CM-P00056794

ELECTROMAGNETIC PROPERTIES OF UNSTABLE PARTICLES

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A B S T R A C T

Charge and current distributions are defined for unstable particles in no-recoil theories. They are in general complex, and oscillate with amplitudes increasing exponentially at large distances. Moments can be defined nevertheless. As with stable particles, electric dipole moments are necessarily zero if time reversibility holds.

I. Introduction

It has been suggested ¹⁾ that even in a time reversible theory ²⁾ an unstable particle might have an electric dipole moment. This is refuted here, and some consequent remarks are made on the definition of moments for such particles. The discussion is confined to simple no-recoil models; i.e., degrees of freedom other than spin are suppressed.

II. Time reversibility and e.d.m.

Let $|\underline{V}\rangle$ denote the quasi-stationary state representing the steady exponential decay. Then there are charge and current distributions

$$\bar{\rho}(\underline{r}) = \langle \underline{V} | \rho(\underline{r}) | \underline{V} \rangle \quad \bar{\underline{j}}(\underline{r}) = \langle \underline{V} | \underline{j}(\underline{r}) | \underline{V} \rangle \quad (1)$$

where ρ and $\underline{j}$ are the charge and current density operators. For a particle of spin $\frac{1}{2}$ there are such distributions for each spin state. Now for a stable particle, and in a time reversible theory, the two spin states are time reverses of one another. Since ρ is even and $\underline{j}$ odd under time reversal, it follows that

$$\bar{\rho}(\underline{r})(\uparrow) = \bar{\rho}(\underline{r})(\downarrow) \quad \bar{\underline{j}}(\underline{r})(\uparrow) = -\bar{\underline{j}}(\underline{r})(\downarrow). \quad (2)$$

It is clear in particular that $\bar{\underline{Q}}$ can have no dipole moment, for this would have to change sign with rotation from one spin direction to the other. However, in the unstable case the time reverse of either decay state is an unphysical state of exponential growth; it is certainly not just the other physical state. One does not then arrive at the relations (2). This is

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the argument of Zel'dovich¹⁾, who shows moreover by an explicit example that $\bar{e}$ can have a dipole moment.

However, there is the following counter argument. An electric dipole moment should cause the spin to precess in an external electric field (which might be induced in practice by motion through a magnetic field³⁾). Suppose for simplicity that the field has an axis of symmetry, which is then the appropriate axis for spin quantization. Let

$$|v \uparrow\rangle \quad |v \downarrow\rangle$$

denote the two eigenstates for zero field and with the decay interaction switched off. Let $|t\rangle$ denote the time dependent state describing the decay process. The spinor

$$\begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix} = \begin{pmatrix} \langle v \uparrow | t \rangle \\ \langle v \downarrow | t \rangle \end{pmatrix} \quad (3)$$

may be used to specify the spin direction and the probability that the particle has not yet decayed. Moreover, the initial state can be supposed to have the form

$$|0\rangle = \alpha_1(0) |v \uparrow\rangle + \alpha_2(0) |v \downarrow\rangle \quad (4)$$

This, of course, really corresponds to switching on field and decay interaction suddenly after the state has been prepared. Strictly, one should instead give some account of the formation of the unstable particle in a reaction. However, we suppose that the error here affects only very small

(and very, very large ⁴) times. We have then a linear relation

$$\alpha_1(t) = u_{11}(t)\alpha_1(0) + u_{12}(t)\alpha_2(0)$$

$$\alpha_2(t) = u_{21}(t)\alpha_1(0) + u_{22}(t)\alpha_2(0)$$

where u is a sub-matrix of the complete evolution operator

$$U(t) = e^{-iHt}$$

Now, time reversibility implies that

$$\langle A | H | B \rangle = \langle A' | H | B' \rangle^*$$

where A' and B' are the time reverses of A and B . If H is also Hermitian we have then the reciprocity relation

$$\langle A | H | B \rangle = \langle B' | H | A' \rangle$$

From that follows the reciprocity relation for U :

$$\langle A | U(t) | B \rangle = \langle B' | U(t) | A' \rangle \quad (5)$$

This does not involve reversing the sign of t . Since $|v\uparrow\rangle$ and $|v\downarrow\rangle$ are time reverses of one another,

$$u_{11}(t) = u_{22}(t).$$

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Moreover, by conservation of angular momentum

$$u_{12}(t) = u_{21}(t) = 0.$$

It follows that

$$\frac{\alpha(t)}{\beta(t)} = \frac{\alpha(0)}{\beta(0)} = \text{constant}.$$

As long as the particle remains undecayed, the spin direction remains fixed.

