthogonality relations for the states. We can deduce conservation laws and define quantum numbers and we obtain branching ratios for processes where several states are involved.

In this paper we started with the knowledge of conservation laws and tried to deduce them from invariance against a transformation group. The conservation laws furnish us with the Lie algebra $(1,1)$, and the corresponding Lie group should be isomorphic to the desired group of transformations, or at least to ^{some} of their sub-groups.

From $(1,1)$ it is very easy to see that the corresponding Lie group is the direct product of two commuting sub-groups, one generated by the elements T , the other by the elements N and U . It follows that the representations are also the direct product of the corresponding representations. The representations of the first group are just the spin representations, the other group is Abelian and has one-dimensional representations only. Thus, from $(1,1)$ alone, we could get no further information beyond that contained in the isospin theory.

But we can choose a larger group, so that $(1,1)$ corresponds to a sub-group. The choice of this group is rather arbitrary and we have to justify our choice of the three-by-three unitary group.

What we essentially want to do is to connect the group operations of the two commuting groups, so that the whole group does

APPENDIX IDegree of the representation $e_a(rs)$:

If we want to know the degree of a representation, we have to count the linear independent vectors in the representation space. Our representation space is spanned by the tensor components

$$\begin{array}{c} \ell_1 \dots \ell_s \\ a \\ k_1 \dots k_r \end{array}$$

that obey the condition (1,9). (V, μ, V^*, μ^*) should be a tensor component with

$$\begin{array}{ll} V & \text{covariant indices} \quad 1 \\ \mu & \text{covariant indices} \quad 2 \end{array}$$

and

$$\begin{array}{ll} V^* & \text{contravariant indices} \quad 1 \\ \mu^* & \text{contravariant indices} \quad 2 \end{array}$$

For one fixed value of V ($0 \leq V \leq r$), μ can be any integer number between 0 and $r - V$, that are $r - V + 1$ different numbers.

When we sum over the values V can take one, we find that there are $\frac{1}{2}(r+1)(r+2)$ different possibilities to choose the covariant indices. Similarly, we find $\frac{1}{2}(s+1)(s+2)$ possibilities for the contravariant indices. Thus, there are $\frac{1}{4}(r+1)(r+2)(s+1)(s+2)$

tensor components, but condition (1,9) was not yet taken into account.

In order to get the degree of the irreducible representation ${}^e_a(rs)$, we have to subtract the degree of the invariant subspace formed by the linear combinations:

$$\sum_i a_i \ell_2 \dots \ell_s \quad (AI,1)$$

Here, only the indices $k_2 \dots k_r, l_2 \dots l_s$ can be chosen freely, so that there are $\frac{1}{4} r(r+1)s(s+1)$ linear independent components in the subspace.

Thus, the representation space of ${}^e_a(rs)$ has the dimension

$$\frac{1}{4} [r(r+1)(r+2)(s+1)(s+2) - r(r+1)s(s+1)] = \frac{1}{2} (r+1)(s+1)(r+s+2) \quad (AI,2)$$

This number is the degree of the representation ${}^e_a(rs)$.

APPENDIX II

In Appendix I we have introduced the tensor components (ν, μ, ν^*, μ^*) . For ν, μ, ν^*, μ^* being any integer number, greater or equal to zero, and satisfying

$$\nu + \mu \leq r \quad \nu^* + \mu^* \leq s \quad (\text{A II}, 1)$$

there is one tensor component.

As a consequence of (1,8) this tensor space is not irreducible, but contains invariant subspaces. In order to obtain an irreducible subspace, the tensor components have to satisfy condition (1,9); consequently, some of them are linearly dependent; these are always components with the same weight.

Now we want to know the range of α, β as defined in (2,4). From (A II,1) and (2,4) follows

$$-s \leq \alpha \leq r, \quad -s \leq \beta \leq r, \quad -s \leq \alpha + \beta \leq r \quad (\text{A II}, 2)$$

so that with α fixed, greater or equal to zero, β can take any value between $-s$ and $r-\alpha$; these are $r+s-\alpha+1$ different values. With α smaller than zero, we find $r+s+\alpha+1$ different values.

Next we sum over α between $-s$ and r and obtain

$$\frac{1}{2} \left[(r+1)(s+1)(r+s+2) - rs(r+s) \right]$$

different pairs.

Let us count now the number $k(\alpha, \beta)$ of linear independent tensor components with the weight determined by α and β . For this end α is kept fixed again and we write down the possible values of v and v^* , so that $v - v^* = \alpha$.

From (A II,1) and (2,4) follows

$$\text{for } \alpha \geq 0 : \begin{cases} v: \alpha & \alpha + 1 \dots \alpha + x \\ v^*: 0 & 1 \dots x \end{cases} \quad (\text{A II,3})$$

with

$$\alpha + x \leq r, \quad 0 \leq x \leq s$$

as a condition for x .

$$\text{for } \alpha \leq 0 : \begin{cases} v: 0 & 1 \dots x \\ v^*: -\alpha & -\alpha + 1 \dots -\alpha + x \end{cases}$$

with

$$0 \leq x \leq r, \quad -\alpha + x \leq s$$

as a condition for x .

The same applies to β and μ, μ^* . We bear in mind that v, v^* are numbers fixed already.

Then we have

$$\text{for } \beta \geq 0 : \begin{cases} \mu : \beta & \beta + 1 \dots \beta + y \\ \mu^* : 0 & 1 \dots y \end{cases}$$

with

$$\beta + y \leq r - v, \quad 0 \leq y \leq s - v^*$$

as a condition for y

$$\text{for } \beta \leq 0 : \begin{cases} \mu : 0 & 1 \dots y \\ \mu^* : -\beta & -\beta + 1 \dots -\beta + y \end{cases}$$

with

$$0 \leq y \leq r - v, \quad -\beta + y \leq s - v^*$$

as a condition for y .