The time reversibility would be spoiled by an external magnetic field, which is odd under time reversal. But the reversibility and reciprocity persist in an external electric field. Such a field, therefore, induces no spin precession. In that sense there is no electric dipole moment.

It is clear, therefore, that the expressions (1) are not directly relevant for the behaviour of the undecayed particle. This is also suggested by two internal difficulties with them. Firstly, it is not clear how $|\underline{V}\rangle$ should be normalised, for the norm of a quasi-stationary state does not exist in the usual sense. Secondly, if the outgoing decay products are charged, the distributions (1) extend to infinity and increase exponentially with distance - because the more distant parts were emitted earlier when the source was stronger.

III. Charge and current distribution

We look now for new definitions of charge and current density. One expects that shortly after formation the system settles into a combination of the quasi-stationary states associated with the two spin directions, and that this description remains good for many lifetimes ⁴⁾. Spin precession occurs if the phase relation between these two components is time dependent, i.e. if they have different energies. So we develop a perturbation theory for the effect of weak fields on the complex energies. We again assume a symmetry axis so that the appropriate direction of spin quantization is immediately known. Then the two spin states can be considered separately. The perturbation of the energy by weak scalar and vector fields has the general form

$$\delta E = \int d\mathbf{r} \left\{ \bar{\rho}(\mathbf{r}) \phi(\mathbf{r}) - \bar{\mathbf{j}}(\mathbf{r}) \cdot \mathbf{A}(\mathbf{r}) \right\} . \quad (6)$$

This defines functions $\bar{\rho}$, $\bar{\mathbf{j}}$, (assumed axially symmetric) which are the same as $\bar{\rho}$, $\bar{\mathbf{j}}$, in the stable case. But in the unstable case they are not the same.

We start from the steady state $|V\rangle$, with real energy E_0 , which would exist in the absence of the decay interaction and with zero applied field. Let H_0 be the part of the Hamiltonian so allowed for, and W the remainder. Then the perturbation expansion of the quasi-stationary state $|\underline{V}\rangle$ with complex energy E can be generated from the following expressions ⁵⁾:

$$|\underline{V}\rangle = \left\{ 1 + GW + GWGW + \dots \right\} |V\rangle \quad (7)$$

$$G = \frac{Q}{i\epsilon + E - H_0} \left\{ 1 + \left(\frac{E_0 - E}{i\epsilon + E_0 - H_0} \right) + \left(\frac{E_0 - E}{i\epsilon + E_0 - H_0} \right)^2 + \dots \right\} \quad (8)$$

$$(E - E_0) \langle V | \underline{V} \rangle = \langle V | W + WGW + WGWGW + \dots | V \rangle \quad (9)$$

where $(1-Q)$ is the projection operator on to $|V\rangle$. Now add to W an increment due to the external field:

$$\delta W = \int d\mathbf{r} \left\{ \rho(\mathbf{r}) \phi(\mathbf{r}) - \mathbf{j}(\mathbf{r}) \cdot \mathbf{A}(\mathbf{r}) \right\}.$$

To first order in δW ,

$$\begin{aligned} \langle V | V \rangle \delta E &= \langle V | \left\{ 1 + WG + WG^2 + \dots \right\} \delta W \left\{ 1 + GW + GW^2 + \dots \right\} | V \rangle \\ &+ \langle V | \left\{ W + WG^2 + \dots \right\} \delta G \left\{ W + WG^2 + \dots \right\} | V \rangle \end{aligned}$$

Moreover

$$\delta G = -G \delta E G$$

So, remembering the projection operator Q in G ,

$$\begin{aligned} \langle V | V \rangle \delta E &= \langle V | \left\{ 1 + WG + \dots \right\} (\delta W - \delta E) \left\{ 1 + GW + \dots \right\} | V \rangle + \\ &+ \langle V | V \rangle \delta E. \end{aligned}$$

It is convenient to introduce a state $|V'\rangle$ obtained by changing the sign of the infinitesimal δ everywhere in (7), (8), (9); $|V'\rangle$ is a solution of the Schrödinger equation with energy E^* and incoming wave boundary conditions. Then

$$\delta E = \langle V' | \delta W | V \rangle / \langle V' | V \rangle \quad (10)$$

and

$$\begin{aligned} \bar{Q} &= \langle V' | Q | V \rangle / \langle V' | V \rangle \\ \bar{j} &= \langle V' | j | V \rangle / \langle V' | V \rangle \end{aligned} \quad (11)$$

These differ from (1) in the explicit normalization, and in the appearance of the conjugate not of $|V\rangle$ but of $|V'\rangle$.

It is important to know how $\bar{\phi}$ and $\bar{j}$ fall off with distance, and whether moments can indeed be defined. Consider first the existence of the norm $\langle \underline{V}' | \underline{V} \rangle$. If there is just one outgoing particle (and that is the most acute case) it is represented asymptotically by a wave function

$$r^{-1} e^{ikr} = r^{-1} e^{ik_1 r + k_2 r}$$

where k_1 and k_2 are positive. We are not concerned with possible angular factors. The corresponding ingoing solution of the Schrödinger equation, with complex conjugate energy, is

$$r^{-1} e^{-ik_1 r + k_2 r}$$

So $\langle \underline{V}' | \underline{V} \rangle$ has a term involving

$$\int_0^{\infty} dr \left\{ e^{-ik_1 r + k_2 r} \right\}^* \left\{ e^{ik_1 r + k_2 r} \right\} = \int_0^{\infty} dr e^{2ik_1 r + 2k_2 r} \quad (12)$$

The integrand here differs from that for $\langle \underline{V} | \underline{V} \rangle$ by the oscillating factor $\exp(2ik_1 r)$. But that is not enough for the integral to exist in an ordinary sense. On the other hand we derived (10) (in so far as it can be called a derivation) by summing perturbation theory⁶⁾. In any finite order of perturbation theory $\exp(2k_2 r)$ is represented by a polynomial of finite degree. Arranging that k_1 is not altered by the perturbation, one meets then only convergent integrals of the type

$$\int dr r^n e^{2ik_1 r - \epsilon r}$$

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where the infinitesimal damping arises from the $i\epsilon$ of energy denominators. The result of summing these is obtained equally by introducing a damping factor $\exp(-\epsilon r)$ into (12), which then converges for big enough ϵ , and putting $\epsilon = 0$ in the resulting analytic expression for the integral. Thus

$$\int_0^{\infty} dr e^{2ikr} = -1/2ik.$$

In this sense the norm exists.

When the outgoing particle is charged $\bar{\rho}$ and $\bar{j}$ themselves involve, from the numerators in (11), parts which oscillate with exponentially increasing amplitude at large distances. Integrals defining moments then exist only in the sense described above for the norm, in which

$$\int dr r^n e^{2ikr} = \left(\frac{d}{d2ik} \right)^n \left(\frac{-1}{2ik} \right).$$

The question arises: are moments defined in this formal way really relevant for the reaction of the system to external fields? That this is so can be made plausible by the following considerations. For brevity we speak only of $\bar{\rho}$.