From the condition for y , we see that we have to distinguish between the following cases :

$$1 -) \quad r - v \geq \beta \Rightarrow (r - v) - (s - v^*) = r - s - \alpha \geq 0$$

then y runs from zero to $r - v - \beta$; these are $r - v - \beta + 1$ different values; all of them yield the same β .

- 2 -) $r-s-\alpha \geq \beta \geq 0$ gives $s-\sqrt{+1}^*$ different values for y .
 3 -) $0 \leq \beta \leq r-s-\alpha$ gives $r-\sqrt{+1}$ different values for y .
 4 -) $0 \leq r-s-\alpha \leq \beta \leq -s+\sqrt{+1}^*$ gives $s-\sqrt{+1}^*+\beta+1$ different values for y .

Next we have to sum over all the values $\sqrt{+1}$ or $\sqrt{+1}^*$ that gives a certain α . The possible values are listed in (A II,3), but we have also

$$r - \sqrt{+1} \geq \beta, \quad \beta \leq -s + \sqrt{+1}^*$$

For β fixed this is a condition for $\sqrt{+1}$ and $\sqrt{+1}^*$. The sum is the number of tensor components with a certain weight, but we want to count only the components in the invariant irreducible subspace determined by (1,9). For this purpose we have to subtract the components with a given weight, which are in the other invariant subspace. From (1,8) it follows that this other subspace is isomorph to the representation space of $(r-1, s-1)$. As a result of the above calculation we obtain the number of vectors in this space also.

Finally, we have to subtract the two numbers, and if we assume $r \geq s$ we find

α	β	$k(\alpha\beta)$
$r - s \leq \alpha \leq r$	$0 \leq \beta \leq r - \alpha$	$r - \alpha - \beta + 1$
	$r - s - \alpha \leq \beta \leq 0$	$r - \alpha + 1$
	$-s \leq \beta \leq r - s - \alpha$	$s + \beta + 1$
$0 \leq \alpha \leq r - s$	$r - s - \alpha \leq \beta \leq r - \alpha$	$r - \alpha - \beta + 1$
	$0 \leq \beta \leq r - s - \alpha$	$s + 1$
	$-s \leq \beta \leq 0$	$s + \beta + 1$
$-s \leq \alpha \leq 0$	$r - s - \alpha \leq \beta \leq r$	$r - \beta + 1$
	$0 \leq \beta \leq r - s - \alpha$	$s + \alpha + 1$
	$-s - \alpha \leq \beta \leq 0$	$s + \alpha + \beta + 1$

Table 8.

This table also contains the character of the representation.

From (2,3) and (2,4) follows

$$u = \alpha + \beta + 2e$$

(A II,5)

$$\tau_3 = \frac{1}{2} (\alpha - \beta)$$

In chapter 2 we have considered the subspace formed by vectors with the same eigenvalue u . For this reason, we have to find all the possible values of α and β , so that $u = \alpha + \beta$ (we put $e = 0$). $k(\alpha\beta)$ then is the number of the linear independent vectors belonging to the pair $\alpha\beta$ and that is also the number of vectors with the eigenvalue $\tau_3 = \frac{1}{2} (\alpha - \beta)$.

Let us write down the possible values of α and β which have the sum $\alpha + \beta = u$. Then $k(\alpha\beta)$ can be found in Table 8.

We have to consider the cases

$$\begin{array}{l}
 1 \rightarrow r - s \leq u \leq r \quad (r \geq s) \\
 \alpha : \quad r \quad \dots \dots \dots u \quad \dots \dots \dots 0 \quad \dots \dots \dots u-r \\
 \beta : \quad u-r \quad \dots \dots \dots 0 \quad \dots \dots \dots u \quad \dots \dots \dots r \\
 \hline
 k(\alpha\beta) : \quad 1 \dots r-\alpha+1 \quad r-u+1 \quad r-\beta+1 \dots 1 \\
 \\
 2 \rightarrow 0 \leq u \leq r - s \\
 \alpha : \quad u+s \quad \dots \dots \dots u \quad \dots \dots \dots 0 \quad \dots \dots \dots s \\
 \beta : \quad -s \quad \dots \dots \dots 0 \quad \dots \dots \dots u \quad \dots \dots \dots u+s \\
 \hline
 k(\alpha\beta) : \quad 1 \dots s+\beta+1 \quad s+1 \quad s+\alpha+1 \dots 1
 \end{array}$$

$$3 \rightarrow -s \leq u \leq 0$$

$$\begin{array}{lcl}
 \alpha & : & u+s \quad \dots\dots\dots 0 \quad \dots\dots\dots u \quad \dots\dots\dots -s \\
 \beta & : & -s \quad \underbrace{\dots\dots\dots u}_{s+\beta+1} \quad \underbrace{\dots\dots\dots 0}_{s+u+1} \quad \underbrace{\dots\dots\dots u+s}_{s+\alpha+1} \\
 k(\alpha\beta) & : & 1 \dots s+\beta+1 \quad s+u+1 \quad s+\alpha+1 \dots 1
 \end{array}$$

Here the conditions (A II,2) are always satisfied.

In the case of $-s \leq u \leq r$ the number of vectors with the eigenvalue $\tau_3 = r - \frac{u}{2}$ is one. It increases steadily by one with decreasing τ_3 and is $r - u + 1$ for $\tau_3 = \frac{u}{2}$. The number is constant until τ_3 becomes $-\frac{u}{2}$, then it decreases and becomes one again for $\tau_3 = -r + \frac{u}{2}$.

In the cases 2 and 3 we get similar results.

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