The origin of the factor $\exp(2ikr)$ in $\bar{\rho}$ is clear: it is the (complex) phase delay associated with the propagation of signals to r and back. It is also clear that in so far as the field at r contributes to δE , the exponential regime to which δE refers cannot be established until the lapse of a corresponding time. Indeed, if the field extends to infinity the exponential regime is never established. A more useful account, therefore,

would involve a "time dependent energy shift", referring to the region near the origin, with a linear relation to the external potential of the general form

$$\delta E(t) = \int dr \, L(\underline{r}, t) \phi(\underline{r}) \quad (13)$$

The function L would be zero for r greater than about $ct/2$ (where c is the velocity of light), small for r greater than about $vt/2$ (where v is the group velocity of the outgoing wave), and then would approach $\bar{\bar{Q}}(r)$. At very large times it would again differ from $\bar{\bar{Q}}$ because even a well localized system eventually decays non-exponentially; we do not consider such very long times. The essential features of (13) are simulated by

$$\delta E(t) = \int_{r < R(t)} dr \, \bar{\bar{Q}}(r) \phi(r) \quad (14)$$

where for the moment we take a sharp upper limit. We study this integral with the asymptotic form for $\bar{\bar{Q}}$, again not bothering about possible angular factors:

$$\int_0^R dr \, e^{2ikr} \phi(r) \quad (15)$$

By partial intergration this can be written

$$-\left(\frac{\phi(0)}{2ik}\right) + \left(\frac{\phi(R)}{2ik}\right) e^{2ikR} - \int_0^R dr \, \frac{e^{2ikr}}{2ik} \frac{d\phi(r)}{dr}.$$

By repeated partial integration in this way, a series can be developed involving derivatives of ϕ at the origin and at R . The latter terms contain oscillatory factors $\exp(2ik_1R)$. Assuming the derivatives at the origin and at R to be of the same order of magnitude, the amplitudes of the oscillatory terms exceed the magnitudes of the corresponding non-oscillatory terms by factors

$$e^{2k_2R} \quad (16)$$

However, these large oscillations come entirely from the artificially sharp cut-off at the upper limit. In a proper theory these oscillations are modified by, among other things, the fact that the system cannot really be formed instantaneously. Correspondingly we should average $\exp(2ikR)$ over some interval of R ⁷⁾. A Gaussian averaging with width d reduces the oscillation amplitude (16) to

$$e^{-k_1^2 d^2} e^{2k_2R}.$$

If d contains several wavelengths this is quite negligible until k_2R is of order $k_1^2 d^2$, i.e. until after many lifetimes. In the meantime ϕE is essentially constant. Moreover, it is given by a series in the derivatives of ϕ at the origin, whose coefficients are just the moments calculated from (11) in the formal way already described. The effective extension of the system, as manifested in the relative magnitudes of successive moments, is k_1^{-1} .

IV. Conclusion

It remains to confirm that with the new expressions (11) there is no electric dipole moment. Introducing spin explicitly, it is clear that in a time reversible theory $|\underline{v}'\uparrow\rangle$ and $|\underline{v}'\downarrow\rangle$ are the time reverses of $|\underline{v}\downarrow\rangle$ and $|\underline{v}\uparrow\rangle$. It then follows that

$$\begin{aligned}\langle \underline{v}'\uparrow | \varrho | \underline{v}\uparrow \rangle &= \langle \underline{v}\downarrow | \varrho | \underline{v}'\downarrow \rangle^* \\ &= \langle \underline{v}'\downarrow | \varrho | \underline{v}\downarrow \rangle.\end{aligned}$$

So we again have the relations (2), with an extra bar.

It should be remarked that in relativistic theory, using Feynman diagrams in a conventional way, one arrives automatically at results analogous to (11) rather than (1)⁸⁾. The non-appearance of electric dipole moments follows there from the reciprocity relation for propagators. However, as is clear from (11), magnetic moments and higher electric moments are in general complex. This means simply that decay rate is affected by applied fields.

In conclusion we repeat that the quantities (1), although in a simple sense the charge and current densities for the complete system, do not govern the dynamics of the undecayed particle. For example, if one computes the rate at which a uniform electric field delivers angular momentum to the system, the result involves the dipole moment of $\bar{\varrho}$. But in a time reversible theory the spin of the undecayed particle does not precess; the angular momentum is delivered entirely to the decay products.

I am indebted to Professors S.M. Berman, J.M. Jauch, and T.D. Lee for discussion of this subject.

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